Higher order Sobolev spaces; Sobolev inequalities (embeddings)

Simple example: $\begin{cases} -u'' + u = f & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$

Find $u \in H_0^1(0,1)$: $\int_0^1 u'v' + \int_0^1 uv = \int_0^1 fv, \quad \forall v \in H_0^1$

Then $\int_0^1 u'v' = -\int_0^1 (u-f)v, \quad \forall v \in H_0^1$ in particular $\forall v \in C_c^\infty$

→ Definition of weak derivative: g is weak derivative of u' if $\int_0^1 u'v' = -\int_0^1 gv, \quad \forall v \in C_c^\infty$

Therefore, $g = u - f \in L^2$ is the weak derivative of u'

$(u')' = u - f$ ⟶ second order weak derivative

Thus the solution u is "more regular" ($u \in H_0^1 \cap H^2$)

Definition Let $u \in L^1_{loc}(\Omega)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ multiindex

A function $u^{(\alpha)} \in L^1_{loc}(\Omega)$ is the α-weak derivative of u if

$$\int_\Omega u D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega u^{(\alpha)} \phi, \quad \forall \phi \in C_c^\infty(\Omega).$$

Remark: Weak derivative is unique (a.e.), (assuming that it exists).

The Sobolev space $W^{k,p}(\Omega)$, $k \geq 0$ integer

$$W^{k,p}(\Omega) = \{ u : \Omega \to \mathbb{R}, \ u \text{ and all its weak derivatives up to order } k \text{ are in } L^p(\Omega) \}.$$

$$W^{k,p}(\Omega) = \{ u : \Omega \to \mathbb{R}, \ u \in L^p(\Omega), \ \forall \alpha, |\alpha| \leq k, \exists g_\alpha \in L^p(\Omega) : \int_\Omega u D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega g_\alpha \phi, \ \forall \phi \in C_c^\infty(\Omega) \}$$

(at $k=0$: $W^{0,p} \equiv L^p$)

Recursive definition

$$W^{k,p}(\Omega) = \{ u \in W^{k-1,p}(\Omega) : D^\alpha u \in W^{k-1,p}(\Omega), \ \forall \alpha, |\alpha| = 1 \}$$

$W^{k,p}$ - norm :

$$\|u\|_{k,p} = \left(\sum_{0 \leq |\alpha| \leq k} \int_\Omega |D^\alpha u|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty .$$

an equivalent norm is

$$\|u\| = \sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|_{L^p} \quad \to \quad 1 \leq p \leq \infty .$$

Denote $W_0^{k,p} = \overline{\{ C_c^\infty(\Omega), \| \ \|_{k,p} \}}$

Thus $u \in W_0^{k,p}$ if and only if there is a sequence $\{u_m\} \in C_c^\infty(\Omega)$ such that $u_m \to u$ in $W^{k,p}$ ($D^\alpha u_m \to D^\alpha u$ in L^p, $\forall \alpha, |\alpha| \leq k$).

$$W_0^{k,p}(\Omega) = \{ u \in W^{k,p}(\Omega) : D^\alpha u = 0 \text{ on } \partial\Omega, \forall \alpha, |\alpha| \le k-1 \}$$

in particular, $H^k = W^{k,2}(\Omega)$, $H_0^k = W_0^{k,2}(\Omega)$

are <u>Hilbert spaces</u> w.r.t. the inner product

$$(u,v)_k = \sum_{0 \le |\alpha| \le k} \int_\Omega D^\alpha u \, D^\alpha v$$

if $\Omega = I$: $\quad (u,v)_k = \int_I uv + \sum_{i=1}^{k} \int_I u^{(i)} v^{(i)}$

if $k=2$: $\quad (u,v)_2 = \int_\Omega uv + \int_\Omega \nabla u \cdot \nabla v + \sum_{i,j=1}^{n} \int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}$

<u>Remark</u> if $u, v \in W^{k,p}(\Omega)$ then for $|\alpha| \le k$

$D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ and

$$D^\beta (D^\alpha u) = D^\alpha (D^\beta u) = D^{\alpha+\beta} u, \quad \forall \alpha, \beta \quad |\alpha| + |\beta| \le k.$$

<u>Proof</u> Let $\phi \in C_c^\infty(\Omega)$. Then $D^\beta \phi \in C_c^\infty(\Omega)$

$$\int_\Omega u D^{\alpha+\beta} \phi = \int_\Omega u D^\alpha [D^\beta \phi] \overset{\text{def } D^\alpha u}{=} (-1)^{|\alpha|} \int_\Omega D^\alpha u \, D^\beta \phi$$

$D^{\alpha+\beta} u$ def $\longrightarrow$ $\Vert$

$$(-1)^{|\alpha+\beta|} \int (D^{\alpha+\beta} u) \phi \implies (-1)^{|\alpha+\beta|} \int_\Omega (D^{\alpha+\beta} u) \phi = (-1)^{|\alpha|} \int_\Omega D^\alpha u \, D^\beta \phi$$

$$\implies \int_\Omega D^\alpha u \, D^\beta \phi = (-1)^{|\alpha|} \cdot (-1)^{|\alpha+\beta|} \int_\Omega (D^{\alpha+\beta} u) \phi = (-1)^{|\beta|} \int_\Omega (D^{\alpha+\beta} u) \phi$$

$$\implies \boxed{D^{\alpha+\beta} u = D^\beta (D^\alpha u)}$$

Sobolev inequalities

Definition Let X, Y Banach spaces.
Let $T: X \to Y$ linear and continuous operator.
T is a compact operator if $T(\bar{B}_X(0,1))$ is a precompact (relatively compact) in Y, i.e. $\overline{T(\bar{B}_X(0,1))}$ is compact set in Y.

Property: Any sequence included in a compact set has a subsequence that is convergent.

Remark: if T is compact operator and $\{x_n\}$ is a bounded sequence in X then $\{T(x_n)\}$ has a subsequence that is convergent in Y.

Example: Projection operator on a finite dimensional subspace of a Hilbert space.

$S = \text{span}\{e_1, \dots, e_m\}$.

$$T(x) = \sum_{i=1}^{m} (x, e_i)\, e_i = \sum_{i=1}^{m} \alpha_i e_i \,, \quad \alpha_i \stackrel{def}{=} (x, e_i)$$

Fundamental example : The unit ball in L^2 is not compact. Let $\Omega = (0, 2\pi)$,

$$f_n(x) = \frac{1}{\sqrt{\pi}} \sin nx \quad \text{is such that}$$

$$\|f_n\|_{L^2} = 1, \quad \|f_n - f_m\|_{L^2} = 2, \quad \forall n \neq m.$$

However, let $\tilde{f}_n(x) = \dfrac{1}{\sqrt{\pi(1+n^2)}} \sin nx$

then $\|\tilde{f}_n\|_1 = 1, \quad \|\tilde{f}_n\|_{L^2} \xrightarrow[n]{} 0$

$\rightarrow$ bounded sequences in H^1 have convergent subsequences in L^2.

Notice that

thus $\|\tilde{f}_n\|_1 = 1$

$$\left(\begin{array}{l} \displaystyle\int_0^{2\pi} \tilde{f}_n^2 = \frac{1}{\pi(1+n^2)} \int_0^{2\pi} \sin^2 nx = \frac{1}{1+n^2} \to 0 \\ \displaystyle\int_0^{2\pi} \tilde{f}_n'^2 = \frac{n^2}{\pi(1+n^2)} \int_0^{2\pi} \cos^2 nx = \frac{n^2}{1+n^2} \to 1 \end{array} \right.$$

$$(\tilde{f}_n, \tilde{f}_m)_1 \approx \int_0^{2\pi} \sin nx \sin mx + \int_0^{2\pi} \cos nx \cos mx = 0$$

thus $\|\tilde{f}_n - \tilde{f}_m\|_1 = \sqrt{2}$

Definition Let X, Y Banach spaces, $X \subset Y$. We say that X is compactly embedded in Y and denote $X \subset\subset Y$ if the injection operator

$T : X \to Y$, $T(x) = x$ is compact:

i) $\exists C : \|x\|_Y \le C\|x\|_X$, $\forall x \in X$ (continuity)

ii) $\forall \{x_n\} \subset X$ bounded sequence in X
~~Then~~ there is a convergent subsequence in Y:
$\{x_{n_k}\} \to x$ in Y.

Definition if $1 \le p < n$, the Sobolev conjugate of p is defined as

$$\boxed{p^* = \frac{np}{n-p}}$$

Notice that $p^* > p$ and

$$\boxed{\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}}$$

Sobolev inequalities (Rellich - Kondrachov) theorem

Let $1 \leq p \leq \infty$. and $\Omega \subset \mathbb{R}^n$ bounded domain, of class C^1.

if $1 \leq p < n$ then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $p \leq q \leq p^*$

if $p = n$ then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $p \leq q < \infty$

if $p > n$ then $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$

are continuous embeddings (injections).

in addition, if $k - \frac{n}{p} > m$ then

$W^{k,p}(\Omega) \hookrightarrow C^m(\bar{\Omega})$ is a continuous injection

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Remark : To the $W^{k,p}(\Omega)$ associate

$\boxed{k - \frac{n}{p}}$ as a coefficient of regularity.

$L^p(\Omega) \longrightarrow -\frac{n}{p}$

$C^m(\Omega) \longrightarrow m$

For $W^{1,p}$: regularity $1 - \frac{n}{p}$

L^q : regularity $-\frac{n}{q}$

$$\boxed{1 - \frac{n}{p} \geq -\frac{n}{q}}$$

for continuous embedding.

Compact embeddings (Rellich - Kondrachov) theorem

Let $\Omega \subset \mathbb{R}^n$ bounded, of class C^1.

if $1 \le p < n$ then $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$, $1 \le q < p^*$

if $p = n$ then $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$, $1 \le q < \infty$

if $p > n$ then $W^{1,p}(\Omega) \subset\subset C(\bar{\Omega})$

are compact embeddings (injections).

in particular, for any $1 \le p \le \infty$,

$W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact injection

Remark for $p = 2$, $2^* = \dfrac{2n}{n-2}$

if $\Omega \subset \mathbb{R}^3$ then $H^1(\Omega) \hookrightarrow L^6(\Omega)$ continuous

for $\varepsilon > 0$: $H^1(\Omega) \subset\subset L^{6-\varepsilon}(\Omega)$ compact

if $\Omega \subset \mathbb{R}^2$ then $H^1(\Omega) \hookrightarrow L^q(\Omega)$ compact $1 \le q < \infty$.

if $\Omega = I$ then $H^1(I) \hookrightarrow C(\bar{I})$ compact.

<u>Theorem</u> (regularity of the weak solution for the Dirichlet problem)

Let $\Omega \subset \mathbb{R}^n$ open and bounded domain with boundary $\partial\Omega$ of class C^2. Let $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ weak solution to the Dirichlet problem:

$$\int_\Omega \nabla u \nabla v + \int_\Omega u v = \int_\Omega f v, \qquad \forall v \in H_0^1(\Omega)$$

Then $u \in H^2(\Omega)$ and $\|u\|_{H^2} \le C \|f\|_{L^2}$ where C is a constant depending on Ω only.

In addition, if Ω is of class C^{m+2} and $f \in H^m(\Omega)$ then $u \in H^{m+2}(\Omega)$ and $\|u\|_{H^{m+2}} \le C \|f\|_{H^m}$

in particular, if $m > \frac{n}{2}$ then <u>$u \in C^2(\bar{\Omega})$</u>

if Ω is of class C^∞ and $f \in C^\infty(\bar{\Omega})$ then $u \in C^\infty(\bar{\Omega})$.

The same conclusions hold for the <u>Neumann problem</u>
$$\int_\Omega \nabla u \nabla v + \int_\Omega u v = \int_\Omega f v, \qquad \forall v \in H^1(\Omega)$$

Remark: The regularity of the weak solution holds for the general uniformly elliptic operator

$$\mathcal{L}u = -\nabla\cdot(K\nabla u) + b\cdot\nabla v + q u$$

under the assumption that the coefficients are such that $K_{ij} \in C^{m+1}(\bar{\Omega})$, $b_i \in C^{m+1}(\bar{\Omega})$,

$$q \in C^{m+1}(\bar{\Omega}), \quad f \in H^m(\Omega)$$

Then $u \in H^{m+2}(\Omega)$