General linear elliptic problems (Ref: Evans, Ch6).

Consider
$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where
$$\begin{aligned} \mathcal{L}u &= -\nabla\cdot(K\nabla u) + b\cdot\nabla u + gu \\ &= -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[K_{ij}(x)\frac{\partial u}{\partial x_j}\right] + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + g(x)u \end{aligned}$$

and $K_{ij}(x) = K_{ji}(x), \quad \forall x \in \Omega$ (K is symmetric matrix)

<u>Definition</u> The partial differential operator $\mathcal{L}$ is <u>uniformly elliptic</u> if there is a constant $\alpha > 0$ such that
$$\sum_{i,j=1}^{n} K_{ij}(x)\,\xi_i\,\xi_j \geq \alpha|\xi|^2$$

a.e. $x \in \Omega$ and $\forall \xi \in \mathbb{R}^n$

in matrix-vector format: $\xi^T K(x)\xi \geq \alpha|\xi|^2$

Equivalently, $K(x)$ is a positive definite matrix, a.e. $x \in \Omega$ and with the smallest eigenvalue
$$\lambda_1(x) \geq \alpha > 0, \quad \text{a.e. } x \in \Omega$$

For the strong (classical) problem the coefficients are functions $g : \Omega \to \mathbb{R}$, $b_i : \Omega \to \mathbb{R}$, $f : \Omega \to \mathbb{R}$
$K_{ij} : \Omega \to \mathbb{R}$
such that $f, g, b_i \in C(\bar{\Omega})$, $K_{ij} \in C^1(\bar{\Omega})$

For the weak (variational) solution is enough to assume $g, b_i, K_{ij} \in L^\infty(\Omega)$, $f \in L^2(\Omega)$.

Variational formulation

$\forall v \in C_c^\infty(\Omega)$:
$$\int_\Omega -\nabla\cdot(K\nabla u)v + (b\cdot\nabla u)v + guv = \int_\Omega fv$$

Notice $\nabla\cdot(K\nabla u)v = \nabla\cdot(K\nabla u v) - (\nabla v)^T K\nabla u$

$\Rightarrow$
$$\int_\Omega \nabla v\cdot K\nabla u + \int_\Omega (b\cdot\nabla u)v + \int_\Omega guv = \int_\Omega fv$$

Notice that $\int_\Omega (b\cdot\nabla u)v = -\int_\Omega u\nabla\cdot(bv)$

such that the left side is not a symmetric expression. Variational problem formulation

(V) Find $u \in H_0^1(\Omega)$ such that for every $v \in H_0^1(\Omega)$
$$\int_\Omega \nabla v\cdot K\nabla u + \int_\Omega (b\cdot\nabla u)v + \int_\Omega guv = \int_\Omega fv$$

<u>Lax-Milgram theorem</u> Let $(H, (\cdot,\cdot))$ a Hilbert space and $a: H\times H \to \mathbb{R}$ satisfy the following

(i) a is bilinear

(ii) a is continuous: $\exists M>0$: $|a(u,v)| \leq M\|u\|\,\|v\|$ $\forall u, v \in H$.

(iii) a is elliptic (coercive) (positive): $\exists \alpha > 0$: $a(v,v) \geq \alpha \|v\|^2$ $\forall v \in H$.

Let $F: H \to \mathbb{R}$ a linear and continuous functional: $|F(v)| \leq K\|v\|$, $K = \|L\|_{H^*} = \text{constant}$ $(K < \infty)$.

Then there is an <u>unique</u> $u \in H$ such that

$$\boxed{(V) \qquad a(u,v) = F(v), \quad \forall v \in H}$$

in addition, <u>if $a(\cdot,\cdot)$ is symmetric</u>, then u is the solution to the minimization problem

$$\boxed{(M) \qquad \min_{v\in H} \frac{1}{2} a(v,v) - F(v)}$$

and the problems (M) and (V) are equivalent: u solves (V) if and only if u solves (M)

Proof: For any fixed $u \in H$ consider the functional $\mathcal{A}: H \to \mathbb{R}$, $\mathcal{A}(v) = a(u,v)$.

Then $\mathcal{A}$ is linear and continuous,

$$|\mathcal{A}(v)| = |a(u,v)| \overset{(ii)}{\leq} M\|u\|\,\|v\|$$

Riesz representation theorem: $\exists!\, Au \in H$ such that

$$\mathcal{A}(v) = a(u,v) = (Au, v), \quad \forall v \in H \quad (*)$$

and $$\|Au\| = \sup_{\substack{v \in H \\ v \neq 0}} \frac{|a(u,v)|}{\|v\|} \leq M\|u\|.$$

Since $F: H \to \mathbb{R}$ is linear and continuous,

Riesz theorem:

$$\exists!\, b \in H : F(v) = (b, v), \quad \forall v \in H \quad (**)$$

Notice that if u in $(*)$ is such that $Au = b$ then from $(*)$ and $(**)$ we have that u solves (V). Therefore, the problem is to show the existence of u : $Au = b$.

We need the <u>Banach fixed point theorem</u>.

Let $\rho > 0$ a fixed constant. Consider the operator $T_\rho : H \to H$ defined as $T_\rho(u) = u - \rho(Au - b)$.

Notice that u is a fixed point to T_ρ, i.e. $T_\rho(u) = u$ if and only if $Au = b$ such that the problem is reduced to finding a fix point of T_ρ

We show that there is $\rho > 0$ such that T_ρ is <u>strictly contractive</u>: $\exists\, 0 < \lambda < 1$:

$$\|T_\rho u_1 - T_\rho u_2\| \le \lambda \|u_1 - u_2\| \quad \forall u_1, u_2 \in H$$

$$\|T_\rho u_2 - T_\rho u_1\|^2 = \|u_2 - u_1 - \rho(Au_2 - Au_1)\|^2 =$$

$$(T_\rho) \qquad = \|u_2 - u_1\|^2 - 2\rho(u_2 - u_1, Au_2 - Au_1) + \rho^2 \|Au_2 - Au_1\|^2$$

<u>Remarks</u>:

$$\left.\begin{aligned} a(u_1, v) &= (Au_1, v) \\ a(u_2, v) &= (Au_2, v) \end{aligned}\right\} \forall v \in H \Rightarrow$$

$$\Rightarrow a(u_1 - u_2, v) = (Au_1 - Au_2, v), \quad \forall v \in H.$$

$$\Rightarrow \left\{ \begin{aligned} &\alpha \|u_1 - u_2\|^2 \le a(u_1 - u_2, u_1 - u_2) = (Au_1 - Au_2, u_1 - u_2) \le \\ &\qquad \le \|Au_1 - Au_2\| \, \|u_1 - u_2\| \\ &\|Au_1 - Au_2\|^2 = a(u_1 - u_2, Au_1 - Au_2) \le M \|u_1 - u_2\| \, \|Au_1 - Au_2\| \end{aligned}\right.$$

Therefore,

$$\alpha \|u_1 - u_2\|^2 \le (u_2 - u_1, Au_2 - Au_1) \xrightarrow{\text{since } \rho > 0}$$

$$\Rightarrow \quad -\rho(u_2 - u_1, Au_2 - Au_1) \le -\alpha\rho \|u_1 - u_2\|^2 \qquad (\xi)$$

and $\quad \|Au_2 - Au_1\|^2 \le M^2 \|u_1 - u_2\|^2 \qquad (\xi\xi)$

Use (ξ) and $(\xi\xi)$ in (T_ρ) equation:

$$\|T_\rho u_2 - T_\rho u_1\|^2 \le \|u_2 - u_1\|^2 - 2\rho\alpha \|u_2 - u_1\|^2 + \rho^2 M^2 \|u_2 - u_1\|^2$$

or

$$\|T_\rho u_2 - T_\rho u_1\|^2 \le \left(1 - 2\rho\alpha + \rho^2 M^2\right) \|u_2 - u_1\|^2 .$$

Let $\rho = \frac{\alpha}{M^2}$: then $1 - 2\rho\alpha + \rho^2 M^2 = \underbrace{1 - \frac{\alpha^2}{M^2}}_{\lambda^2} < 1$

($0 < \rho < \frac{2\alpha}{M^2}$ will do)

Therefore $\|T_\rho u_2 - T_\rho u_1\| \le \lambda \|u_2 - u_1\|$ with $\underline{\underline{\lambda < 1}}$

such that T_ρ is <u>strictly contractive</u>.

$\exists!$ fixed point: $\overline{T_\rho(u) = u}$

such that u solves (V).

Uniqueness of the solution to (V):

Let u_1, u_2 solutions.

$$\left.\begin{aligned} a(u_1, v) &= F(v) \\ a(u_2, v) &= F(v) \end{aligned}\right\} \Rightarrow a(u_1 - u_2, v) = 0, \quad \forall v \in H$$

$$\Rightarrow a(u_1 - u_2, u_1 - u_2) = 0 \Rightarrow \alpha \|u_1 - u_2\|^2 \leq 0$$

$$\Rightarrow u_1 = u_2 .$$

Equivalence (V) $\Leftrightarrow$ (M) when $a(\cdot,\cdot)$ is symmetric

(V) $\Rightarrow$ (M) Assume $a(u,v) = F(v)$, $\forall v \in H$

$$J(v) = J(u + v - u) = \frac{1}{2} a(u + v - u, u + v - u) - F(u + v - u)$$

$$= \frac{1}{2} a(u,u) - F(u) + \underbrace{a(u, v-u) - F(v-u)}_{=0} + \frac{1}{2} a(v-u, v-u)$$

$$= J(u) + \frac{1}{2} a(v-u, v-u) \geq J(u) + \frac{1}{2} \alpha^2 \|v-u\|^2$$

$\Rightarrow J(v) \geq J(u)$ and $J(v) = J(u)$ iff $u = v$.

(M) $\Rightarrow$ (V) : Consider $g(\varepsilon) = J(u + \varepsilon v)$.

if u solves (M) then $g'(0) = 0$

$$g(\varepsilon) = \frac{1}{2} a(u+\varepsilon v, u+\varepsilon v) - F(u+\varepsilon v) =$$

$$= J(u) + \varepsilon[a(u,v) - F(v)] + \frac{1}{2}\varepsilon^2 a(v,v).$$

$g'(0) = a(u,v) - F(v) = 0$, therefore u solves (V)

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Theorem (stability of the solution w.r.t. data)
Under the hypothesis of Lax-Milgram theorem the solution u satisfies

$$\|u\|_H \le \frac{K}{\alpha}, \quad \text{where } K = \|F\|_{H'}$$

Proof: $a(u,v) = F(v)$ let $v = u$

$$\alpha \|u\|^2 \le a(u,u) = F(u) \le K\|u\|$$

$$\Rightarrow \boxed{\|u\| \le \frac{K}{\alpha}}$$

(the problem may become ill-conditioned if α is small)

<u>Remark</u> : if $a(\cdot,\cdot)$ is symmetric then $a(u,v)$ is an inner product on $H \times H$ and the induced norm $\|v\|_a^2 = a(v,v)$ is equivalent with the H-norm :

$$\alpha \|v\|^2 \leq a(v,v) = \|v\|_a^2 \leq M\|v\|^2$$

since $a(v,v) \leq M\|v\|^2$

$$\Rightarrow \quad \sqrt{\alpha}\,\|v\| \leq \|v\|_a \leq \sqrt{M}\,\|v\|$$

Therefore, $\big(H, a(\cdot,\cdot)\big)$ is Hilbert and existence and uniqueness of the solution to (V) follows directly from Riesz' theorem.

However, Lax-Milgram applies (extends the theory) ~~even~~ when a is not necessarily symmetric.

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Existence and uniqueness of the solution to the elliptic problem

$$\mathcal{L}u = f \quad \text{where}$$

$$\mathcal{L}u = -\nabla\cdot(K\nabla u) + b\cdot\nabla u + g u$$

is an elliptic operator $\left(\xi^T K \xi \geq \alpha|\xi|^2 \quad \text{a.e. } x\in\Omega,\ \alpha>0 \right)$

Theorem Assume $K_{ij}\in L^\infty(\Omega)$

$b_i\in L^\infty(\Omega)$, $g\in L^\infty(\Omega)$

Then there is a constant $\delta\geq 0$ such that for each $\mu\geq\delta$ and any $f\in L^2(\Omega)$ there is a unique weak solution to $\begin{cases} \mathcal{L}u+\mu u = f & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{cases}$

$$\boxed{\begin{array}{l} \exists!\ u\in H_0^1(\Omega) : \forall v\in H_0^1(\Omega) \\ \underbrace{\int_\Omega (\nabla v)^T K\nabla u + \int_\Omega (b\cdot\nabla u) v + \int_\Omega g u v + \int_\Omega \mu u v}_{a(u,v)} = \int_\Omega f v \end{array}}$$

Proof: we show that there is a constant $\delta\geq 0$ such that $a(u,v)$ satisfies the hypothesis of Lax-Milgram theorem for each $\mu\geq\delta$

Proof : Notice that $a : H_0^1 \times H_0^1 \to \mathbb{R}$ is well-defined since the coefficients K_{ij}, b_i, q are assumed to be $L^\infty(\Omega)$.

(i) $a(\cdot,\cdot)$ is bilinear : obvious.

(ii) $a(\cdot,\cdot)$ is continuous : $\exists M$ such that

$$|a(u,v)| \le M \|u\|_1 \|v\|_1 \quad , \forall u, v \in H_0^1$$

$$|a(u,v)| \le \max_{1 \le i,j \le n} \|K_{ij}\|_\infty \int_\Omega \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right| \sum_j \left|\frac{\partial v}{\partial x_j}\right| + \|b\|_\infty \int_\Omega \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right| |v|$$

$$+ \left(\mu + \|q\|_\infty\right) \|u\|_{L^2} \|v\|_{L^2}$$

Cauchy-Schwarz

$$\le n \|K\|_\infty \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \sqrt{n} \|b\|_\infty \|\nabla u\|_{L^2} \|v\|_{L^2} +$$

$$+ \left(\mu + \|q\|_\infty\right) \|u\|_{L^2} \|v\|_{L^2}$$

$$\le \underbrace{\left(n\|K\|_\infty + \sqrt{n}\, \|b\|_\infty + \mu + \|q\|_\infty\right)}_{M} \|u\|_1 \|v\|_1$$

So $a(\cdot,\cdot)$ is continuous for any $\mu \ge 0$.

iii) $a(\cdot,\cdot)$ is elliptic : $\exists \alpha > 0$ such that

$$a(v,v) \geq \alpha \|v\|_1^2 , \quad \forall v \in H_0^1(\Omega)$$

$$a(v,v) = \int_\Omega (\nabla v)^T K \nabla v + \int_\Omega (b \cdot \nabla v) v + \int_\Omega (q + \mu) v^2$$

<u>Remark</u> : Young's inequality : $\forall a, b \geq 0$

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 , \quad \forall \varepsilon > 0 .$$

(since $(2\varepsilon a - b)^2 \geq 0$) .

Then,

$$\left| \int_\Omega (b \cdot \nabla v) v \right| \leq \|b\|_\infty \int_\Omega \sum_{i=1}^n \left| \frac{\partial v}{\partial x_i} \right| |v| \leq$$

$$\leq \sqrt{n} \|b\|_\infty \int_\Omega |\nabla v| |v| \leq \sqrt{n} \|b\|_\infty \int_\Omega \left[\varepsilon |\nabla v|^2 + \frac{1}{4\varepsilon} |v|^2 \right]$$

$$a(v,v) \geq \alpha \|\nabla v\|_{L^2}^2 - \sqrt{n} \|b\|_\infty \left[\varepsilon \|\nabla v\|_{L^2}^2 + \frac{1}{4\varepsilon} \|v\|_{L^2}^2 \right]$$

$$- \|q\|_\infty \|v\|_{L^2}^2 + \mu \|v\|_{L^2}^2 =$$

$$= \left(\alpha - \sqrt{n}\, \varepsilon \|b\|_\infty \right) \|\nabla v\|_{L^2}^2 +$$

$$+ \left(\mu - \sqrt{n} \|b\|_\infty \cdot \frac{1}{4\varepsilon} - \|q\|_\infty \right) \|v\|_{L^2}^2 .$$

Let $\varepsilon = \dfrac{\alpha}{2\sqrt{n}\,\|b\|_\infty}$ then

$$a(v,v) \geq \frac{1}{2}\alpha \|\nabla v\|^2_{L^2} + \left[\mu - \frac{n\|b\|^2_\infty}{2\alpha} - \|q\|_\infty\right]\|v\|^2_{L^2}$$

Let $\mu \geq \gamma \overset{\text{def}}{=} \dfrac{n\|b\|^2_\infty}{2\alpha} + \|q\|_\infty$.

Then

$$a(v,v) \geq \frac{1}{2}\alpha\|\nabla v\|^2_{L^2} \geq \frac{\alpha}{2[1+C(\Omega)]}\|v\|^2_1 .$$

thus $a(\cdot,\cdot)$ is elliptic.

<u>Remark</u>: for $\mu = 0$,

$$a(v,v) \geq \left(\alpha - \varepsilon\sqrt{n}\|b\|_\infty\right)\|\nabla v\|^2_{L^2} - \left[\|q\|_\infty + \frac{\sqrt{n}}{4\varepsilon}\|b\|_\infty\right]\|v\|^2_{L^2}$$

$$\geq \underbrace{\left[\alpha - \varepsilon\sqrt{n}\|b\|_\infty - C(\Omega)\left(\|q\|_\infty + \frac{\sqrt{n}}{4\varepsilon}\|b\|_\infty\right)\right]}_{g(\varepsilon)}\|\nabla v\|^2_{L^2}$$

$g(\varepsilon)$, $g : \mathbb{R}^+ \to \mathbb{R}$

has max value at $\boxed{\varepsilon^* = \frac{1}{2}\sqrt{C(\Omega)}}$

if $g(\varepsilon^*) > 0$ then the original problem $Lu = f$ has a unique weak solution