Weak derivatives, Sobolev spaces

Definition Let $u \in L^1_{loc}(\Omega)$. A function $g \in L^1_{loc}(\Omega)$ is the i-directional "weak derivative" of u if

$$\int_\Omega u \phi_{x_i} \, dx = - \int_\Omega g \phi \, dx , \quad \forall \phi \in C_0^\infty(\Omega)$$

Remark: Notice that we do not require u to be continuously differentiable

Example:
$$u(x) = \begin{cases} x, & 0 < x \leq 1 \\ 1, & 1 \leq x < 2 \end{cases}$$

then $u'(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ is the weak derivative

i.e., $\int_0^2 u \phi' \, dx = - \int_0^2 u' \phi \, dx , \quad \forall \phi \in C_0^\infty(0,2)$.

Indeed,
$$\int_0^2 u\phi' = \int_0^1 x \phi'(x) + \int_1^2 \phi'(x) =$$
$$= x\phi(x)\Big|_0^1 - \int_0^1 \phi(x) + \underbrace{\phi(2) - \phi(1)}_{=0}$$
$$= - \int_0^1 \phi(x) \, dx , \quad \forall \phi \in C_0^\infty(0,2)$$

and $-\int_0^2 u'\phi = - \int_0^1 \phi(x) \, dx , \quad \forall \phi \in C_0^\infty(0,2)$ $\Bigg) =$

<u>Remark</u>: The weak derivative is unique up to a set of measure zero.

<u>Proof</u> Let $g_i, \tilde{g}_i$ such that

$$\int_\Omega u\phi_{x_i} = -\int_\Omega g_i\phi = -\int_\Omega \tilde{g}_i\phi, \quad \forall \phi \in C_c^\infty(\Omega)$$

$$\Rightarrow \int_\Omega (g_i - \tilde{g}_i)\phi = 0; \ \forall \phi \in C_c^\infty(\Omega) \quad \Rightarrow \quad g_i = \tilde{g}_i \text{ a.e.}$$

(since C_c^∞ is dense)

<u>Remark</u>: if $u \in C^1(S)$, $S \subset \Omega$ then the weak derivatives coincide with the strong derivatives

$$g|_S = \nabla u|_S \quad (\text{a.e.}).$$

$$\int_S u\phi_{x_i} = -\int_S u_{x_i}\phi = -\int_S g_i\phi, \quad \forall \phi \in C_c^\infty(\Omega)$$

$$\Rightarrow \quad u_{x_i} = g_i \quad (\text{a.e.}).$$

<u>Remark</u>: Not all functions have weak derivatives

Let $u(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2, & 1 < x \le 2 \end{cases}$ then $\int_0^2 u\phi' = -\int_0^1 \phi\, dx - \phi(1)$

if $g(x)$ is the weak derivative then $g(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$

$$-\int_0^2 g\phi = -\int_0^1 \phi\, dx \neq \int_0^2 u\phi' \text{ if } \phi(1) \neq 0.$$

Definition Let $1 \le p \le \infty$. The Sobolev space $W^{1,p}(\Omega)$ is defined as the set of ~~th~~ those $L^p(\Omega)$ functions whose weak derivatives exist and are L^p functions.

$$W^{1,p}(\Omega) = \Big\{ u \in L^p(\Omega) : \exists u_{x_1}, \ldots, u_{x_n} \in L^p(\Omega) \text{ such that}$$
$$\int_\Omega u \phi_{x_i} = - \int_\Omega u_{x_i} \phi, \quad \forall \phi \in C_0^\infty(\Omega) \Big\}$$

We denote $\boxed{H^1(\Omega) = W^{1,2}(\Omega)}$

The norm on $W^{1,p}$ is defined as

$$\|u\|_{W^{1,p}} = \left[\|u\|_{L^p}^p + \sum_{i=1}^n \|u_{x_i}\|_{L^p}^p \right]^{\frac{1}{p}} =$$
$$= \left[\int_\Omega |u|^p + \sum_{i=1}^n \int_\Omega |u_{x_i}|^p \right]^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

$$\|u\|_{W^{1,\infty}} = \|u\|_{L^\infty} + \sum_{i=1}^n \|u_{x_i}\|_{L^\infty}$$

in particular, $H^1(\Omega)$ is equipped with the inner product

$$(u, v)_{H^1} = \int_\Omega u v + \sum_{i=1}^n \int_\Omega u_{x_i} v_{x_i}$$

such that $\|u\|_{H^1} = (u, u)_{H^1}^{\frac{1}{2}}$

<u>Property</u>: $(W^{1,p}, \| \, \|_{1,p})$ is a Banach space, $1 \le p \le \infty$

<u>Proof</u>: Let $\{u_n\}$ Cauchy sequence in $W^{1,p}$

then $\|u_n - u_m\|_{L^p} + \sum_{i=1}^{n} \|u_{n,x_i} - u_{m,x_i}\|_{L^p} < \varepsilon, \; n,m \ge N(\varepsilon)$

$\Rightarrow \begin{cases} \{u_n\} \text{ Cauchy in } L^p \Rightarrow u_n \longrightarrow u \in L^p \\ \{u_{n,x_i}\} \text{ Cauchy in } L^p \Rightarrow u_{n,x_i} \to u_{x_i} \in L^p \end{cases}$

Need to verify that u_{x_i} are the weak derivatives

$\forall \phi \in C_c^\infty(\Omega)$ $\quad \int_\Omega u_n \phi_{x_i} = -\int_\Omega u_{n,x_i} \phi$

$$\Big\downarrow u_n \to u \qquad\qquad \Big\downarrow u_{n,x_i} \to u_{x_i}$$

$$\int_\Omega u \phi_{x_i} = -\int_\Omega u_{x_i} \phi$$

So, $u_{x_i} \in L^p$ are weak derivatives.

in particular, $(H^1, \| \, \|_1)$ is <u>Hilbert</u>.

Definition Let $1 \le p \le \infty$. The Sobolev space $W_0^{1,p}(\Omega)$ is defined as

$$W_0^{1,p}(\Omega) = \overline{\{C_c^\infty(\Omega), \| \cdot \|_{1,p}\}}$$

(the closure of $C_c^\infty(\Omega)$ in $W^{1,p}$ w.r.t. $\| \ \|_{1,p}$).

$u \in W_0^1(\Omega)$ if there is $\{u_n\} \subset C_c^\infty(\Omega)$:

$$\| u_n - u \|_{W^{1,p}} \longrightarrow 0 .$$

Remark : $W_0^{1,p}(\Omega)$ are those functions in $W^{1,p}$ with zero values on the boundary $\partial\Omega$.

$$u|_{\partial\Omega} = 0 .$$

---

in particular, $(H_0^1(\Omega), \| \ \|_1)$ is Hilbert.

Variational formulation of the BVP

$$(*)\quad \begin{cases} -\Delta u + u = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

For all $v \in C_0^\infty(\Omega)$: $\int_\Omega \nabla u \nabla v + \int_\Omega uv = \int_\Omega fv$

For all $v \in H_0^1(\Omega)$, let $\{v_n\} \subset C_0^\infty(\Omega)$, $v_n \xrightarrow[\|\,\|_{H^1}]{} v$, then

(**) Find $u \in H_0^1(\Omega)$ such that

$$\boxed{\int_\Omega \nabla u \nabla v + \int_\Omega uv = \int_\Omega fv, \quad \forall v \in H_0^1(\Omega)}$$

(*) is called the "strong" (classical) problem

(**) is called the "weak" (variational) problem

Application of the Riesz theorem:

Theorem (existence and uniqueness) Let $f \in L^2(\Omega)$ then there is an unique solution $u \in H_0^1(\Omega)$ to the variational problem. In addition, $T: L^2(\Omega) \to H_0^1(\Omega)$, $T(f) = u$ is a linear and continuous operator.