The general solution to the PDE

Definition A function $\phi : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$, $\phi \in C^1(\Omega)$ is called a first integral to the PDE

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u) \qquad (*)$$

if

$$\boxed{a\phi_x + b\phi_y + c\phi_u = 0} \qquad (**)$$

Property if ϕ is a first integral and satisfies $\phi_u(\bar{x},\bar{y},\bar{u}) \neq 0$ then any level surface

$$\phi(x,y,u) = K \quad \text{(constant)}$$

defines a solution to the PDE $(*)$ in a vicinity of $(\bar{x},\bar{y},\bar{u})$.

Proof implicit function theorem

$\phi_u(\bar{x},\bar{y},\bar{u}) \neq 0 \Rightarrow u = f(x,y)$ in a vicinity of $(\bar{x},\bar{y})$

$$\phi(x,y,f(x,y)) = K \Rightarrow \left.\begin{array}{l} \phi_x + \phi_u f_x = 0 \\ \phi_y + \phi_u f_y = 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow a\phi_x + b\phi_y + \phi_u(af_x + bf_y) = 0 \overset{(**)}{\Longrightarrow} \boxed{au_x + bu_y = c}$$

Property: if ϕ is a first integral to the PDE then for any function $F: \mathbb{R} \to \mathbb{R}$ continuously differentiable and such that $F' \neq 0$ we have that

$F(\phi)$ is a first integral and therefore

$F(\phi) = K$ (constant) defines a solution.

Proof: Let $\psi(x,y,u) = F(\phi(x,y,u))$

Then $a\psi_x + b\psi_y + c\psi_u = F'(\phi(x,y,u))[a\phi_x + b\phi_y + c\phi_u]$
$= 0$.

Definition: Two functions φ_1 and φ_2 are functionally independent ~~if~~ in $\Omega \subset \mathbb{R}^3$ if

$$\nabla\varphi_1 \times \nabla\varphi_2 \neq 0 \quad \text{in } \Omega .$$

Property: There are only two functionally independent first integrals to the PDE.

Proof: By contradiction

Let $\varphi_1, \varphi_2, \varphi_3$ first integrals.

$$\begin{cases} a\varphi_{1,x} + b\varphi_{1,y} + c\varphi_{1,u} = 0 \\ a\varphi_{2,x} + b\varphi_{2,y} + c\varphi_{2,u} = 0 \\ a\varphi_{3,x} + b\varphi_{3,y} + c\varphi_{3,u} = 0 \end{cases}$$

assume $a^2 + b^2 + c^2 \neq 0$.

then we must have

$$\begin{vmatrix} \varphi_{1,x} & \varphi_{1,y} & \varphi_{1,u} \\ \varphi_{2,x} & \varphi_{2,y} & \varphi_{2,u} \\ \varphi_{3,x} & \varphi_{3,y} & \varphi_{3,u} \end{vmatrix} = 0$$

$$\Rightarrow \quad \nabla\varphi_3 = k_1 \nabla\phi_1 + k_2 \nabla\phi_2 \quad \text{in } \Omega.$$

$$\varphi_3 = k_1 \phi_1 + k_2 \varphi_2 + k.$$

<u>Definition</u> The general solution to the PDE is expressed as $F(\varphi_1, \varphi_2) = 0$

where φ_1, φ_2 are functionally independent first integrals to the PDE and $F : \mathbb{R}^2 \to \mathbb{R}$ is an <u>arbitrary</u> $C^1(\mathbb{R}^2)$ function such that $F_\xi^2 + F_\eta^2 \neq 0$

<u>Remark</u>: Along the characteristics

$$\frac{dx}{d\tau} = a, \quad \frac{dy}{d\tau} = b, \quad \frac{du}{d\tau} = c$$

we have $$\frac{d\phi}{d\tau} = \phi_x x'(\tau) + \phi_y y'(\tau) + \phi_u u'(\tau) =$$
$$= a\phi_x + b\phi_y + c\phi_u = 0.$$

First integrals are constant along the characteristics: $\phi(\tau) = \phi(x(\tau), y(\tau), u(\tau)) = \text{const}$.

<u>Example</u>: Find the solution to $\begin{cases} u_x + 2u_y = u^2 \\ u(x,0) = x \end{cases}$

$$\frac{dx}{1} = \frac{dy}{2} = \frac{du}{u^2} \Rightarrow \begin{cases} 2dx = dy \Rightarrow 2x - y = k \Rightarrow \boxed{\phi_1(x,y,u) = 2x-y} \\ -\frac{1}{u} = x + k \Rightarrow x + \frac{1}{u} = k \end{cases}$$

$$\Rightarrow \boxed{\phi_2(x,y,u) = x + \frac{1}{u}}$$

General Solution: $\boxed{F\left(2x-y, x+\frac{1}{u}\right) = 0}$

Solution to Cauchy problem:

$$x + \frac{1}{u} = f(2x-y) \Rightarrow u(x,y) = \frac{1}{f(2x-y) - x}$$

$$u(x,0) = \frac{1}{f(2x) - x} = x \Rightarrow f(2x) = x + \frac{1}{x} \Rightarrow f(x) = \frac{x}{2} + \frac{2}{x}$$

$$\Rightarrow \boxed{u(x,y) = \frac{2(2x-y)}{4 + y^2 - 2xy}}$$

Example : Burgers Equation

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = h(x) \end{cases}$$

First integrals: $\dfrac{dt}{1} = \dfrac{dx}{u} = \dfrac{du}{0}$

$u = \text{const} \Rightarrow \varphi_1(x,t,u) = u$

$dx = u\,dt \xrightarrow{u = ct} \varphi_2(x,t,u) = x - ut$.

General solution $F(u, x-ut) = 0$

$$\left.\begin{aligned} u(x,t) &= f(x - u(x,t)t) \\ u(x,0) &= h(x) = f(x) \end{aligned}\right\} \Rightarrow \boxed{u = h(x-ut)}$$

implicit solution

may exist only locally near $t = 0$.

$F(x,t,u) = u - h(x-ut) = 0$.

$F_u = 1 + t h'(x-ut)$

$F_u(x,t,u)\big|_{t=0} = 1 \neq 0$ a local solution exists

Singularity may appear if

$$1 + t h'(x-ut) = 0$$

if $h'(s) \geq 0 \ \forall s$ then the solution exists for $\forall t \geq 0$

Find the solution to the Cauchy problem

$$\begin{cases} (y+u)\,u_x + y\,u_y = x-y \\ u(x,1) = 1+x \end{cases}$$

Existence and uniqueness

Γ: $x_0(s) = s$
$y_0(s) = 1$
$u_0(s) = 1+s$

$$\begin{pmatrix} x_0' & y_0' \\ a & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2+s & 1 \end{pmatrix}$$

there is an unique solution

First integrals :

$$\frac{dx}{y+u} = \frac{dy}{y} = \frac{du}{x-y} = \frac{d(x+u)}{x+u}$$

$$\frac{x+u}{y} = k_1 \quad : \quad \boxed{\varphi_1 = \frac{x+u}{y}}$$

$$\frac{d(x-y)}{u} = \frac{du}{x-y} \Rightarrow u^2 - (x-y)^2 = k_2 \quad : \quad \boxed{\varphi_2 = u^2 - (x-y)^2}$$

General integral : $\boxed{F\left(\frac{x+u}{y},\ u^2 - (x-y)^2\right) = 0}$

$$u^2 = (x-y)^2 + f\left(\frac{x+u}{y}\right)$$

Find f: $u(x,1) = 1+x \Rightarrow (1+x)^2 = (x-1)^2 + f(2x+1)$

$$\Rightarrow 4x = f(2x+1) \Rightarrow f(x) = 2(x-1)$$

$$\Rightarrow u^2(x,y) = (x-y)^2 + 2\cdot\frac{x-y+u(x,y)}{y}$$

$$(u-x+y)(u+x-y) = \frac{2}{y}\left[u+x-y\right]$$

$$\Rightarrow \left\{ u = x-y+\frac{2}{y} \right.$$

Consider the continuity equation

$$\begin{cases} u_t + [\phi(u)]_x = 0 \\ u(x,0) = f(x) \end{cases} \Rightarrow \begin{cases} u_t + g(u)\, u_x = 0 \\ u(x,0) = f(x) \end{cases}$$

Characteristic equations

$$\frac{dx}{dt} = g(u) \Rightarrow x(t) = x_0 + t\, g(u_0) = x_0 + t\, g[f(x_0)]$$

$$\frac{du}{dt} = 0 \Rightarrow u(x,t) = u(x_0,0) = f(x_0)$$

$$u(\underbrace{x_0 + t\, g(f(x_0))}_{x}, t) = f(x_0).$$

$$\Rightarrow x_0 + t\, g(u) = x$$

$$\boxed{u(x,t) = f(x - t\, g(u))} \quad \text{implicit solution}$$

$$F(x,t,u) = u - f(x - t\, g(u))$$

$$F_u \big|_{t=0} = 1 + t\, f'(x - t\, g(u))\, g'(u) \big|_{t=0} = 1 \neq 0$$

$\Rightarrow$ local solution exists.

Particular case : Burgers equation

$$u_t + u u_x = 0 \qquad g(u) = u.$$

$$\boxed{x(t) = x_0 + t\, f(x_0)} \qquad u(x(t), t) = f(x_0).$$

Solution exists for $t > 0$ if f increasing.
(characteristics do not intersect)

$$\frac{dt}{1} = \frac{dx}{g(u)} = \frac{du}{0}$$

$u = k_1 \qquad g(u)\,dt = dx$

$\qquad\qquad x - t g(u) = k_2$

$\varphi(x,t,u) = g \quad x - t g(u)$

$-g(u) + g(u) = 0.$

$u = F(x - t g(u))$

$F(x) = f(x).$