# First order PDEs

A first order PDE for the unknown function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is an equation of the form

$$(*) \quad F(x, u, Du) = 0, \quad \text{where } F : \mathcal{D}(F) \subset \mathbb{R}^{2n+1} \to \mathbb{R}$$

A solution $u(x)$ is a function $u \in C^1(\Omega)$ such that

1) $\forall x \in \Omega$ : $(x, u(x), Du(x)) \in \mathcal{D}(F)$
2) The equation $(*)$ is satisfied $\forall x \in \Omega$.

## Classification of the first order PDEs

<u>Linear</u> $\quad \alpha(x) \cdot \nabla u(x) + \beta(x) u(x) = f(x)$

<u>Semilinear</u> $\quad \alpha(x) \cdot \nabla u(x) = f(x, u)$

<u>Quasi-linear</u> $\quad \alpha(x, u) \cdot \nabla u(x) = f(x, u)$

Otherwise, <u>nonlinear</u> e.g., $u_x^2 + u_y^2 = 1$

Examples in $\mathbb{R}^2$ : $u(x, y)$

$x u_x + y u_y = u^2 \longrightarrow$ semilinear

$u_x + u_y = u + f(x, y) \longrightarrow$ linear

$u u_x + u_y = f(x, y, u) \longrightarrow$ quasi-linear

## The method of characteristics : first illustration

Consider $u(x,t)$ and the PDE

$$u_t + a(x,t)\,u_x + b(x,t)\,u = c(x,t) \qquad (*)$$

→ look for curves $x = x(t)$ in the $(x,t)$ plane such that along $(x(t),t)$ the PDE $(*)$ is reduced to an ODE for $u(x(t),t)$

$$\frac{du}{dt} = u_x\, x'(t) + u_t$$

Let $\boxed{x'(t) = a(x(t),t)}$ → ODE for $x(t)$.

Then

$$\frac{du}{dt}(x(t),t) + b(x(t),t)\,u(x(t),t) = c(x(t),t)$$

ODE for $U(t) = u(x(t),t)$.

Example Solve the initial-value problem

$$\begin{cases} u_t + x\,u_x = \sin t \\ u(x,0) = x+1 \end{cases}$$

$$x'(t) = x \Rightarrow x(t) = x(0)\,e^t = x_0 e^t$$

$$\frac{d}{dt}u(x(t),t) = \sin t \Rightarrow u(x_0 e^t, t) = -\cos t + \underbrace{u(x_0,0)}_{+1}$$

$$= -\cos t + x_0 + 2$$

Then, $\boxed{u(x,t) = xe^{-t} - \cos t + 2}$

## The method of characteristics for quasilinear equations

Consider $$\boxed{a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)} \quad (*)$$

where $a, b, c$ continuous functions.

**Geometrical interpretation**

Consider the vector field in $\mathbb{R}^3$ defined as

$$\vec{V}(x,y,z) = \big( a(x,y,z),\ b(x,y,z),\ c(x,y,z) \big)$$

The solution to $(*)$ is a surface $u(x,y)$ in the 3D space $x, y, z$, $z = u(x,y)$.

Tangent plane to the surface at $(x_0, y_0, u_0)$ is

$$u - u_0 = u_x(x_0,y_0)(x-x_0) + u_y(x_0,y_0)(y-y_0)$$

The normal vector at $(x_0, y_0, u_0)$ is

$$\nu_0 \left( -u_x(x_0,y_0),\ -u_y(x_0,y_0),\ 1 \right)$$

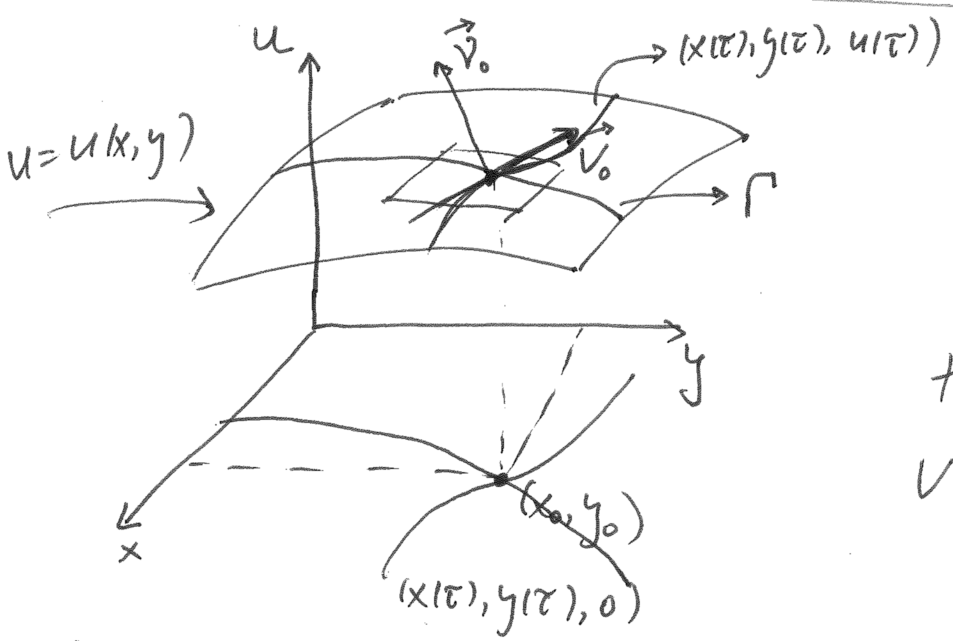
such that $\boxed{\vec{V}_0 \cdot \vec{\nu}_0 = 0}$

A solution $u(x,y)$ is a surface such that the vector field $V\big(a(x,y,u(x,y)), b(x,y,u(x,y)), c(x,y,u(x,y))\big)$ is tangent to the surface

The Cauchy problem: in addition, we require that the solution surface $u(x,y)$ contains a given curve $\Gamma \subset \mathbb{R}^3$ with parametric equations $\Gamma : (x_0(s), y_0(s), u_0(s))$

Find $u(x,y)$ such that

$$\begin{cases} a(x,y,u)\, u_x + b(x,y,u)\, u_y = c(x,y,u) \\ u(x_0(s), y_0(s)) = u_0(s) \end{cases}$$




Characteristic curves

Solution curves that are tangent to $V(a,b,c)$ at $(x,y,u)$

Parametric curves $(x(\tau), y(\tau), u(\tau))$ that satisfy

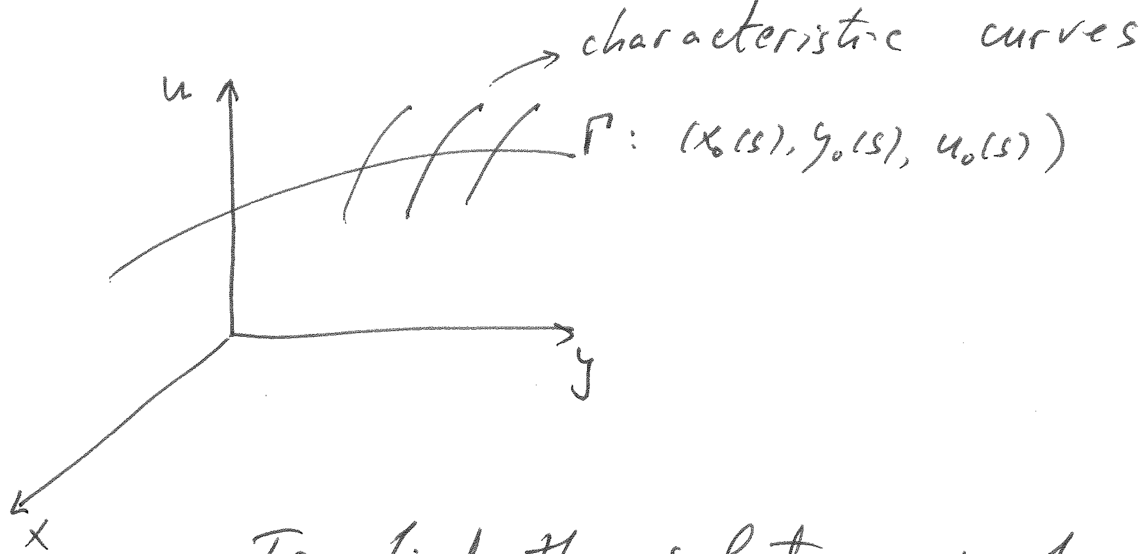
$$\frac{dx}{d\tau} = a(x,y,u)$$

$$\frac{dy}{d\tau} = b(x,y,u)$$

$$\frac{du}{d\tau} = c(x,y,u)$$

then $U(\tau) = u(x(\tau), y(\tau))$

$$\frac{du}{d\tau} = u_x x' + u_y y' = a u_x + b u_y = c$$




To find the solution surface, solve the characteristics equations:

$$\frac{dx}{d\tau} = a(x,y,u)$$
$$\frac{dy}{d\tau} = b(x,y,u)$$
$$\frac{du}{d\tau} = c(x,y,u)$$

with the initial conditions

$$\left.\begin{aligned} x(0,s) &= x_0(s) \\ y(0,s) &= y_0(s) \\ u(0,s) &= u_0(s) \end{aligned}\right\} \Gamma$$

Solution $x = x(s,\tau)$, $y = y(s,\tau)$, $u = u(s,\tau)$.

if $\left.\frac{\partial(x,y)}{\partial(s,\tau)}\right|_{\tau=0}$ is nonsingular matrix i.e.,

$$\left.\begin{vmatrix} x_s & x_\tau \\ y_s & y_\tau \end{vmatrix}\right|_{\tau=0} \neq 0 \quad : \quad \begin{vmatrix} x_0'(s) & a \\ y_0'(s) & b \end{vmatrix} \neq 0 .$$

then we may solve for $s, \tau$ in terms of $x, y$

$\Rightarrow$ $u = u(\tau(x,y), s(x,y))$ unique solution

The PDE must be satisfied along $\Gamma$, such that

$\Gamma$: $u_0(s) = u(x_0(s), y_0(s))$ implies

$$\left.\begin{aligned} x_0'(s)\,u_x + y_0'(s)\,u_y &= u_0'(s) \\ a\,u_x + b\,u_y &= c \end{aligned}\right\} \text{ system for } u_x,\ u_y$$

No solution, infinite number of solutions, or unique solution.

<u>Definition</u> The curve $\Gamma$ is non-characteristic for the PDE if the matrix

$$\begin{pmatrix} x_0'(s) & y_0'(s) \\ a(x_0(s), y_0(s), u_0(s)) & b(x_0(s), y_0(s), u_0(s)) \end{pmatrix}$$

is nonsingular : $b(x_0(s), y_0(s), u_0(s))\,x_0'(s) \neq a(x_0(s), y_0(s), u_0(s))\,y_0'(s)$

<u>Theorem</u> Let $\Omega \subset \mathbb{R}^3$, $a, b, c \in C^1(\Omega)$ given functions and $\Gamma$: $(x_0(s), y_0(s), u_0(s))$ a $C^1$ curve in $\Omega$. If $\Gamma$ is non-characteristic then the Cauchy problem $\begin{cases} a u_x + b u_y = c \\ u(x_0(s), y_0(s)) = u_0(s) \end{cases}$ has an unique solution defined in a vicinity of $\Gamma$

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if $bx_0' = ay_0'$ and $cx_0' = au_0'$ there are an infinite number of solutions

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if $bx_0' = ay_0'$ and $cx_0' \neq au_0'$ there are no solutions

Example 1 $\quad x u_x + (x+y) u_y = u+1$

$$u(x,0) = x^2 \quad : \quad \boxed{\begin{aligned} \Gamma : \quad & x_0(s) = s \\ & y_0(s) = 0 \\ & u_0(s) = s^2 \end{aligned}}$$

Characteristic equations

$$\frac{dx}{d\tau} = x \quad \Rightarrow \quad x(\tau, s) = x_0(s) e^{\tau} = s e^{\tau}$$

$$\frac{dy}{d\tau} = x+y \quad \Rightarrow \quad y(\tau, s) = s \tau e^{\tau}$$

$$\frac{du}{d\tau} = u+1 \quad \Rightarrow \quad u(\tau, s) = (s^2+1) e^{\tau} - 1$$

$$\left.\begin{aligned} x &= s e^{\tau} \\ y &= s \tau e^{\tau} \end{aligned}\right\} \Rightarrow \tau = \frac{y}{x}, \quad s = x e^{-\frac{y}{x}}$$

$$\Rightarrow \quad u(x,y) = \left(x^2 e^{-\frac{2y}{x}} + 1\right) e^{\frac{y}{x}} - 1$$

$$\boxed{u(x,y) = x^2 e^{-\frac{y}{x}} + e^{\frac{y}{x}} - 1}$$

Example 2 Solve $\begin{cases} uu_x + yu_y = x \\ u(x,1) = 2x \end{cases}$

$\Gamma$: $x_0(s) = s$ $\qquad \frac{x_0'}{a} = \frac{1}{2s}$

$y_0(s) = 1$

$u_0(s) = 2s$ $\qquad \frac{y_0'}{b} = 0$

Characteristic equations

$$\begin{cases} \frac{dx}{d\tau} = u \\ \frac{dy}{d\tau} = y \\ \frac{du}{d\tau} = x \end{cases}$$

initial conditions

$$\begin{cases} x(s,0) = x_0(s) = s \\ y(s,0) = y_0(s) = 1 \\ u(s,0) = u_0(s) = 2s \end{cases}$$

Solution:

$$\left.\begin{aligned} x(s,\tau) &= \frac{3s}{2}e^{\tau} - \frac{s}{2}e^{-\tau} \\ y(s,\tau) &= e^{\tau} \\ u(s,\tau) &= \frac{3s}{2}e^{\tau} + \frac{s}{2}e^{-\tau} \end{aligned}\right\} \Rightarrow \boxed{u = \frac{x(1+3y^2)}{3y^2-1}}$$

Example 3 Cauchy problem with no solution

$$\begin{cases} x u_x + y u_y = u + 1 \\ u(x, x) = x^2 \end{cases}$$

$\Gamma$: $x_0(s) = s$, $y_0(s) = s$, $u_0(s) = s^2$

$$\begin{vmatrix} x_0'(s) & x_0(s) \\ y_0'(s) & y_0(s) \end{vmatrix} = 0$$

Along $\Gamma$: $\begin{cases} u_x + u_y = 2s \\ s u_x + s u_y = s^2 + 1 \end{cases}$ $\longrightarrow$ no solution

Example 4 Cauchy problem with infinite number of solutions

$$\begin{cases} x u_x + y u_y = u + 1 \\ u(x, x) = 2x - 1 \end{cases}$$

Along $\Gamma$: $\left.\begin{array}{l} u_x + u_y = 2 \\ s u_x + s u_y = 2s \end{array}\right\}$ infinite # of solutions

$\left.\begin{array}{l} u(x,y) = x + y - 1 \\ u(x,y) = 2x - 1 \\ u(x,y) = 2y - 1 \end{array}\right\}$ for example, solution functions

$$\boxed{u = x f\left(\frac{y}{x}\right) - 1}$$