Poisson's problem, Poincaré inequality

Consider $\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^n \text{ bounded domain} \\ u = 0 & \text{on } \partial\Omega \end{cases}$

Weak (variational) formulation

$$\boxed{\text{Find } u \in H_0^1(\Omega) : \int_\Omega \nabla u \nabla v \, dx = \int_\Omega f v \, dx, \quad \forall v \in H_0^1(\Omega)}$$

The left side defines an inner product on $H_0^1(\Omega)$

$$(u,v)_1 = \int_\Omega \nabla u \nabla v$$

and the induced norm is $|v|_1^2 = \int_\Omega |\nabla v|^2$

Next we show that $|\ |_1$ and $\|\ \|_1$ are equivalent norms on $H_0^1(\Omega)$, where

$$\|v\|_1^2 = \int_\Omega |\nabla v|^2 + \int_\Omega |v|^2$$

therefore $(H_0^1(\Omega), |\ |_1)$ is Hilbert and

$v \longrightarrow \int_\Omega f v$ is linear and continuous

operator from $(H_0^1(\Omega), |\ |_1)$ to $\mathbb{R}$.

The answer is given by Poincaré inequality

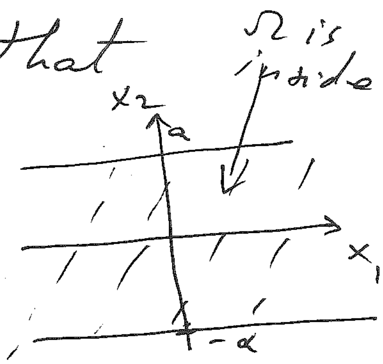
Poincaré inequality Let $\Omega \subset \mathbb{R}^n$ bounded domain. Then there is a constant $C(\Omega)$ such that

$$(*) \qquad \boxed{\int_\Omega |u|^2\,dx \leq C(\Omega)\int_\Omega |\nabla u|^2\,dx, \qquad \forall u \in H_0^1(\Omega)}$$

Proof: it is enough to assume that Ω is bounded in one direction, say x_n-direction:

There is a constant $a > 0$ such that

$$\boxed{\Omega \subset \{x \in \mathbb{R}^n : |x_n| < a\}}$$



(Ω is inside the region determined by hyperplanes $x_n = a$ and $x_n = -a$).

We prove that $(*)$ holds for all $u \in C_c^\infty(\Omega)$ then extend the result to $H_0^1(\Omega)$ by density.

Let $u \in C_c^\infty(\Omega)$:

$$\int_\Omega |u|^2 = \int_\Omega 1 \cdot u^2 = \int_\Omega \frac{\partial}{\partial x_n}(x_n)\, u^2 = -\int_\Omega x_n \frac{\partial}{\partial x_n} u^2$$

$$= -\int_\Omega 2 x_n u \frac{\partial u}{\partial x_n} \leq 2\int_\Omega |x_n|\,|u|\,\left|\frac{\partial u}{\partial x_n}\right| \leq$$

use $|x_n| < a$

$$\leq 2a \int_\Omega |u|\,\left|\frac{\partial u}{\partial x_n}\right| \leq 2a \|u\|_{L^2} \|\nabla u\|_{L^2}$$

$$\Rightarrow \|u\|_{L^2} \leq 2a \|\nabla u\|_{L^2} \Rightarrow \|u\|_{L^2}^2 \leq \underbrace{4a^2}_{C(\Omega)} \|\nabla u\|_{L^2}^2$$

Extension to $H_0^1(\Omega)$:

Let $u \in H_0^1(\Omega)$: there is a sequence $\{u_k\} \subset C_c^\infty(\Omega)$

$u_k \longrightarrow u$ in $\| \; \|_1$. Then

$$\left.\begin{array}{l} u_k \to u \text{ in } L^2(\Omega) \Rightarrow \|u_k\|_{L^2} \to \|u\|_{L^2} \\ \dfrac{\partial u_k}{\partial x_i} \to \dfrac{\partial u}{\partial x_i} \text{ in } L^2(\Omega) \Rightarrow \left\|\dfrac{\partial u_k}{\partial x_i}\right\|_{L^2} \to \left\|\dfrac{\partial u}{\partial x_i}\right\|_{L^2} \\ \text{since } \{u_k\} \subset C_c^\infty(\Omega) : \|u_k\|_{L^2}^2 \le C(\Omega) \|\nabla u_k\|_{L^2}^2 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \boxed{\|u\|_{L^2}^2 \le C(\Omega) \|\nabla u\|_{L^2}^2} \quad \text{with } \underline{C(\Omega) = 4a^2}$$

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<u>Consequence</u>

$$|u|_1^2 \le \|u\|_1^2 \le [1 + C(\Omega)] |u|_1^2$$

such that $| \; |_1$ and $\| \; \|_1$ are equivalent norms on $H_0^1(\Omega)$.

In particular, convergence in $| \; |_1$-norm implies convergence in $\| \; \|_{L^2}$-norm.

Remark : Poincaré inequality does not hold on $H^1(\Omega)$, for example $u(x) \equiv 1$ has

$$\|u\|_{L^2}^2 = |\Omega|$$

$$\|\nabla u\|_{L^2}^2 = 0$$

Remark : Show that there is no constant C such that

$$\|\nabla u\|_{L^2}^2 \le C \|u\|_{L^2}^2 , \quad \forall u \in H_0^1(\Omega)$$

(in other words, convergence in L^2-norm does not imply convergence in $\| \ \|_1$-norm).

Remark : Existence and uniqueness of the weak solution to the Poisson problem follows from Poincaré and Riesz Theorem.

$v \to \int_\Omega f v$ is a linear and continuous operator from $(H_0^1, | \ |_1)$ to $\mathbb{R}$.

$(u,v)_1 = \int_\Omega \nabla u \nabla v$ is the inner product on H_0^1

thus $\forall f \in L^2$: $\exists! \, u \in H_0^1$: $\int_\Omega \nabla u \nabla v = \int_\Omega f v , \quad \forall v \in H_0^1$

<u>Remark</u> Poincaré inequality : There is $C(\Omega)$:

$$\int_\Omega u^2 \leq C(\Omega) \int_\Omega |\nabla u|^2, \quad \forall u \in H_0^1(\Omega).$$

The <u>smallest constant</u> $C^*(\Omega)$ is called the <u>Poincaré constant</u>. Notice that for the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u \\ u|_{\partial\Omega} = 0 \end{cases} \quad \text{we have} \quad \lambda = \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} \geq \frac{1}{C(\Omega)}$$

Later we will show that $\boxed{C^*(\Omega) = \frac{1}{\lambda_1}}$ where λ_1 is the first (principal) eigenvalue of the Laplacian with Dirichlet boundary conditions.

$$C^*(\Omega) = \inf \left\{ C(\Omega) : \int_\Omega u^2 \leq C(\Omega) \int_\Omega |\nabla u|^2, \ \forall u \in H_0^1(\Omega) \right\}$$