Trace theorem (ref. Evans)

Remark: $\Omega \subset \mathbb{R}^n$ then $\partial\Omega$ is of n-dimensional Lebesgue measure zero. Since the elements of $H^1(\Omega)$ are defined a.e. in Ω, it is not clear what $v|_{\partial\Omega}$ means. To give a meaning to the boundary values on $H^1(\Omega)$ we introduce the "trace operator".

Trace theorem Assume $\Omega \subset \mathbb{R}^n$ is bounded with $\partial\Omega$ of class C^1. Then there is a linear and continuous operator

$$T : H^1(\Omega) \longrightarrow L^2(\partial\Omega)$$

such that

i) $Tu = u|_{\partial\Omega}$ if $u \in H^1(\Omega) \cap C(\bar{\Omega})$

ii) $\|Tu\|_{L^2(\Omega)} \leq C\|u\|_{H^1(\Omega)}$, $\forall u \in H^1(\Omega)$

Definition We call Tu the trace of u on $\partial\Omega$

denote $Tu = u|_{\partial\Omega}$

Let $u \in H^1(\Omega)$, $\exists \{u_k\} \subset C^1(\bar{\Omega})$: $u_k \to u$ in H^1

Then $Tu_k = u_k\big|_{\partial\Omega}$ is Cauchy sequence in $L^2(\partial\Omega)$

$Tu_k \longrightarrow Tu \stackrel{def}{=} u\big|_{\partial\Omega}$

<u>Simple example</u> Let $\Omega = (0,1)\times(0,1)$. Show that there is a constant c such that $\forall v \in C^1(\bar{\Omega})$

$$\left[\int_{\partial\Omega} v^2\,ds\right]^{\frac{1}{2}} \le c\left[\int_\Omega v^2 + \int_\Omega |\nabla v|^2\right]^{\frac{1}{2}} \qquad (*)$$

<u>Proof</u> Let $v \in C^1(\bar{\Omega})$.

$v(x,y) = v(x,0) + \int_0^y v_y(x,s)\,ds \Rightarrow |v(x,0)| \le |v(x,y)| + \int_0^y |v_y(x,s)|\,ds$

$\Rightarrow |v(x,0)| \le |v(x,y)| + \int_0^1 |v_y(x,s)|\,ds$

$\Rightarrow |v(x,0)|^2 \le 2\left[\,|v(x,y)|^2 + \left[\int_0^1 |v_y(x,s)|\,ds\right]^2\right] \le$

$\le 2\left[\,|v(x,y)|^2 + \int_0^1 v_y^2(x,s)\,ds\,\right]$

$\Rightarrow \int_0^1 |v(x,0)|^2\,dx \le 2\left[\int_0^1 |v(x,y)|^2\,dx + \int_0^1\int_0^1 v_y^2(x,y)\,dy\,dx\right]$

$\Rightarrow \int_0^1 |v(x,0)|^2\,dx \le 2\,\|v\|_1^2 \Rightarrow \|v\|_{L^2(\partial\Omega)} \le 2\sqrt{2}\,\|v\|_1$

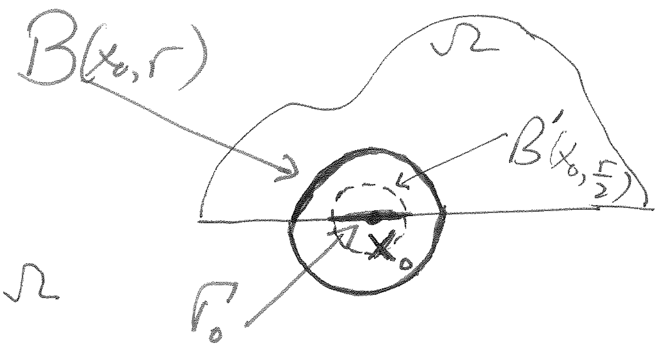
By density of $C^1(\bar{\Omega})$ in $H^1(\Omega)$, $(*)$ holds in $H^1(\Omega)$.

Proof of the trace theorem

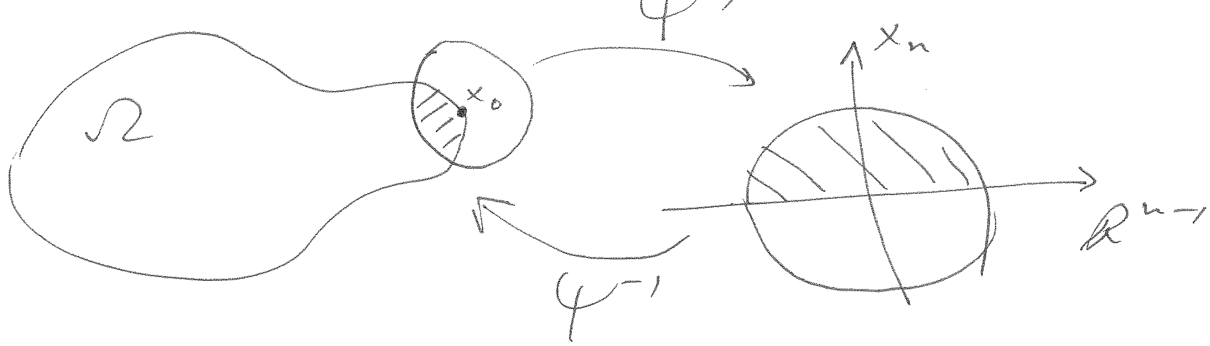
Since $\partial\Omega$ is C^1, we may assume that for any $x_0 \in \partial\Omega$ there is an open ball $B(x_0, r)$ such that

$$B^+ = B \cap \{x_n \geq 0\} \subset \bar{\Omega}$$

$$B^- = B \cap \{x_n < 0\} \subset \mathbb{R}^n \setminus \Omega$$



[$\partial\Omega$ is flat near x_0, lying in the plane $x_n = 0$, otherwise we can transform the domain.



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Let $B' = B(x_0, \frac{r}{2})$

By Urysohn's lemma, there is a function $\xi \in C_c^\infty(B)$ such that $\xi(x) \geq 0$ in B and $\xi(x) \equiv 1$ on B'.

Let $\Gamma_0 \overset{def}{=} \partial\Omega \cap B'$, $x' \overset{not}{=} (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$

$$\int_{\Gamma_0} |u|^2\,dx' \le \int_{\{x_n=0\}} \xi |u|^2\,dx' = -\int_{B^+} \frac{\partial}{\partial x_n}\left[\xi |u|^2\right] dx$$

$$= -\int_{B^+} |u|^2 \frac{\partial \xi}{\partial x_n} + 2u \frac{\partial u}{\partial x_n} \xi \, dx$$

$$\le c \int_{B^+} |u|^2 dx + c_1 \int_{B^+} u^2 + [u_{x_n}]^2\,dx \le \tilde{c} \int_{B^+} u^2 + |Du|^2$$

Since $\partial\Omega$ is compact set, there are a finite number of points $x_i \in \partial\Omega$ and open subsets $\Gamma_i \subset \partial\Omega$, $i = 1:N$ such that

$$\partial\Omega = \bigcup_{i=1}^{N} \Gamma_i$$

and $\|u\|_{L^2(\Gamma_i)} \le C_i \|u\|_{H^1(\Omega)}$

Let $C = \sqrt{C_1^2 + \cdots + C_N^2}$ and $Tu = u|_{\partial\Omega}$

Then $\|Tu\|_{L^2(\partial\Omega)} \le C\|u\|_{H^1(\Omega)}$

By density of $C^\infty(\bar{\Omega})$ in $H^1(\Omega)$ the result holds in $H^1(\Omega)$.

Let $u \in H^1(\Omega)$. There is a sequence $u_m \in C^\infty(\bar{\Omega})$ such that $u_m \to u$ in $H^1(\Omega)$.

Then $\|Tu_m - Tu_k\|_{L^2(\partial\Omega)} \leq C \|u_m - u_k\|_{H^1(\Omega)}$

$\Rightarrow \{Tu_m\}_m$ is Cauchy sequence in $L^2(\partial\Omega)$

Define $Tu = \lim_m Tu_m \in L^2(\partial\Omega)$.

$$\begin{array}{ccc} \|Tu_m\|_{L^2(\partial\Omega)} & \leq C & \|u_m\|_{H^1(\Omega)} \\ \downarrow & & \downarrow \\ \|Tu\|_{L^2(\partial\Omega)} & \leq C & \|u\|_{H^1(\Omega)} \end{array}$$

Notice that the definition is independent on the choice of $\{u_m\}$.

For any $w_m \in C^\infty(\bar{\Omega})$, $w_m \to u$ in $H^1(\Omega)$,

$$\|Tw_m - Tu_k\|_{L^2(\partial\Omega)} \leq C \|w_m - u_k\|_{H^1(\Omega)} \xrightarrow[m,k\to\infty]{} 0$$