The obstacle problem. Stampacchia's Theorem

Variational inequalities

Consider an elastic membrane attached to a flat wire frame that encloses a region $\Omega \subset \mathbb{R}^2$. Assume that there is a load force $f(x,y)$. Equilibrium solution is given by the Poisson problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

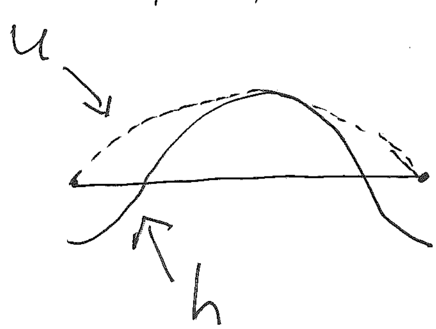
Variational formulation:

Find $u \in H_0^1(\Omega)$: $\int_\Omega \nabla u \nabla v = \int_\Omega f v, \quad \forall v \in H_0^1(\Omega)$

Equivalent to the minimization problem

$$u = \arg \min_{H_0^1(\Omega)} J(v), \quad J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega f v$$

The obstacle problem Assume that an obstacle given by the surface $h(x,y)$ is placed under the membrane. Assume that h is smooth function such that $h|_{\partial\Omega} < 0$ (if $f \equiv 0$ then assume $\max_{\Omega} h > 0$)



Consider the set

$$K = \{ v \in H_0^1(\Omega) : v \geq h \text{ a.e. in } \Omega \}$$

Minimization problem: $\min_{v \in K} J(v)$

<u>Property</u> K is a <u>closed</u> and <u>convex</u> subset of $H_0^1(\Omega)$.

<u>Proof</u> i) K is closed in $H_0^1(\Omega)$.

Let $\{v_n\} \subset K$, $v_n \to v$ in $H_0^1(\Omega)$.

By contradiction, <u>assume $v \notin K$</u>.

Let $A = \{x \in \Omega : v(x) < h(x)\}$

Then $\mu(A) > 0$ (A has positive measure)

Then there is $\varepsilon > 0$ and a set $A_\varepsilon \subset \Omega$ such that $\mu(A_\varepsilon) > 0$ and $v \le h - \varepsilon$ on A_ε

<u>Remark</u> $A = \bigcup_{k=1}^{\infty} \{x \in \Omega : v(x) < h(x) - \frac{1}{k}\} = \bigcup_{k=1}^{\infty} A_k$

$\Rightarrow \exists k : \mu(A_k) > 0$.

Then $\int_\Omega (v_n - v)^2 \ge \int_{A_\varepsilon} (v_n - v)^2 \ge \int_{A_\varepsilon} \varepsilon^2 = \varepsilon^2 \mu(A_\varepsilon) > 0$

contradiction with $v_n \to v$ in $H_0^1(\Omega)$.

So <u>K is closed</u> in $H_0^1(\Omega)$.

ii) <u>K is convex</u>: <u>obvious</u>

$v_1 \ge h$, $v_2 \ge h$ a.e. $\Rightarrow t v_1 + (1-t) v_2 \ge h$ a.e. $t \in [0,1]$

Stampacchia's theorem

Let H Hilbert space, $a(\cdot,\cdot) : H \times H \to \mathbb{R}$

i) bilinear

ii) continuous $|a(u,v)| \leq M\|u\| \|v\|$, $\forall u,v \in H$

iii) elliptic : $\exists \alpha > 0 : a(v,v) \geq \alpha \|v\|^2$, $\forall v \in H$

Let $K \subset H$ convex and closed subset (non empty)

Then for any function $f : H \to \mathbb{R}$ linear and continuous there is a unique $\underline{u \in K}$ such that

$$a(u, v-u) \geq f(v-u), \quad \underline{\forall v \in K} \quad (V)$$

in addition, if $a(\cdot,\cdot)$ is symmetric then (V) is equivalent with the minimization problem

$$\begin{cases} u \in K \\ \frac{1}{2} a(u,u) - f(u) = \min\limits_{v \in K} \left\{ \frac{1}{2} a(v,v) - f(v) \right\} \end{cases}$$

Corollary (Lax-Milgram) if $K = H$

Let $\varepsilon \in \mathbb{R}$ arbitrary, $w \in H$ arbitrary.

Let $v = u + \varepsilon w$

$a(u, \varepsilon w) \geq f(\varepsilon w) \Rightarrow \varepsilon \left[a(u,w) - f(w) \right] \geq 0$

since ε arbitrary $\Rightarrow a(u,w) = f(w)$, $\forall w \in H$

Proof of Stampacchia's theorem

Uniqueness: if u_1, u_2 solutions then

$$a(u_1, u_2-u_1) \geq f(u_2-u_1)$$
$$a(u_2, u_1-u_2) \geq f(u_1-u_2)$$

(+) $\quad a(u_2-u_1, u_1-u_2) \geq 0$

(iii)

$$\Rightarrow \quad a(u_1-u_2, u_1-u_2) \leq 0 \quad \Rightarrow u_1 = u_2$$

Existence Riesz theorem implies

$\exists!\, b \in H : f(v) = (b, v)$

Let $u \in H$ arbitrary, fixed.

$v \longrightarrow a(u,v)$ linear and continuous

$\exists!\, Au \in H : \quad a(u,v) = (Au, v), \quad \forall v \in H.$

Let $A : H \to H, \quad A(u) = Au$

Then A is linear and continuous operator

$\|Au\| \leq M\|u\| \quad$ (since $|(Au,v)| \leq M\|u\|\|v\|$)

$(Au, u) \geq \alpha \|u\|^2$

(v) problem is equivalent to

(*) Find $u \in K$: $(Au, v-u) \geq (b, v-u), \; \forall v \in K$

Let $\rho > 0$ constant, arbitrary fixed.
Then $(*)$ is equivalent to

$u \in K$: $\quad (\rho(b - Au) + u - u, v - u) \leq 0, \forall v \in K$

Therefore we must have (<u>projection theorem</u>)

$$u = P_K(\rho(b - Au) + u)$$

and u is a fixed point to the operator $T(v) = P_K(\rho(b - Av) + v)$

We show that there is $\rho > 0$ such that T <u>strict</u> contraction

$$\|T(v_1) - T(v_2)\|^2 \overset{\text{by } P_K \text{ contraction property}}{\leq} \|v_1 - v_2 - \rho(Av_1 - Av_2)\|^2$$

$$= \|v_1 - v_2\|^2 - 2\rho(v_1 - v_2, Av_1 - Av_2) + \rho^2 \|Av_1 - Av_2\|^2$$

Recall $\|Av_1 - Av_2\| \leq M \|v_1 - v_2\|$

$$(Av_1 - Av_2, v_1 - v_2) = a(v_1 - v_2, v_1 - v_2) \geq \alpha \|v_1 - v_2\|^2 ;$$

Then,

$$\|T(v_1)-T(v_2)\|^2 \le \overbrace{\left(1-2\rho\alpha+\rho^2 M^2\right)}^{\gamma} \|v_1-v_2\|^2$$

Let $\rho = \frac{\alpha}{M^2}$ (ok $0 < \rho < \frac{2\alpha}{M^2}$)

then $\gamma = 1 - \frac{\alpha^2}{M^2} < 1$ thus

T is strictly contractive $\Rightarrow \exists!$ fixed point u.

Equivalence $(V) \Leftrightarrow (M)$ if $a(\cdot,\cdot)$ is symmetric

Then $a(\cdot,\cdot)$ defines an inner product $(u,v)_a = a(u,v)$ with induced norm

$$\alpha\|u\|^2 \le \|u\|_a^2 = a(u,u) \le M\|u\|^2$$

thus $\|\cdot\|_a \sim \|\ \|$ equivalent norms.

Then $\exists!\, g \in H : f(v) = a(g,v),\ \forall v \in H$ such that

$a(u, v-u) \geq f(v-u)$ is equivalent to

$$a(u, v-u) \geq a(g, v-u)$$

and (V) problem is equivalent to

$$\begin{cases} a(g-u,\, v-u) \leq 0, & \forall v \in K \\ u \in K \end{cases}$$

thus $u = P_K g$ w.r.t. $(\cdot,\cdot)_a$ and (V) equivalent to

$$\begin{cases} u \in K \\ \|g-u\|_a = \min\limits_{v\in K} \|g-v\|_a \end{cases}$$

$$\|g-u\|_a^2 = \min_{v\in K} \|g-v\|_a^2$$

equivalent to $u \in K$ minimizes on K

$$a(v,v) - 2\underbrace{a(g,v)}_{f(v)} + \underbrace{a(g,g)}_{\text{constant}}$$

thus $u \in K$ minimizes on K

$$\frac{1}{2} a(v,v) - f(v).$$

Stampacchia's theorem shows existence and uniqueness of the minimization problem associated with the obstacle problem.

[A direct proof may be given using convexity and coercivity of $J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega f v$ to be discussed later ...]

What boundary value problem corresponds to the minimization problem?

if u is smooth ($u \in H^2(\Omega)$) the BVP is

thus continuous since in 2D: $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$

$$\begin{cases} -\Delta u \geq f & \text{in } \Omega \\ u \geq h & \text{in } \Omega \\ (-\Delta u - f)(u-h) = 0 & \text{in } \Omega \\ u = h & \text{on } \Gamma \\ \nabla u = \nabla h & \text{on } \Gamma \end{cases}$$

where $\Gamma = \{x \in \Omega : u(x) = h(x)\} \cap \{x \in \Omega : u(x) > h(x)\}$ is the boundary of the set $\{x \in \Omega : u(x) = h(x)\}$ and is unknown. free boundary problem!

<u>Proof</u> Show that $-\Delta u \geq f$ a.e. in Ω

Let $v \in C_0^\infty(\Omega)$, $v \geq 0$ and $\varepsilon > 0$ arbitrary.

Then $w = u + \varepsilon v \in K$

$$\Rightarrow \varepsilon \left[\int_\Omega \nabla u \nabla v - \int_\Omega f v \right] \geq 0$$

$$\Rightarrow \int_\Omega (-\Delta u - f) v \geq 0, \quad \forall v \in C_0^\infty(\Omega), \; v \geq 0$$

$$\Rightarrow -\Delta u - f \geq 0 \quad \text{a.e. in } \Omega.$$

Show that $\nabla u = \nabla h$ on $\Gamma = \partial\{x : u(x) = h(x)\}$

Remark: $\Gamma \cap \partial\Omega = \emptyset$ since $h|_{\partial\Omega} < 0$.

Let $x_0 \in \Gamma$, define $v = u - h$

then $v(x_0) = u(x_0) - h(x_0) = 0$

$$\left.\begin{array}{l} \\ \text{and } v(x) = u(x) - h(x) \geq 0, \; \forall x \in \Omega \end{array}\right\} \Rightarrow$$

$\Rightarrow$ x_0 is min point to $v : \Omega \to \mathbb{R}$ in the <u>interior</u> of Ω, thus

$$\nabla v(x_0) = 0 \quad \text{i.e.} \quad \nabla u(x_0) = \nabla h(x_0)$$

Prove that $(-\Delta u - f)(u-h) = 0$ in Ω

Let $\mathcal{O}_h = \{x \in \Omega : u(x) > h(x)\}$

Then $\mathcal{O}_h \subset \Omega$ open set with boundary

$\partial\mathcal{O}_h = \{x \in \Omega : u(x) = h(x)\} \cap \bar{\mathcal{O}}_h$ is closed

subset of Ω (preimage of $u - h = 0$)

Show that

$$-\Delta u = f \text{ in } \mathcal{O}_h$$

(if the obstacle is not active then PDE is satisfied!)

Let $v \in C_0^\infty(\mathcal{O}_h)$, denote $M = \|v\|_\infty$ (max norm)

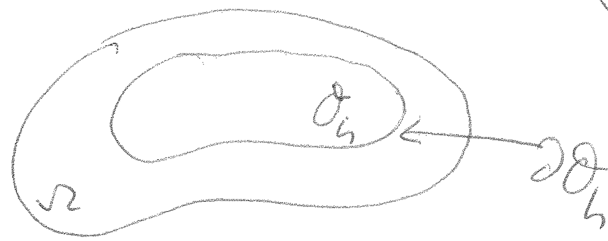
Since u is continuous on $\underbrace{\operatorname{supp}(v)}_{\text{(compact)}} \subset \mathcal{O}_h$

$u - h > 0$ on $\operatorname{supp}(v) = \overline{\{x \in \mathcal{O}_h : v(x) \neq 0\}}$

thus

$$\underbrace{\min_{x \in \operatorname{supp}(v)} \{u(x) - h(x)\}}_{\text{denote } m} > 0$$

Define $\varepsilon_0 = \dfrac{m}{M}$.



For any ε with $|\varepsilon| \le \varepsilon_0$ we have

$$(u+\varepsilon v) - h = u - h + \varepsilon v \ge m - |\varepsilon| M \ge 0$$

thus $w_\varepsilon \stackrel{def}{=} u + \varepsilon v \in K$ (admissible test function)

$$\int_\Omega \nabla u \nabla (w_\varepsilon - u) - \int_\Omega f (w_\varepsilon - u) \ge 0$$

$\Rightarrow \varepsilon \int_\Omega [\nabla u \nabla v - f v] \ge 0$ but ε can be $\pm$

$\Rightarrow \int_\Omega \nabla u \nabla v = \int_\Omega f v, \quad \forall v \in C_0^\infty(\mathcal{O}_h)$

thus u is weak solution to

$$\begin{cases} -\Delta u = f & \text{in } \mathcal{O}_h \\ u = h & \text{on } \partial\mathcal{O}_h \end{cases}$$