Chapter 9: **Green's functions for time-independent problems**

Introductory examples

One-dimensional heat equation

Consider the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{1}$$

with boundary conditions

$$u(0,t) = 0 \tag{2}$$

$$u(L,t) = 0 \tag{3}$$

and initial condition

$$u(x,0) = f(x) \tag{4}$$

We already know that the solution of this problem is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t} \tag{5}$$

where a_n are the Fourier sine series coefficients of $f(x)$.

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \tag{6}$$

After replacing (6) in (5) we may express the solution as

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(x_0)\sin\frac{n\pi x_0}{L}dx_0\right]\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t}$$

which may be written (after interchanging $\int$ and $\sum$)

$$u(x,t) = \int_0^L f(x_0)\left(\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t}\right)dx_0 \tag{7}$$

We define the quantity

$$\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t}$$

as the *influence function for the initial condition.* For every point x_0 this quantity shows the influence of the initial temperature at x_0 on the temperature at position x and time t.
Further insight may be obtained by considering the heat equation with sources

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \tag{8}$$

with boundary conditions (2-3) and the initial condition (4).

Q: Show that the solution of (8), (2-3), (4) may be expressed as

$$u(x,t) = \int_0^L f(x_0) G(x,t;x_0,0)dx_0 + \int_0^L \int_0^t Q(x_0,t_0) G(x,t;x_0,t_0) dt_0 dx_0 \tag{9}$$

where $G(x,t;x_0,t_0)$ is given by

$$G(x,t;x_0,t_0) = \sum_{n=1}^{\infty} \frac{2}{L} \sin \frac{n\pi x_0}{L} \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2(t-t_0)} \tag{10}$$

The function $G(x,t;x_0,t_0)$ defined by (10) is called the **Green's function** for the heat equation problem (8), (2-3), (4).
At $t_0 = 0$, $G(x,t;x_0,t_0)$ expresses the influence of the initial temperature at x_0 on the temperature at position x and time t. In addition, $G(x,t;x_0,t_0)$ shows the influence of the source/sink term $Q(x_0,t_0)$ at position x_0 and time t_0 on the temperature at position x and time t.

Notice that the Green's function depends only on the elapsed time $t - t_0$ since

$$G(x,t;x_0,t_0) = G(x,t-t_0;x_0,0)$$

Green's functions for boundary value problems for ODE's

In this section we investigate the Green's function for a Sturm-Liouville nonhomogeneous ODE

$$L(u) = f(x)$$

subject to two homogeneous boundary conditions.
The simplest example is the steady-state heat equation

$$\frac{d^2x}{dx^2} = f(x)$$

with homogeneous boundary conditions

$$u(0) = 0, \;\; u(L) = 0$$

The method of variation of parameters

Consider the linear nonhomogeneous problem

$$L(u) = f(x) \tag{11}$$

where $u = u(x), a < x < b$ satisfies homogeneous boundary conditions and L is the Sturm-Liouville operator

$$L \equiv \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$$

If u_1 and u_2 are two linearly independent solutions of the homogeneous problem $L(u) = 0$, the general solution of the homogeneous problem is

$$u = c_1 u_1 + c_2 u_2$$

where c_1 and c_2 are arbitrary constants. To solve the nonhomogeneous problem, we use *the method of variation of parameters* and search for a particular solution of (11) of the form

$$u(x) = v_1(x)u_1(x) + v_2(x)u_2(x) \tag{12}$$

where $v_1(x)$ and $v_2(x)$ are functions to be determined such that (12) satisfies (11). Since we have only one equation to be satisfied and two unknown functions, we impose an additional constraint

$$\frac{dv_1}{dx}u_1 + \frac{dv_2}{dx}u_2 = 0 \tag{13}$$

Using (13), we obtain from (12)

$$\frac{du}{dx} = v_1\frac{du_1}{dx} + v_2\frac{du_2}{dx}$$

and the equation (11) is satisfied if

$$\frac{dv_1}{dx}p\frac{du_1}{dx} + \frac{dv_2}{dx}p\frac{du_2}{dx} = f(x) \tag{14}$$

From (13) and (14) we obtain

$$\frac{dv_1}{dx} = \frac{-fu_2}{c} \tag{15}$$

$$\frac{dv_2}{dx} = \frac{fu_1}{c} \tag{16}$$

where

$$c = p\left(u_1\frac{du_2}{dx} - u_2\frac{du_1}{dx}\right) \tag{17}$$

Remark: The quantity

$$W = u_1\frac{du_2}{dx} - u_2\frac{du_1}{dx} = \begin{vmatrix} u_1 & u_2 \\ \frac{du_1}{dx} & \frac{du_2}{dx} \end{vmatrix}$$

is called the *Wronskian* of u_1 and u_2 and satisfies the differential equation

$$\frac{dW}{dx} = -\frac{1}{p}\frac{dp}{dx}W$$

Q: Using the Wronskian, show that expression (17) is constant.
Then, integrating (15) and (16) we obtain

$$v_1(x) = -\frac{1}{c}\int_a^x f(x_0)u_2(x_0)dx_0 + c_1 \tag{18}$$

$$v_2(x) = \frac{1}{c}\int_a^x f(x_0)u_1(x_0)dx_0 + c_2 \tag{19}$$

such that the general solution of the nonhomogeneous problem (11) is

$$\begin{aligned} u(x) &= v_1(x)u_1(x) + v_2(x)u_2(x) &(20)\\ &= c_1u_1(x) + c_2u_2(x) - \frac{u_1(x)}{c}\int_a^x f(x_0)u_2(x_0)dx_0 + \frac{u_2(x)}{c}\int_a^x f(x_0)u_1(x_0)dx_0 &(21)\end{aligned}$$

The constants c_1 and c_2 are determined by the boundary conditions.

A simple example. Consider the following problem

$$\frac{d^2u}{dx^2} = f(x)$$

with homogeneous boundary conditions

$$u(0) = 0, \;\; u(L) = 0$$

Two linearly independent solutions of the homogeneous differential equation are

$$u_1(x) = x \tag{22}$$

$$u_2(x) = L - x \tag{23}$$

Then, $W = u_1 \frac{du_2}{dx} - u_2 \frac{du_1}{dx} = -L$ such that from (18) and (19) we obtain

$$v_1(x) = \frac{1}{L} \int_0^x f(x_0)(L - x_0) dx_0 + c_1 \tag{24}$$

$$v_2(x) = -\frac{1}{L} \int_0^x f(x_0) x_0 dx_0 + c_2 \tag{25}$$

Using the boundary conditions,

$$u(0) = 0 \Rightarrow c_2 = 0$$

$$u(L) = 0 \Rightarrow c_1 = -\frac{1}{L} \int_0^L f(x_0)(L - x_0) dx_0$$

and replacing in (24), (25) it follows that

$$v_1(x) = \frac{1}{L} \int_0^x f(x_0)(L - x_0) dx_0 - \frac{1}{L} \int_0^L f(x_0)(L - x_0) dx_0 = -\frac{1}{L} \int_x^L f(x_0)(L - x_0) dx_0 \tag{26}$$

$$v_2(x) = -\frac{1}{L} \int_0^x f(x_0) x_0 dx_0 \tag{27}$$

and the solution of the nonhomogeneous problem is

$$\begin{aligned} u(x) &= v_1(x) u_1(x) + v_2(x) u_2(x) & (28) \\ &= -\frac{x}{L} \int_x^L f(x_0)(L - x_0) dx_0 - \frac{L - x}{L} \int_0^x f(x_0) x_0 dx_0 & (29) \end{aligned}$$

This solution may be written as

$$u(x) = \int_0^L f(x_0) G(x, x_0) dx_0 \tag{30}$$

where the *Green's function* $G(x, x_0)$ is given by

$$G(x, x_0) = \begin{cases} \frac{-x(L - x_0)}{L}, & x < x_0 \\ \frac{-x_0(L - x)}{L}, & x > x_0 \end{cases} \tag{31}$$

The method of eigenfunction expansion for Green's functions

Consider a general Sturm-Liouville nonhomogeneous ODE

$$L(u) = f(x), \quad a < x < b$$

subject to two homogeneous boundary conditions. We know that the eigenfunctions $\Phi_n(x)$ of the related eigenvalue problem

$$L(\Phi) = -\lambda \sigma \Phi$$

(for an arbitrary weight function σ) subject to the same homogeneous boundary conditions form a "complete set", such that $u(x)$ may be expressed as a generalized Fourier series of eigenfunctions

$$u(x) = \sum_{n=1}^{\infty} a_n \Phi_n(x) \tag{32}$$

Term-by-term differentiation of (32) implies (together with the linearity of L)

$$L(u) = L\left(\sum_{n=1}^{\infty} a_n \Phi_n(x)\right) = \sum_{n=1}^{\infty} a_n L(\Phi_n) = -\sum_{n=1}^{\infty} a_n \lambda_n \sigma \Phi_n$$

In the last relation above we used $L(\Phi_n) = -\lambda_n \sigma \Phi_n$. Therefore,

$$L(u) = f(x) \Rightarrow -\sum_{n=1}^{\infty} a_n \lambda_n \sigma \Phi_n = f(x)$$

and using the orthogonality of the eigenfunctions (with weight σ) we obtain the coefficients

$$a_n = \frac{\int_a^b f(x)\Phi_n dx}{-\lambda_n \int_a^b \phi_n^2 \sigma dx} \tag{33}$$

Replacing (33) in (32) we obtain (after interchanging $\int$ and $\sum$)

$$u(x) = \sum_{n=1}^{\infty} \frac{\int_a^b f(x)\Phi_n(x)dx}{-\lambda_n \int_a^b \phi_n^2 \sigma dx} \Phi_n(x) = \int_a^b f(x_0) \sum_{n=1}^{\infty} \frac{\Phi_n(x)\Phi_n(x_0)}{-\lambda_n \int_a^b \phi_n^2 \sigma dx} dx_0 \tag{34}$$

For this problem,

$$u(x) = \int_a^b f(x_0) G(x, x_0) dx_0$$

where the Green's function $G(x, x_0)$ is

$$G(x, x_0) = \sum_{n=1}^{\infty} \frac{\Phi_n(x)\Phi_n(x_0)}{-\lambda_n \int_a^b \phi_n^2 \sigma dx} \tag{35}$$

Remark: Formula (35) implies that the Green's function is symmetric:

$$G(x, x_0) = G(x_0, x)$$

The Dirac Delta Function and its relationship to Green's function

In the previous section we proved that the solution of the nonhomogeneous problem

$$L(u) = f(x)$$

subject to homogeneous boundary conditions is

$$u(x) = \int_a^b f(x_0) G(x, x_0) dx_0$$

In this section we want to give an interpretation of the Green's function. Let $x_s, a < x_s < b$ represent an arbitrary fixed point. We consider a perturbation of the source term

$$f(x) + \delta_\epsilon(x)$$

where $\delta_\epsilon(x)$ is a continuous function with the following properties:

1. $\int_a^b \delta_\epsilon(x) dx = 1$
2. $\delta_\epsilon(x) = 0$ if $x \geq x_s + \epsilon$ or $x \leq x_s - \epsilon$

A possible choice for $\delta_\epsilon(x)$ is the hat function

$$\delta_\epsilon(x) = \left\{ \begin{array}{cc} 0, & a < x \leq x_s - \epsilon \\ \frac{1}{\epsilon^2}(x - x_s + \epsilon), & x_s - \epsilon < x < x_s \\ -\frac{1}{\epsilon^2}(x - x_s - \epsilon), & x_s < x < x_s + \epsilon \\ 0, & x_s + \epsilon \leq x < b \end{array} \right. \tag{36}$$

Denote $u_\epsilon(x)$ the solution of the perturbed problem

$$L(u_\epsilon) = f(x) + \delta_\epsilon(x)$$

subject to the same homogeneous boundary conditions as u. The variation $\Delta u = u_\epsilon - u$ in the solution u due to the perturbation of the source term satisfy the following problem:

$$L(\Delta u) = \delta_\epsilon(x)$$

with homogeneous boundary conditions, such that we have

$$\Delta u(x) = \int_a^b \delta_\epsilon(x_0)G(x, x_0)dx_0 = \int_{x_s-\epsilon}^{x_s+\epsilon} \delta_\epsilon(x_0)G(x, x_0)dx_0 = G(x, x_\epsilon) \int_{x_s-\epsilon}^{x_s+\epsilon} \delta_\epsilon(x_0)dx_0$$

where x_ϵ is a point in the interval $(x_s - \epsilon, x_s + \epsilon)$. The existence of such point is a consequence of the mean value theorem, since $G(x, x_0)$ is continuous. Notice that

$$\int_{x_s-\epsilon}^{x_s+\epsilon} \delta_\epsilon(x_0)dx_0 = \int_a^b \delta_\epsilon(x_0)dx_0 = 1$$

If we let $\epsilon \longrightarrow 0$, we obtain

$$\Delta u(x) = G(x, x_s)$$

The limit

$$\lim_{\epsilon \longrightarrow 0} \delta_\epsilon = \delta(x - x_s) \tag{37}$$

represents an infinitely concentrated pulse at x_s, that is zero everywhere, except at $x = x_s$

$$\delta(x - x_s) = \left\{ \begin{array}{cc} 0, & x \neq x_s \\ \infty, & x = x_s \end{array} \right. \tag{38}$$

We define the **Dirac delta function** $\delta(x - x_s)$ as an operator with the property (38) such that for every continuous function $f(x)$ we have

$$f(x) = \int_{-\infty}^{\infty} f(x_s)\delta(x - x_s)dx_s$$

The Dirac delta function has unit area

$$\int_{-\infty}^{\infty} \delta(x - x_s)dx_s = 1$$

and is even in the argument $x - x_s$

$$\delta(x - x_s) = \delta(x_s - x)$$

Then $G(x, x_s)$ satisfies

$$L[G(x, x_s)] = \delta(x - x_s)$$

and the homogeneous boundary conditions at $x = a$ and $x = b$, such that we obtain the following interpretation: **the Green's function $G(x, x_s)$ is the response at x due to a concentrated source at x_s**. The symmetry of the Green's function

$$G(x, x_s) = G(x_s, x)$$

implies that **the response at x due to a concentrated source at x_s is the same as the response at x_s due to a concentrated source at x.** This property is known as *Maxwell's reciprocity.*

Jump conditions

Next we show how the Green's function may be obtained by directly solving

$$L[G(x,x_0)] = \delta(x-x_0) \tag{39}$$

subject to homogeneous boundary conditions.

Example consider the steady-state problem

$$u''(x) = f(x)$$

$$u(0) = 0, u(L) = 0$$

The Green's function to this problem is then a solution to

$$\frac{d^2G(x,x_0)}{dx^2} = \delta(x-x_0) \tag{40}$$

$$G(0,x_0) = 0,\ G(L,x_0) = 0 \tag{41}$$

It follows then that $\frac{d^2G(x,x_0)}{dx^2} = 0, x \neq x_0$, therefore $G(x,x_0)$ must be a linear function on each interval $x < x_0$ and $x > x_0$

$$G(x,x_0) = \begin{cases} a + bx, & x < x_0 \\ c + dx, & x > x_0 \end{cases} \tag{42}$$

The constants a, b, c, d need to be determined. From the boundary conditions we have

$$G(0,x_0) = 0 \Rightarrow a = 0$$

$$G(L,x_0) = 0 \rightarrow c = -dL$$

such that we obtain

$$G(x,x_0) = \begin{cases} bx, & x < x_0 \\ d(x-L), & x > x_0 \end{cases} \tag{43}$$

To find b and d, first we require that $G(x,x_0)$ must be continuous at x_0

$$G(x_0^-,x_0) = G(x_0^+,x_0) \Rightarrow bx_0 = d(x_0 - L) \tag{44}$$

Next, for any $\epsilon > 0$, integrating (40) in $(x_0-\epsilon, x_0+\epsilon)$

$$\frac{d}{dx}G(x_0+\epsilon,x_0) - \frac{d}{dx}G(x_0-\epsilon,x_0) = 1$$

then passing to the limit as $\epsilon \rightarrow 0$, we get the *jump condition*

$$\frac{d}{dx}G(x_0^+,x_0) - \frac{d}{dx}G(x_0^-,x_0) = 1 \Rightarrow d - b = 1 \tag{45}$$

Solving (44-45) we get

$$d = \frac{x_0}{L}, \quad b = \frac{x_0 - L}{L}$$

such that the Green's function (43) is

$$G(x,x_0) = \begin{cases} -\frac{x}{L}(L-x_0), & x < x_0 \\ -\frac{x_0}{L}(L-x), & x > x_0 \end{cases} \tag{46}$$

which is the same expression we obtained using the variation of parameters.

Nonhomogeneous boundary conditions

The Green's function may be used to solve problems with nonhomogeneous boundary conditions e.g.,

$$u'' = f(x) \tag{47}$$

$$u(0) = \alpha,\ u(L) = \beta \tag{48}$$

The solution $u(x)$ may be written as the sum

$$u = u_1 + u_2$$

where $u_1(x)$ solves the nonhomogenous ODE with homogeneous boundary conditions

$$u_1'' = f(x), \quad u_1(0) = u_1(L) = 0$$

and $u_2(x)$ solves the homogenous ODE with nonhomogeneous boundary conditions

$$u_2'' = 0, \quad u_2(0) = \alpha, u_2(L) = \beta$$

Then,

$$u_1(x) = \int_0^L f(x_0)G(x, x_0)\, dx_0$$

$$u_2(x) = \alpha + \frac{\beta - \alpha}{L}x$$

such that

$$u(x) = u_1(x) + u_2(x) = \int_0^L f(x_0)G(x, x_0)\, dx_0 + \alpha + \frac{\beta - \alpha}{L}x$$

is the solution to the nonhomogeneous problem (47-48).