Modern theory for linear second-order PDEs

insight : Consider a Sturm-Liouville problem

$$\boxed{(*)}\quad \begin{cases} -[p(x)u'(x)]' + q(x)u(x) = f(x), & a<x<b \\ u(a) = u(b) = 0 \end{cases}$$

where $p \in C'[a,b]$, $p(x) \geq \alpha > 0$, $x \in [a,b]$,

$f, q \in C[a,b]$

Let $V = \{v : [a,b] \to \mathbb{R},\ v \in C'(a,b),\ v(a) = v(b) = 0\}$

if $u \in C^2(a,b)$ solves $(*)$, then

$\forall v \in V$:

$$\boxed{(**)}\quad \int_a^b p u' v' \, dx + \int_a^b q u v \, dx = \int_a^b f v \, dx$$

Conversely, if $u \in V$ satisfies $(**)$ and $u \in C^2(a,b)$ then we can "undo" the integration by parts

$$\int_a^b p u' v' dx = \underbrace{p u' v \Big|_a^b}_{=0} - \int_a^b [p u']' v \, dx$$

such that $(**)$ becomes

$$\int_a^b \left(-[p u']' v + q u v\right) dx = \int_a^b f v \, dx$$

and using the "density" of the space V we may conclude that u solves $(*)$.

we may be able to prove the existence and uniqueness of a solution u to $(**)$ and if the solution is $C^2(a,b)$ (i.e. more regular) then u is the solution to $(*)$.

Notice that $(**)$ may be also formulated as the solution to a minimization problem

$$\boxed{(***)}\quad J: V \to \mathbb{R}, \qquad J(u) = \frac{1}{2}\int_a^b p u'^2 dx + \frac{1}{2}\int_a^b q u^2 dx - \int_a^b f u\, dx$$

if $u \in V$ is a minimum point for J,

i.e., $J(u) \le J(v)$, $\forall v \in V$

then $g: \mathbb{R} \to \mathbb{R}$, $g(\varepsilon) = J(u + \varepsilon v)$ has a min point at $\underline{\varepsilon = 0}$

$$g(\varepsilon) = \frac{1}{2}\int_a^b p(u+\varepsilon v)'^2 + \frac{1}{2}\int_a^b q(u+\varepsilon v)^2 - \int_a^b f(u+\varepsilon v)$$

$$g'(0) = \int_a^b \left(p u' v' + q u v - f v\right) dx = 0 .$$

Therefore, if we are able so show that J has an unique min point then there is an unique solution to $(**)$.

in higher dimensions : $\Omega \subset \mathbb{R}^n$ bounded set

$$(*) \quad \begin{cases} -\Delta u + u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$V = \{ v \in C^1(\Omega) : v|_{\partial\Omega} = 0 \}$

if $u \in C^2(\Omega)$ solves $(*)$ then

$$(**) \qquad \int_\Omega \nabla u \nabla v + \int_\Omega uv = \int_\Omega fv, \quad \forall v \in V$$

The minimization problem is

$$(***) \quad \min_{u \in V} J(u) \quad : \quad J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega u^2 - \int_\Omega fu$$

if $u \in V$ is the minimizer then

$g(\varepsilon) = J(u + \varepsilon v)$ has min point at $\varepsilon = 0$, for any $v \in V$.

$$g(\varepsilon) = \frac{1}{2}\int_\Omega \sum_{i=1}^n \left([u + \varepsilon v]_{x_i} \right)^2 + \frac{1}{2}\int_\Omega (u + \varepsilon v)^2 - \int_\Omega f(u + \varepsilon v)$$

$$g'(0) = \int_\Omega \nabla u \nabla v + \int_\Omega uv - \int_\Omega fv = 0$$

$\boxed{Q:}$ show that $(**)$ implies $(***)$

Review of functional analysis concepts

Banach space

Consider $(X, \|\cdot\|)$ a normed vector space.

A sequence $\{x_n\} \subset X$ is called Cauchy sequence if $\forall \varepsilon > 0$, $\exists N(\varepsilon)$: $\|x_n - x_m\| < \varepsilon$, $\forall n, m \geq N(\varepsilon)$

$(X, \|\cdot\|)$ is a Banach space (complete space) if every Cauchy sequence is convergent in X.

Example: Consider $C[a,b]$ with the norm $\|f\|_1 = \int_a^b |f(x)|\,dx$. Then $(C[a,b], \|\cdot\|_1)$ is not Banach.

Counter-example: Let $[a,b] = [0, 2]$.

$$f_n(x) = \begin{cases} x^n, & 0 \leq x \leq 1 \\ 1, & 1 \leq x \leq 2 \end{cases} \longrightarrow \begin{cases} 0, & 0 \leq x < 1 \\ 1, & 1 \leq x \leq 2 \end{cases}$$

and $\{f_n\}$ is Cauchy w.r.t. $\|\ \|_1$.

→ require the "sup" (max) norm

$$\|f\| = \max_{x \in [a,b]} |f(x)|$$

then $C[a,b]$ is complete (Banach)

Spaces of functions on $\Omega \subset \mathbb{R}^n$ open

$$\left.\begin{aligned} L^1(\Omega) &= \{ f : \Omega \to \mathbb{R},\ \int_\Omega |f(x)|\,dx < \infty \} \\ \|f\|_{L^1} &= \int_\Omega |f(x)|\,dx \end{aligned}\right\} \text{Banach}$$

$$\left.\begin{aligned} L^p(\Omega) &= \{ f : \Omega \to \mathbb{R} : \int_\Omega |f(x)|^p\,dx < \infty \} \\ \|f\|_{L^p} &= \left[\int_\Omega |f(x)|^p \right]^{\frac{1}{p}} \end{aligned}\right\} \begin{matrix} 1 \le p < \infty \\ \text{Banach} \end{matrix}$$

$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R},\ \exists c \geq 0 : |f(x)| \le c$ almost everywhere in $\Omega \}$

$$\|f\|_\infty = \inf \{ c : |f(x)| \le c \text{ a.e. in } \Omega \}$$

$\rightarrow$ essentially bounded functions.

$C_c(\Omega) = \{ f : \Omega \to \mathbb{R},$ continuous, with compact support $\}$.

$$\operatorname{supp}(f) = \overline{\{ x : \Omega : f(x) \neq 0 \}}$$

$$C^\infty(\Omega) = \bigcap_{k \geq 0} C^k(\Omega)$$

$$C_c^\infty(\Omega) = C^\infty(\Omega) \cap C_c(\Omega) \overset{\text{notation}}{=} C_0^\infty(\Omega) = \mathcal{D}(\Omega)$$

$\rightarrow$ Example: $\rho(x) = \begin{cases} e^{\frac{1}{\|x\|^2 - 1}}, & \|x\| < 1 \\ 0, & \|x\| \geq 1 \end{cases}$

Let $\rho_k(x) = C k^n \rho(nx)$, $C = [\int \rho]^{-1}$ then $\operatorname{supp}(\rho_n) = \overline{B}(0, \frac{1}{n})$

Definition Let $(X, \|\cdot\|)$ a normed space.
A subspace of X, $S \subset X$ is said to be dense in X if $\forall x \in X, \forall \varepsilon > 0, \exists y \in S:$

$$\|x - y\| < \varepsilon.$$

(equivalently: $\forall x \in X, \exists \{x_n\} \subset S : x_n \to x$)

Property: $C_c(\Omega)$ is dense in $L^p(\Omega)$, $1 \le p < \infty$
for any Ω (not necessary bounded)

Property: if $\Omega \subset \mathbb{R}^n$ is bounded then

$$L^1(\Omega) \supset L^2(\Omega) \supset \cdots \supset L^\infty(\Omega) \supset C(\bar{\Omega})$$

and the inclusion is continuous.

$f_n \to f$ in $L^q(\Omega)$ then $f_n \to f$ in $L^p(\Omega)$ for $p < q$

Conjugate exponent:

Let $1 \le p \le \infty$. The conjugate exponent of p is p' such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Hölder inequality: Let $f \in L^p$, $g \in L^{p'}$, $1 \le p \le \infty$

Then $f \cdot g \in L^1$ and

$$\int_\Omega |fg| \le \|f\|_{L^p} \|g\|_{L^{p'}}$$

Proof: Young's inequality

$$\boxed{ab \le \frac{1}{p} a^p + \frac{1}{p'} b^{p'} \quad \forall\, a, b \ge 0.}$$

use $\ln(x)$ is concave $\longrightarrow$

$$\ln\left(\frac{1}{p} a^p + \frac{1}{p'} b^{p'}\right) \geq \ln ab$$

Therefore, $|f(x)| \cdot |g(x)| \le \frac{1}{p} |f(x)|^p + \frac{1}{p'} |g(x)|^{p'}$

$$\Rightarrow \boxed{\int_\Omega |fg| \le \frac{1}{p} \|f\|^p_{L^p} + \frac{1}{p'} \|g\|^{p'}_{L^{p'}}} \qquad (*)$$

Let $\tilde{f}(x) = \dfrac{f(x)}{\|f\|_{L^p}}$, $\tilde{g}(x) = \dfrac{g(x)}{\|g\|_{L^{p'}}}$

Then $\|\tilde{f}\|_{L^p} = 1$, $\|\tilde{g}\|_{L^{p'}} = 1$

Apply $(*)$ to $\tilde{f}, \tilde{g}$:

$$\frac{1}{\|f\|_{L^p} \|g\|_{L^{p'}}} \int_\Omega |f(x) g(x)| \le \frac{1}{p} + \frac{1}{p'} = 1$$

$$\Rightarrow \int_\Omega |f(x) g(x)| \le \|f\|_{L^p} \|g\|_{L^{p'}}$$

Hilbert Spaces

Let H vector space,

$\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ scalar product (inner product)

1) bilinear $\langle u, c_1 v_1 + c_2 v_2 \rangle = c_1 \langle u, v_1 \rangle + c_2 \langle u, v_2 \rangle$

$\langle c_1 u_1 + c_2 u_2, v \rangle = c_1 \langle u_1, v \rangle + c_2 \langle u_2, v \rangle$

2) symmetric $\langle u, v \rangle = \langle v, u \rangle$

3) positive definite: $\langle u, u \rangle \geq 0$

$\langle u, u \rangle = 0 \Rightarrow u = 0$.

Cauchy-Schwarz inequality:

$$|\langle u, v \rangle| \leq (u,u)^{\frac{1}{2}} \cdot (v,v)^{\frac{1}{2}}$$

$\to$ Norm induced by $(\cdot, \cdot)$:

$$\|u\| = (u, u).$$

Remark: $\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2(u,v) \leq$

$\leq \|u\|^2 + \|v\|^2 + 2\|u\|\,\|v\|$

$= (\|u\| + \|v\|)^2$

such that $\|u+v\| \leq \|u\| + \|v\|$

Definition A Hilbert space is a vector space H equipped with a scalar product $(\cdot, \cdot)$ and that is complet (Banach) for the norm $\|u\| = (u,u)^{\frac{1}{2}}$ induced the ~~the~~ scalar product

Remark : Any finite dimensional space is Hilbert with an appropriate norm. (but not with any norm)

Remark : $L^2(\Omega)$ is Hilbert for the inner product $\boxed{(f,g) = \int_\Omega f g}$

Theorem if $S \subset H$ is a dense subspace of the Hilbert space H and $(u,v) = 0$, $\forall v \in S$ then $u = 0$

Proof : Let $\{u_n\} \subset S$, $u_n \to u$

Then $|(u_n - u, u)| \leq \|u_n - u\| \cdot \|u\| \longrightarrow 0$.

$(u_n, u) = 0 \longrightarrow \|u\|^2$

$\Rightarrow \|u\| = 0 \quad \Rightarrow u = 0$.

Theorem $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$, $\forall 1 \leq p < \infty$.
(Ω open, ~~bounded~~ in $\mathbb{R}^n$)

Consequence : $\int_\Omega u v \, dx = 0$, $\forall v \in C_c^\infty(\Omega) \Rightarrow u = 0$ (a.e.)

The dual space of a normed vector space

Definition Given $(X, \|\cdot\|)$ normed vector space

consider $X^* = \{ f : X \to \mathbb{R}, \text{ linear and continuous} \}$

Define the dual norm on X^* as

$$\|f\|_{X^*} = \sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{\|x\| \leq 1} |f(x)|$$

X^* is the dual space of X,

Examples: $X = C[a,b]$,

(1) integral operator $\mathcal{F}[f] = \int_a^b f(x)\, dx$

is an element of the dual space

$\mathcal{F} \in \left(C[a,b]\right)^*$

(5) $u \in L^p$

$T_u : L^{p'} \to \mathbb{R}$

$T_u(v) = \int_\Omega uv$

$T_u \in (L^{p'})'$

$\|T_u\| = \|u\|$

$\to u^*$ $v = |u|^{p-2} u \in L^{p'}$

(2) $X = L^2(a,b)$: Given $u \in L^2(a,b)$ define

$F(v) = \int_a^b uv\, dx$ then $F \in \left[L^2(a,b)\right]^*$

(3) **Remark**: $Y \xrightarrow[\text{continuous}]{} X$ then $X^* \subset Y^*$

$F \in X^* \Rightarrow F \in Y^*$

$\|y_n - y\|_Y \to 0 \Rightarrow$

$\Rightarrow \|y_n - y\|_X \to 0$

(4) Remark: $T \in X^*$, $E \subset X$ dense, $T(E) = 0 \Rightarrow T \equiv 0$

Theory of Hilbert spaces: Riesz-Fréchet Representation Theorem

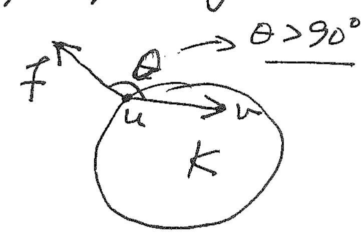
Theorem (projection on a closed convex set)

Let $(H, (\cdot,\cdot))$ Hilbert space, $K \subset H$ closed, convex set. Then for every $f \in H$ there is a unique $u \in K$ such that

$$\|f-u\| = \inf_{v\in K} \|f-v\| \qquad (*)$$

in addition, u is defined by the property

$$(**) \quad \boxed{\begin{array}{l} u \in K \\ (f-u, v-u) \le 0, \ \forall v \in K \end{array}}$$



Proof: Existence: Let $\{v_n\} \subset K$ such that

$$d_n = \|v_n - f\| \longrightarrow d \overset{\text{not}}{=} \inf_{v\in K} \|f-v\|$$

we show that $\{v_n\}$ is Cauchy

Recall the parallelogram identity:

$$\|a+b\|^2 + \|a-b\|^2 = 2\|a\|^2 + 2\|b\|^2$$

Apply it for $f-v_n$, $f-v_m$:

$$\|f-v_n + f-v_m\|^2 + \|f-v_n-(f-v_m)\|^2 = 2\|f-v_n\|^2 + 2\|f-v_m\|^2$$

$$\Rightarrow \underbrace{4\left\|f-\frac{v_n+v_m}{2}\right\|^2}_{\ge d^2 \text{ since } \frac{v_n+v_m}{2} \in K} + \|v_n-v_m\|^2 = 2\|f-v_n\|^2 + 2\|f-v_m\|^2$$

$$\Rightarrow \|v_n - v_m\|^2 \le 2\|f - v_n\|^2 + 2\|f - v_m\|^2 - 4d^2 \xrightarrow[n,m]{} 0 .$$

$$\Rightarrow \|v_n - v_m\| \longrightarrow 0 \Rightarrow v_n \longrightarrow u \in K$$

$$\Rightarrow \boxed{\|u - f\| = d}$$

Equivalence of $(*)$ and $(**)$:

$(**) \Rightarrow (*)$

Let u satisfy $(**)$. Then, $\forall v \in K$:

$$\|u-f\|^2 - \|v-f\|^2 = 2(f-u, v-u) - \|u-v\|^2 \le 0$$

$\Rightarrow \|u-f\| \le \|v-f\|$, $\forall v \in K$, so $(*)$ is verified.

$(*) \Rightarrow (**)$

Let u satisfy $(*)$, let $v \in K$ arbitrary, fixed.

Consider $w_\varepsilon = (1-\varepsilon)u + \varepsilon v \in K$, $\forall \varepsilon \in [0,1]$

Then $\|f-u\| \le \|f - w_\varepsilon\| = \|f - u - \varepsilon(v-u)\|$

$$\Rightarrow \|f-u\|^2 \le \|f-u\|^2 - 2\varepsilon(f-u, v-u) + \varepsilon^2\|v-u\|^2$$

$$\left.\begin{array}{l} \Rightarrow 2(f-u, v-u) \le \varepsilon\|v-u\|^2 \\ \text{let } \varepsilon \to 0 \end{array}\right\} \Rightarrow (f-u, v-u) \le 0$$

so $(**)$ is verified.

Uniqueness: Assume $(f-u_1, v-u_1) \le 0$, $\forall v \in K$
$(f-u_2, v-u_2) \le 0$, $\forall v \in K$

Then $\left.\begin{array}{l}(f-u_1, u_2-u_1) \le 0 \\ (f-u_2, u_1-u_2) \le 0\end{array}\right\} \Rightarrow \|u_2 - u_1\| \le 0 \Rightarrow u_1 = u_2$

We denote $P_K f = u$ $\rightarrow$ the projection of f on K

Then $P_K : H \rightarrow K \subset H$ is a well-defined operator since u is unique.

<u>Property</u>: The projection operator is contractive.

$$\forall f_1, f_2 \in H : \quad \| P_K f_1 - P_K f_2 \| \le \| f_1 - f_2 \|$$

<u>Proof</u>: Let $u_1 = P_K f_1$, $u_2 = P_K f_2$

$$\left.\begin{aligned} (f_1 - u_1, v - u_1) \le 0 &\overset{v = u_2}{\Longrightarrow} (f_1 - u_1, u_2 - u_1) \le 0 \\ (f_2 - u_2, v - u_2) \le 0 &\overset{v = u_1}{\Longrightarrow} (f_2 - u_2, u_1 - u_2) \le 0 \end{aligned}\right\} \overset{(+)}{\Longrightarrow}$$

$$\Rightarrow (f_1 - u_1 + u_2 - f_2, u_2 - u_1) \le 0 \Rightarrow$$

$$\Rightarrow (f_1 - f_2, u_2 - u_1) + \| u_2 - u_1 \|^2 \le 0$$

$$\Rightarrow \| u_2 - u_1 \|^2 \le (f_1 - f_2, u_1 - u_2) \le \| f_1 - f_2 \| \| u_1 - u_2 \|$$

$$\Rightarrow \| u_1 - u_2 \| \le \| f_1 - f_2 \|$$

* * *

<u>Corrolary</u> Let $M \subset H$ closed subspace and $f \in H$.

Then $u = P_M f$ <u>if and only if</u> $\begin{cases} u \in M \\ (f-u, v) = 0, \ \forall v \in M \end{cases}$

in addition, P_M is a <u>linear operator</u>.

<u>Proof</u>: "$\Rightarrow$" Let $u = P_M f$. Then $(f-u, v-u) \le 0, \ \forall v \in M$

Let $w \in M$ arbitrary, then $v = \varepsilon w + u \in M$, $\varepsilon \in \mathbb{R}$

$(f-u, \varepsilon w + u - u) \le 0 \Rightarrow \varepsilon (f-u, w) \le 0$

$\Rightarrow (f-u, w) = 0$. since ε is arbitrary.

"$\Leftarrow$" Let $u \in M$: $(f-u, v) = 0, \ \forall v \in M$.

Let $w \in M$ arbitrary, define $v = w - u \in M$.

Then $(f-u, w-u) = 0$, so $\underline{u = P_M f}$.

<u>Linearity of P_M</u>:

$$\left.\begin{aligned} (f - P_M f, v) &= 0 \\ (g - P_M g, v) &= 0 \end{aligned}\right\} \Rightarrow \Big(f + g - (P_M f + P_M g), v\Big) = 0$$

$$\Rightarrow \boxed{P_M f + P_M g = P_M (f+g)}$$

Also, $(f - P_M f, v) = 0 \Rightarrow (cf - cP_M f, v) = 0 \Rightarrow \boxed{P_M(cf) = cP_M f}$

Riesz - Fréchet representation theorem

Let $(H, (\cdot,\cdot))$ Hilbert space and $F : H \to \mathbb{R}$ linear and continuous functional. Then there is a unique $u \in H$ such that

$$F(v) = (u, v), \quad \forall v \in H$$

in addition, $\|F\|_{H^*} = \|u\|$

Proof: Uniqueness $(u_1, v) = (u_2, v)$, $\forall v \in H$
$\Rightarrow u_1 = u_2$.

Existence Let $M = F^{-1}(\{0\}) = \{v \in H : F(v) = 0\}$

Then M is a closed subspace of H.

Assume $M \neq H$, (otherwise $F \equiv 0$ and choose $u=0$)

Let $w \in H \setminus M$. Then $(w - P_M w, v) = 0$, $\forall v \in M$

Notice that $w - P_M w \notin M$ since $w \notin M$.

Let $\bar{w} = w - P_M w$, then $F(\bar{w}) \neq 0$ and
$(\bar{w}, v) = 0$, $\forall v \in M$

For any $v \in H$ we have $F\left(v - \frac{F(v)}{F(\bar{w})}\bar{w}\right) = 0$

$$\Rightarrow \quad v - \frac{F(v)}{F(\bar{w})}\bar{w} \in M$$

Define $u = \frac{F(\bar{w})}{\|\bar{w}\|^2}\bar{w} = \alpha \bar{w}, \quad \alpha = \frac{F(\bar{w})}{\|\bar{w}\|^2}$

Then $\left(u,\ v - \frac{F(v)}{F(\bar{w})}\bar{w}\right) = \alpha\left(\bar{w},\ v - \frac{F(v)}{F(\bar{w})}\bar{w}\right) = 0$

$$\Rightarrow (u, v) = \frac{F(v)}{F(\bar{w})}\cdot(u, \bar{w}) = \frac{F(v)}{F(\bar{w})}\cdot\frac{F(\bar{w})}{\|\bar{w}\|^2}\|\bar{w}\|^2$$

$$\Rightarrow \boxed{(u, v) = F(v)}$$

Show that $\|F\|_{H^*} = \|u\|$ use double inequality

$$\|F\|_* = \sup_{v \neq 0} \frac{|F(v)|}{\|v\|} \geq \frac{|F(u)|}{\|u\|} = \frac{\|u\|^2}{\|u\|} = \|u\|$$

$$|F(v)| = |(u, v)| \leq \|u\|\,\|v\|$$

$$\Rightarrow \sup_{v \neq 0} \frac{|F(v)|}{\|v\|} \leq \|u\| \Rightarrow \|F\|_* \leq \|u\|$$

$$\Rightarrow \boxed{\|F\|_* = \|u\|}$$