Method of characteristics for 1^{st} order nonlinear PDE

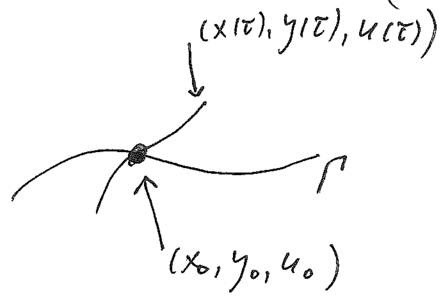
$$F(x, y, u, u_x, u_y) = 0 \qquad (1)$$

Notation: $p = u_x \qquad q = u_y$

Cauchy problem: $\Gamma: \ (x_0(s), y_0(s), u_0(s))$

$$u(x_0(s), y_0(s)) = u_0(s) \qquad (2)$$

For a generic initial point (x_0, y_0, u_0) on Γ we look for a curve $(x(\tau), y(\tau), u(\tau))$ on the surface u such that $(x(0), y(0), u(0)) = (x_0, y_0, u_0)$.



in the quasi-linear case

$$a u_x + b u_y = c$$

the curve was tangent to $V(a, b, c)$

in the nonlinear case, the derivatives

$$p(\tau) = u_x(x(\tau), y(\tau))$$

$$q(\tau) = u_y(x(\tau), y(\tau))$$

must be evaluated along with $x(\tau), y(\tau), u(\tau)$.

$\longrightarrow$ 5 dimensional ODE system.

Derivation of the characteristic equations

PDE $\quad F(x,y,u,u_x,u_y)=0$: take x-, y- derivatives

$$F_x + F_u u_x + F_p u_{xx} + F_q u_{xy} = 0 \qquad (3)$$

$$F_y + F_u u_y + F_p u_{xy} + F_q u_{yy} = 0 \qquad (4)$$

Along the curve $\mathcal{C}$: $(x(\tau), y(\tau), u(\tau))$

$$\left.\begin{aligned} p(\tau) &= u_x(x(\tau), y(\tau)) \\ q(\tau) &= u_y(x(\tau), y(\tau)) \end{aligned}\right\} \text{ differentiate}$$

$$p'(\tau) = u_{xx}\, x'(\tau) + u_{xy}\, y'(\tau) \qquad (5)$$

$$q'(\tau) = u_{xy}\, x'(\tau) + u_{yy}\, y'(\tau) \qquad (6)$$

To eliminate the second order derivatives, impose

$$\boxed{\begin{aligned} x'(\tau) &= F_p(x,y,u,p,q) \qquad (7) \\ y'(\tau) &= F_q(x,y,u,p,q) \qquad (8) \end{aligned}}$$

Then (5), (6), (3), (4) provide

$$\boxed{\begin{aligned} p'(\tau) &= -\left[F_x(x,y,u,p,q) + p F_u(x,y,u,p,q)\right] \qquad (10) \\ q'(\tau) &= -\left[F_y(x,y,u,p,q) + q F_u(x,y,u,p,q)\right] \qquad (11) \end{aligned}}$$

in addition, $u(x(\tau), y(\tau))$ provides

(9) $\quad \boxed{u'(\tau) = u_x\, x'(\tau) + u_y\, y'(\tau) = p F_p(x,y,u,p,q) + q F_q(x,y,u,p,q)}$

in short notation, the characteristic equations are:

$$(*)\quad \boxed{\begin{aligned} x'(\tau) &= F_p \\ y'(\tau) &= F_q \\ u'(\tau) &= pF_p + qF_q \\ p'(\tau) &= -[F_x + pF_u] \\ q'(\tau) &= -[F_y + qF_u] \end{aligned}}$$

initial conditions are specified as follows:

$$\boxed{\left\{\begin{aligned} x(0) &= x_0 \\ y(0) &= y_0 \\ u(0) &= u_0 \end{aligned}\right.}\qquad (**)$$

The initial values $p(0)$ and $q(0)$ are determined by:

$$\begin{aligned} &\text{PDE}: \\ &\Gamma: \end{aligned}\ \boxed{\left\{\begin{aligned} &F(x_0, y_0, u_0, p_0, q_0) = 0 \\ &p_0 x_0' + q_0 y_0' = u_0' \end{aligned}\right.}\qquad (***)$$

For a generic initial point $(x_0(s), y_0(s), u_0(s))$ we have initial conditions $(**)$ and $(***)$

$$\boxed{\begin{aligned} &x(s,0) = x_0(s) \quad y(s,0) = y_0(s), \quad u(s,0) = u_0(s) \\ &p(s,0) = p_0(s) \quad q(s,0) = q_0(s) \end{aligned}}$$

<u>Example</u> Solve the Cauchy problem

$$\begin{cases} u_x u_y = u \\ u(0,y) = y^2 \end{cases} \qquad F(x,y,u,p,q) = pq - u$$

Characteristic equations

$$(*) \begin{cases} x'(\tau) = F_p = q \\ y'(\tau) = F_q = p \\ u'(\tau) = pF_p + qF_q = 2pq \overset{\text{PDE}}{=} 2u \\ p'(\tau) = -[F_x + pF_u] = -[0 + p(-1)] = p \\ q'(\tau) = -[F_y + qF_u] = -[0 + q(-1)] = q \end{cases}$$

General solution to $(*)$

$$\begin{cases} p(s,\tau) = p_0(s)e^{\tau} \\ q(s,\tau) = q_0(s)e^{\tau} \\ u(s,\tau) = u_0(s)e^{2\tau} \\ x(s,\tau) = q_0(s)e^{\tau} + [x_0(s) - q_0(s)] \\ y(s,\tau) = p_0(s)e^{\tau} + [y_0(s) - p_0(s)] \end{cases}$$

initial conditions $(**)$

$$\Gamma: \boxed{x_0(s) = 0, \quad y_0(s) = s, \quad u_0(s) = s^2}$$

initial conditions $(***)$
$$\left.\begin{matrix} \text{PDE}: & p_0 q_0 = u_0 = s^2 \\ \Gamma: & p_0 \cdot 0 + q_0 = 2s \end{matrix}\right\} \boxed{\begin{matrix} q_0(s) = 2s \\ p_0(s) = \frac{s}{2} \end{matrix}}$$

$$
\left.\begin{aligned}
x(s,\tau) &= 2se^{\tau} - 2s = 2s(e^{\tau}-1) \\
y(s,\tau) &= \frac{s}{2}e^{\tau} + \frac{s}{2} = \frac{s}{2}(e^{\tau}+1)
\end{aligned}\right\} se^{\tau} = y + \frac{x}{4}
$$

$$
u(s,\tau) = s^2 e^{2\tau} = (se^{\tau})^2 \Rightarrow \boxed{\left(y+\frac{x}{4}\right)^2 = u}
$$

Remarks: Existence and uniqueness of the solution requires

i) Existence and uniqueness to (***)
(initial data is compatible with the PDE)

ii) inversion $\left\{\begin{array}{l} x(s,\tau) \\ y(s,\tau) \end{array}\right. \rightarrow \begin{array}{l} s(x,y) \\ \tau(x,y) \end{array}$ at $\tau = 0$.

$$
\left.\frac{\partial(x,y)}{\partial(s,\tau)}\right|_{\tau=0} = \begin{pmatrix} x_0' & F_p(x_0,y_0,u_0,p_0,q_0) \\ y_0' & F_q(x_0,y_0,u_0,p_0,q_0) \end{pmatrix} \quad \text{nonsingular}
$$

$$
\boxed{x_0' F_q - y_0' F_p \neq 0}
$$

Theorem (existence and uniqueness)

Consider the PDE $F(x,y,u,u_x,u_y)=0$, denote $p=u_x$, $q=u_y$, $F: \Omega \subset \mathbb{R}^5 \to \mathbb{R}$, $F \in C^2(\Omega)$ and $F_p^2 + F_q^2 \neq 0$ in Ω. Consider the parametric curve $\Gamma: (x_0(s), y_0(s), u_0(s))$, continuously differentiable

Assume that the system

$$(***) \quad \begin{cases} F(x_0, y_0, u_0, p_0, q_0) = 0 \\ x_0' p_0 + y_0' q_0 = u_0' \end{cases}$$

is solvable for (p_0, q_0) and

$$\begin{vmatrix} x_0' & F_p(x_0,y_0,u_0,p_0,q_0) \\ y_0' & F_q(x_0,y_0,u_0,p_0,q_0) \end{vmatrix} \neq 0$$

Then there is a solution to the Cauchy problem

$$\begin{cases} F(x,y,u,u_x,u_y) = 0 \\ u(x_0(s), y_0(s)) = u_0(s) \end{cases}$$ defined in a vicinity of Γ

if the system $(***)$ is uniquely solvable for (p_0, q_0) then the solution is unique.

Particular case :

$$p = u_x, \quad q = u_y \qquad \begin{cases} q = G(x, y, u, p) \\ u(x,0) = f(x) \end{cases}$$

Then: Γ: $\begin{cases} x_0(s) = s \\ y_0(s) = 0 \\ u_0(s) = f(s) \end{cases}$

(***) becomes: $\rightarrow$ PDE : $q_0(s) = G(s, 0, f(s), p_0(s))$

$\rightarrow \Gamma$: $p_0(s) = f'(s)$

has unique solution:

in addition, $\begin{vmatrix} x_0' & F_p \\ y_0' & F_q \end{vmatrix} = \begin{vmatrix} 1 & -G_p \\ 0 & 1 \end{vmatrix} = 1$

such that there is an unique solution to the Cauchy problem. $\begin{cases} u_y = G(x, y, u, u_x) \\ u(x,0) = f(x) \end{cases}$

in practice, $y = $ time

$$\boxed{\begin{cases} u_t = G(x, t, u, u_x) \\ u(x,0) = f(x) \end{cases}}$$

Example 2 Eikonal equation (optics)

$$u_x^2 + u_y^2 = 1$$

with initial curve $\Gamma : \begin{cases} x_0(s) = \cos(s) \\ y_0(s) = \sin(s) \\ u_0(s) = 0 \end{cases}$

Characteristic equations : $\boxed{F(x,y,u,p,q) = p^2 + q^2 - 1}$

$$x'(\tau) = F_p = 2p$$

$$y'(\tau) = F_q = 2q$$

$$u'(\tau) = pF_p + qF_q = 2p^2 + 2q^2 = 2 \Rightarrow \underline{u(s,\tau) = 2\tau + \overbrace{u_0(s)}^{=0}} = 2\tau$$

$$p'(\tau) = -[F_x + pF_u] = 0 \Rightarrow p(s,\tau) = p_0(s)$$

$$q'(\tau) = -[F_y + qF_u] = 0 \Rightarrow q(s,\tau) = q_0(s)$$

$$\Rightarrow x(s,\tau) = 2\tau p_0(s) + x_0(s)$$

$$y(s,\tau) = 2\tau q_0(s) + y_0(s)$$

Find $p_0(s)$, $q_0(s)$:

$$\left.\begin{array}{ll} \text{PDE :} & p_0^2(s) + q_0^2(s) = 1 \\ \Gamma : & -p_0(s)\sin s + q_0(s)\cos s = 0 \end{array}\right\} \to \text{2 solutions}$$

$$\boxed{\begin{array}{l} p_0 = \cos s \\ q_0 = \sin s \end{array}} \quad \text{or} \quad \boxed{\begin{array}{l} p_0 = -\cos s \\ q_0 = -\sin s \end{array}}$$

$$\left.\begin{aligned} x(s,\tau) &= \pm 2\tau\cos(s) + \cos(s) \\ y(s,\tau) &= \pm 2\tau\sin(s) + \sin(s) \end{aligned}\right\} \quad x^2+y^2 = (1 \pm 2\tau)^2$$

$$u(s,\tau) = 2\tau$$

$$u(x,y) = -1 \pm \sqrt{x^2+y^2} \quad , \quad u(x,y) = 1 \pm \sqrt{x^2+y^2}$$

$$x(s,\tau) = (1+2\tau)\cos(s) \qquad \text{or} \qquad x(s,\tau) = (1-2\tau)\cos(s)$$

$$y(s,\tau) = (1+2\tau)\sin(s) \qquad\qquad y(s,\tau) = (1-2\tau)\sin(s)$$

$$1+2\tau = \pm\sqrt{x^2+y^2} \qquad\qquad 1-2\tau = \pm\sqrt{x^2+y^2}$$

$$u = -1 \pm \sqrt{x^2+y^2} \qquad\qquad u = 1 \pm \sqrt{x^2+y^2}$$

$$\boxed{u = -1 + \sqrt{x^2+y^2}} \quad \text{or} \quad \boxed{u = 1 - \sqrt{x^2+y^2}}$$

Example 3 $\quad \begin{cases} u = u_x^2 - 3u_y^2 \\ u(x,0) = x^2 \end{cases} \qquad F(x,y,u,p,q) = p^2 - 3q^2 - u$

$$x' = 2p$$
$$y' = -6q$$
$$u' = 2p^2 - 6q^2 = 2u \Rightarrow u = u_0 e^{2\tau}$$
$$p' = -[F_x + pF_u] = -[-p] = p \Rightarrow p = p_0 e^{\tau}$$
$$q' = -[F_y + qF_u] = -[-q] = q \Rightarrow q = q_0 e^{\tau}$$

$$x = 2p_0 e^{\tau} + x_0 - 2p_0$$
$$y = -6q_0 e^{\tau} + 6q_0 + y_0$$

$x_0(s) = s$
$y_0(s) = 0$
$u_0(s) = s^2$

$$\left.\begin{aligned} u_0(s) &= p_0(s)^2 - 3q_0(s)^2 = s^2 \\ 1 \cdot p_0(s) &= 2x_0(s)\cdot 1 = 2s \end{aligned}\right\} \Rightarrow \boxed{p_0(s) = 2s} \quad \boxed{q_0(s) = \pm s}$$

$$u(s,\tau) = s^2 e^{2\tau}$$
$$\left.\begin{aligned} x(s,\tau) &= 4se^{\tau} - 3s \\ y(s,\tau) &= \pm s(6 - 6e^{\tau}) \end{aligned}\right\}$$

(1)
$$x = 4se^{\tau} - 3s$$
$$y = -6se^{\tau} + 6s$$
$$2x + y = 2se^{\tau} \Rightarrow \boxed{u = \left(x + \frac{y}{2}\right)^2}$$

(2)
$$x = 4se^{\tau} - 3s$$
$$y = 6se^{\tau} - 6s$$
$$y - 2x = -2se^{\tau} \Rightarrow \boxed{u = \left(x - \frac{y}{2}\right)^2}$$