MTH 621: Homework # 1, due 10/15, in class

To receive full credit, present complete answers that show all work.

Problem 1 (10 points). Consider the continuity equation for $\rho(x,t)$,

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{1}$$

where $x \in \mathcal{R}^n, t > 0$, and $\mathbf{v}(x,t) \in \mathcal{R}^n$ is a given vector field on Ω. Let $\rho_A(x,t)$ and $\rho_B(x,t)$ solutions to (1), and assume that $\rho_B(x,t) > 0$. Show that the ratio ρ_A/ρ_B solves the PDE

$$\rho_t + \mathbf{v} \cdot \nabla \rho = 0 \tag{2}$$

Problem 2 (10 points). Let Ω an open, bounded domain in $\mathcal{R}^n$ with smooth boundary partitioned as $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Consider the Poisson's boundary value problem for $u : \Omega \to \mathcal{R}$

$$\begin{aligned} -\Delta u(x) &= f(x), \ \ x \in \Omega & (3)\\ u(x) &= g_1(x), \ \ x \in \Gamma_1 & (4)\\ -\nabla u(x) \cdot \nu(x) &= g_2(x), \ \ x \in \Gamma_2 & (5)\\ -\nabla u(x) \cdot \nu(x) &= h(x)[u(x) - u_B(x)], \ \ x \in \Gamma_3 & (6) \end{aligned}$$

where f, g_1, g_2, h, u_B are given functions and ν denotes the outward unit vector normal to boundary. In addition, assume that $h(x) \geq 0, x \in \Gamma_3$. Show that there is at most one smooth solution to the problem (3) - (6).

Problem 3 (15 points). Let Ω an open, bounded domain in $\mathcal{R}^n$ with smooth boundary partitioned as $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Consider the heat equation for $u : \Omega \times [0, \infty) \to \mathcal{R}$, $u = u(x,t)$:

$$u_t(x,t) - \Delta u(x,t) = f(x,t), \ \ (x,t) \in \Omega \times (0, \infty) \tag{7}$$

with the initial condition

$$u(x,0) = u_0(x), \ \ x \in \Omega \tag{8}$$

and boundary conditions

$$\begin{aligned} u(x,t) &= g_1(x,t), \ \ (x,t) \in \Gamma_1 \times [0,\infty) & (9)\\ -\nabla u(x,t) \cdot \nu(x) &= g_2(x,t), \ \ (x,t) \in \Gamma_2 \times [0,\infty) & (10)\\ -\nabla u(x,t) \cdot \nu(x) &= h(x)[u(x,t) - u_B(x,t)], \ \ (x,t) \in \Gamma_3 \times [0,\infty) & (11) \end{aligned}$$

where f, g_1, g_2, h, u_B are given functions and ν denotes the outward unit vector normal to boundary. In addition, assume that $h(x) \geq 0, x \in \Gamma_3$. Show that there is at most one smooth solution to the problem (7) - (11).

Problem 4 (15 points). Let Ω an open, bounded domain in $\mathcal{R}^n$ with smooth boundary partitioned as $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Consider the wave equation for $u : \Omega \times [0, \infty) \to \mathcal{R}$, $u = u(x, t)$:

$$u_{tt}(x,t) - \Delta u(x,t) = f(x,t), \ (x,t) \in \Omega \times (0,\infty) \tag{12}$$

with the initial conditions

$$u(x,0) = u_0(x), \ x \in \Omega \tag{13}$$
$$u_t(x,0) = v_0(x), \ x \in \Omega \tag{14}$$

and boundary conditions

$$u(x,t) = g_1(x,t), \ (x,t) \in \Gamma_1 \times [0,\infty) \tag{15}$$
$$-\nabla u(x,t) \cdot \nu(x) = g_2(x,t), \ (x,t) \in \Gamma_2 \times [0,\infty) \tag{16}$$
$$-\nabla u(x,t) \cdot \nu(x) = h(x)[u(x,t) - u_B(x,t)], \ (x,t) \in \Gamma_3 \times [0,\infty) \tag{17}$$

where f, g_1, g_2, h, u_B are given functions and ν denotes the outward unit vector normal to boundary. In addition, assume that $h(x) \geq 0, x \in \Gamma_3$. Show that there is at most one smooth solution to the problem (12) - (17).