Review of fundamental concepts and notation

$\mathbb{R}^n$ = n dimensional real Euclidean space, $\mathbb{R} = \mathbb{R}^1$

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow \text{row } i \\ \\ \\ \\ \end{matrix}$$ the i^{th} standard coordinate vector

$x \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $x = \sum_{i=1}^{n} x_i e_i$

Often $t = x_{n+1}$ $\rightarrow$ time dimension, $(x,t) \in \mathbb{R}^{n+1}$

$\Omega \subset \mathbb{R}^n$ denotes an open connected (domain) subset of $\mathbb{R}^n$.

$\bar{\Omega}$ denotes the closure of Ω (all limit points)

$\Gamma = \partial\Omega$ = boundary of Ω

$$\partial\Omega = \bar{\Omega} \setminus \Omega$$

$B_r(x) = B(x,r) = \{ y \in \mathbb{R}^n : \|y - x\| < r \}$ is the open ball in $\mathbb{R}^n$ with center at x and radius r.

$\overline{B}(x,r) = \{ y \in \mathbb{R}^n : \|y-x\| \leq r \}$ closed ball

Notation for functions $u : \Omega \to \mathbb{R}$

$u(x) = u(x_1, \ldots, x_n)$, $x \in \Omega$

Derivatives: Partial derivative w.r.t. x_i

$$\frac{\partial u}{\partial x_i}(x) = \lim_{h \to 0} \frac{u(x + h e_i) - u(x)}{h}, \quad \text{provided the limit exists}$$

$$u_{x_i}(x) \equiv \frac{\partial u}{\partial x_i}(x) \quad \to \text{notation}$$

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = u_{x_i x_j}(x)$$

Multiindex notation A vector of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$ where each component α_i is a nonnegative integer is called multiindex.
The order of the multiindex is defined as

$$|\alpha| = \alpha_1 + \cdots + \alpha_n = \sum_{i=1}^{n} \alpha_i$$

Given a multiindex α, define the α-partial derivative operator (α- derivative)

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \quad \to \text{order } |\alpha|$$

if $k > 0$ is an integer, denote

$$D^k u(x) \equiv \{ D^\alpha u(x) \mid |\alpha| = k \} =$$ collection of all partial derivatives of order k.

<u>Gradient</u>: $$Du(x) \equiv \nabla u(x) = \begin{bmatrix} u_{x_1}(x) \\ \vdots \\ u_{x_n}(x) \end{bmatrix}$$ vector in $\mathbb{R}^n$

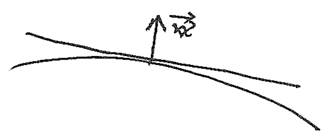
<u>Directional derivative</u> Let $\xi = (\xi_1, \dots, \xi_n)$ a vector in $\mathbb{R}^n$ of length 1, $\|\xi\| = 1$.
The ξ-directional derivative of u at x is defined as

$$u_\xi(x) = \lim_{h \to 0} \frac{u(x + h\xi) - u(x)}{h}$$

if the partial derivatives $u_{x_i}(x)$ exists and are continuous functions of x, $i = 1:n$ then

$$u_\xi(x) = \nabla u(x) \cdot \xi = \sum_{i=1}^n \xi_i \frac{\partial u}{\partial x_i}(x)$$

$\vec{n}$ = outward unit vector <u>normal to the boundary</u> <u>$\partial\Omega$</u>



$$\frac{\partial u}{\partial \vec{n}} = \sum_{i=1}^n n_i u_{x_i} = \nabla u \cdot \vec{\nu}$$

often denote $\vec{\nu} = \vec{n}$

Smooth functions: $u: \Omega \to \mathbb{R}$ is continuously differentiable, $u \in C^1(\Omega)$, if $u(x)$ and $u_{x_i}(x)$, $i = 1:n$ are continuous.

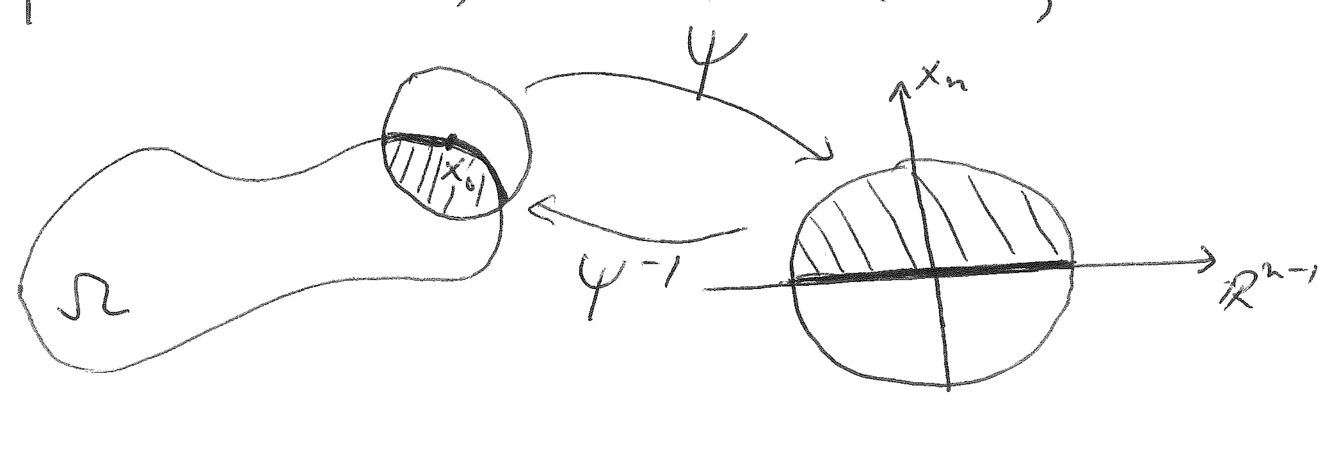
$u \in C^k(\Omega)$ if $D^\alpha u(x)$ is continuous for all α with $|\alpha| \leq k$

A smooth function is a function u such that $u \in C^k(\Omega)$, $\forall k \geq 0$. We write $u \in C^\infty(\Omega)$.

Definition Let $\Omega \subset \mathbb{R}^n$ be open and bounded domain. We say that the boundary is C^1 (or simply Ω is C^1) if for every $x_0 \in \partial\Omega$ there is an open ball $B(x_0, r)$ and a C^1- diffeomorfism $\psi: B(x_0,r) \to B(0,1)$ such that

i) $\psi(\partial\Omega \cap B(x_0,r)) = B(0,1) \cap \{x_n = 0\}$

ii) $\psi(\Omega \cap B(x_0,r)) = B(0,1) \cap \{x_n > 0\}$



Partial differential equations (PDEs)

<u>Definition</u> A PDE is an equation involving an unknown function of two or more variables and certain of its partial derivatives.

<u>Definition</u> An equation of the form

$$F(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x) = 0,$$

$$x \in \Omega$$

where $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \dots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ is given and $u : \Omega \to \mathbb{R}$ is the unknown function is called <u>a PDE of order k</u>.

The PDE is <u>linear</u> if it has the form

$$\boxed{Lu = f} \quad \boxed{\sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u(x) = f(x)}$$

for certain functions a_α and f

a_α are the coefficients

f is the forcing (nonhomogeneous) term.

Otherwise, the PDE is <u>nonlinear</u>.

Examples : $u_t = K(x) u_{xx} + 1 \rightarrow 2^{nd}$ order linear nonhomogeneous

$u_t + u u_x = \varepsilon u_{xx} \rightarrow 2^{nd}$ order nonlinear

A classical (strong) solution to the PDE is a function $u : \Omega \rightarrow \mathbb{R}$, $u \in C^k(\Omega)$ that satisfies ~~solves~~ the PDE at all $x \in \Omega$.

The PDE is well-posed if

(a) it has a solution

(b) the solution is unique

(c) the solution depends continuously on the data given in the problem.

$\rightarrow$ additional constraints may be required on the solution to have a well-posed problem $\rightarrow$ boundary conditions

$\rightarrow$ initial condition.

(boundary value problems)

Divergence theorem $F : \Omega \to \mathbb{R}^n$

$$\operatorname{div} F(x) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x) = (\nabla \cdot F)(x) \qquad \text{scalar}$$

Laplacian : $u : \Omega \to \mathbb{R}$, $\Delta u = \nabla^2 u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$

Remark :
$$\Delta u = \operatorname{div}(\nabla u)$$

Theorem (Gauss) Assume $\Omega \subset \mathbb{R}^n$ is bounded domain with smooth boundary $\partial\Omega$.
if $F : \Omega \to \mathbb{R}^n$ is $C^1(\Omega)$ and $C^0(\bar{\Omega})$
then
$$\int_\Omega \operatorname{div} F \, dx = \int_{\partial\Omega} F \cdot n \, ds$$

Green's identities Remark $\operatorname{div}(v \nabla u) = v \Delta u + \nabla u \cdot \nabla v$

1st Green's ident
$$\boxed{\int_\Omega (v \Delta u + \nabla u \cdot \nabla v) dx = \int_{\partial\Omega} v \nabla u \cdot n \, ds = \int_{\partial\Omega} v \frac{\partial u}{\partial n} ds}$$

2nd identity
$$\boxed{\int_\Omega (v \Delta u - u \Delta v) dx = \int_{\partial\Omega} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds}$$

3rd Green's identity
$$\boxed{\int_\Omega \Delta u = \int_{\partial\Omega} \nabla u \cdot n \, ds = \int_{\partial\Omega} \frac{\partial u}{\partial n} ds}$$