Nonhomogeneous boundary conditions

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \qquad (*)$$

Weak (variational) problem: Find $u \in H^1(\Omega)$ such that

$$\begin{cases} \int_\Omega \nabla u \nabla v = \int_\Omega f v, & \forall v \in H_0^1(\Omega) \\ u = g & \text{on } \partial\Omega \end{cases}$$

To formulate the variational problem in $H_0^1(\Omega)$, assume that there is $\tilde{g} : \Omega \to \mathbb{R}$ such that $\tilde{g}_{|\partial\Omega} = g$.
For the strong solution $(*)$ we require $\tilde{g} \in C^2(\Omega)$ whereas for the weak solution we require $\tilde{g} \in H^1(\Omega)$

Let $w = u - \tilde{g}$ then $w_{|\partial\Omega} = 0$ and

$$-\Delta w = -\Delta u + \Delta \tilde{g} = f + \Delta \tilde{g}$$

Weak solution: Find $w \in H_0^1(\Omega)$ such that

$$\boxed{\int_\Omega \nabla w \nabla v = \int_\Omega f v - \int_\Omega \nabla \tilde{g} \nabla v, \quad \forall v \in H_0^1(\Omega)}$$

Existence and uniqueness of $w \in H_0^1(\Omega)$ with Riesz need to show that $v \xrightarrow{F} \int_\Omega f v - \int_\Omega \nabla \tilde{g} \nabla v$ is linear (straightforward) and continuous

$$|F(v_n)-F(v)| \le \|f\|_{L^2}\|v_n-v\|_{L^2} + \|\nabla\tilde{g}\|_{L^2}\|\nabla(v_n-v)\|_{L^2}$$

So $F : H_0^1 \to \mathbb{R}$ is continuous.

Return to the classical (strong) solution:

Assume that $w \in C^2(\Omega) \cap H_0^1(\Omega)$ and $\tilde{g} \in C^2(\Omega)$.

$$\int_\Omega \nabla w \nabla v = \int_\Omega f v - \int_\Omega \nabla\tilde{g}\nabla v \quad \Rightarrow$$

$$\Rightarrow \quad \int_\Omega -(\Delta w) v = \int_\Omega f v + \int_\Omega (\Delta\tilde{g}) v, \quad \forall v \in C_c^\infty(\Omega)$$

$$\Rightarrow \int_\Omega -\Delta(w+\tilde{g}) v = \int_\Omega f v \quad \Rightarrow \quad -\Delta(w+\tilde{g}) = f.$$

Thus, $u \overset{\text{def}}{=} w+\tilde{g}$ satisfies $-\Delta u = f$

and $u|_{\partial\Omega} = w|_{\partial\Omega} + \tilde{g}|_{\partial\Omega} = g$.

* * *

For the general elliptic operator

$$\mathcal{L}u = -\nabla\cdot(K\nabla u) + b\cdot\nabla u + gu$$

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$$

we have $\mathcal{L}(u-\tilde{g}) = \mathcal{L}(u) - \mathcal{L}(\tilde{g}) = f - \mathcal{L}(\tilde{g})$.

such that, $w = u-\tilde{g}$ satisfies $\mathcal{L}(w) = f - \mathcal{L}(\tilde{g})$

Variational problem: Find $w \in H_0^1(\Omega)$ such that

$$a(w,v) = (f,v)_{L^2} - a(\tilde{g},v), \quad \forall v \in H_0^1(\Omega)$$

where $a(w,v) = \int_\Omega (\nabla v)\cdot(K\nabla w) + (b\cdot\nabla w)v + gwv$

<u>Remark</u>: if $a(\cdot,\cdot)$ is continuous then

$$v \xrightarrow{F} (f,v)_{L^2} - a(\tilde{g},v) \quad \text{is continuous}$$

$$|F(v_n) - F(v)| \le \|f\|_{L^2}\|v_n - v\|_{L^2} + M\|\tilde{g}\|_1\|v_n - v\|_1$$

So $F: H_0^1(\Omega) \to \mathbb{R}$ is continuous under the assumption that K, b, g are L^∞

Neumann boundary conditions

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

Variational formulation:

$$\int_\Omega \nabla u \nabla v - \int_{\partial\Omega} v \frac{\partial u}{\partial \vec{\nu}} + \int_\Omega u v = \int_\Omega f v, \quad \forall v \in C^1(\bar{\Omega})$$

Find $u \in H^1(\Omega)$: $\int_\Omega \nabla u \nabla v + \int_\Omega u v = \int_\Omega f v, \ \forall v \in H^1(\Omega)$

Existence and uniqueness: $\forall f \in L^2(\Omega)$, $\exists! u \in H^1(\Omega)$ weak solution

if $u \in C^2(\Omega)$ then is a solution to the (strong) PDE.

$$\int_\Omega \nabla u \nabla v = -\int_\Omega \Delta u \, v + \int_{\partial\Omega} v \frac{\partial u}{\partial \vec{\nu}}$$

$$\Rightarrow \int_\Omega (-\Delta u + u) v + \int_{\partial\Omega} v \frac{\partial u}{\partial \vec{\nu}} = \int_\Omega f v, \quad \forall v \in H^1(\Omega).$$

First, let $v \in C_c^\infty$ (or H_0^1)

$$\int_\Omega (-\Delta u + u) v = \int_\Omega f v \quad \Rightarrow \quad \underline{-\Delta u + u = f}$$

$$\Rightarrow \int_{\partial\Omega} v \frac{\partial u}{\partial \vec{\nu}} = 0, \quad \forall v \in H^1(\Omega) \quad \Rightarrow \quad \frac{\partial u}{\partial \vec{\nu}} = 0.$$

Remark: Neumann B.C. are "natural" $\overline{\text{need not to be}}$ included in the variational formulation.
Dirichlet B.C. are "essential" $\rightarrow$ require $H_0^1(\Omega)$

Nonhomogeneous Neumann problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \vec{\nu}} = g & \text{on } \partial\Omega \end{cases}$$

Weak formulation: find $u \in H^1(\Omega)$ such that

$$\underbrace{\int_\Omega \nabla u \nabla v + \int_\Omega u v}_{a(u,v)} = \underbrace{\int_\Omega f v + \int_{\partial\Omega} g v}_{F(v)}, \quad \forall v \in H^1(\Omega)$$

Notice that $F : H^1(\Omega) \to \mathbb{R}$ is linear and continuous due to the trace theorem.

$$|F(v)| \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)}$$

$$\leq \left[\|f\|_{L^2(\Omega)} + c \|g\|_{L^2(\partial\Omega)} \right] \|v\|_1 .$$