Chapter 9: Green’s functions for time-independent problems

Introductory examples

One-dimensional heat equation

Consider the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$  \hspace{1cm} (1)

with boundary conditions

$$u(0, t) = 0$$  \hspace{1cm} (2)
$$u(L, t) = 0$$  \hspace{1cm} (3)

and initial condition

$$u(x, 0) = f(x)$$  \hspace{1cm} (4)

We already know that the solution of this problem is given by

$$u(x, t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} e^{-k \left(\frac{n\pi}{L}\right)^2 t}$$  \hspace{1cm} (5)

where $a_n$ are the Fourier sine series coefficients of $f(x)$.

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$  \hspace{1cm} (6)

After replacing (6) in (5) we may express the solution as

$$u(x, t) = \sum_{n=1}^{\infty} \left[ \frac{2}{L} \int_0^L f(x_0) \sin \frac{n\pi x_0}{L} \, dx_0 \right] \sin \frac{n\pi x}{L} e^{-k \left(\frac{n\pi}{L}\right)^2 t}$$

which may be written (after interchanging $\int$ and $\sum$)

$$u(x, t) = \int_0^L f(x_0) \left( \sum_{n=1}^{\infty} \frac{2}{L} \sin \frac{n\pi x_0}{L} \sin \frac{n\pi x}{L} e^{-k \left(\frac{n\pi}{L}\right)^2 t} \right) \, dx_0$$  \hspace{1cm} (7)

We define the quantity

$$\sum_{n=1}^{\infty} \frac{2}{L} \sin \frac{n\pi x_0}{L} \sin \frac{n\pi x}{L} e^{-k \left(\frac{n\pi}{L}\right)^2 t}$$

as the influence function for the initial condition. For every point $x_0$ this quantity shows the influence of the initial temperature at $x_0$ on the temperature at position $x$ and time $t$.

Further insight may be obtained by considering the heat equation with sources

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q(x, t)$$  \hspace{1cm} (8)

with boundary conditions (2-3) and the initial condition (4).
Q: Show that the solution of (8), (2-3), (4) may be expressed as

\[ u(x, t) = \int_0^L f(x_0)G(x, t; x_0, 0)dx_0 + \int_0^L \int_0^t Q(x_0, t_0)G(x, t; x_0, t_0)dt_0dx_0 \]  \tag{9} \]

where \(G(x, t; x_0, t_0)\) is given by

\[ G(x, t; x_0, t_0) = \sum_{n=1}^{\infty} 2 \, \frac{n \pi x_0}{L} \sin \frac{n \pi x}{L} e^{-k \left( \frac{n \pi}{L} \right)^2 (t-t_0)} \]  \tag{10} \]

The function \(G(x, t; x_0, t_0)\) defined by (10) is called the Green’s function for the heat equation problem (8), (2-3), (4).

At \(t_0 = 0\), \(G(x, t; x_0, t_0)\) expresses the influence of the initial temperature at \(x_0\) on the temperature at position \(x\) and time \(t\). In addition, \(G(x, t; x_0, t_0)\) shows the influence of the source/sink term \(Q(x_0, t_0)\) at position \(x_0\) and time \(t_0\) on the temperature at position \(x\) and time \(t\).

Notice that the Green’s function depends only on the elapsed time \(t - t_0\) since \(G(x, t; x_0, t_0) = G(x, t - t_0; x_0, 0)\)

Green’s functions for boundary value problems for ODE’s

In this section we investigate the Green’s function for a Sturm-Liouville nonhomogeneous ODE

\[ L(u) = f(x) \]

subject to two homogeneous boundary conditions.

The simplest example is the steady-state heat equation

\[ \frac{d^2x}{dx^2} = f(x) \]

with homogeneous boundary conditions

\[ u(0) = 0, \quad u(L) = 0 \]

The method of variation of parameters

Consider the linear nonhomogeneous problem

\[ L(u) = f(x) \]  \tag{11} \]

where \(u = u(x), a < x < b\) satisfies homogeneous boundary conditions and \(L\) is the Sturm-Liouville operator

\[ L \equiv \frac{d}{dx} \left( p \frac{d}{dx} \right) + q \]

If \(u_1\) and \(u_2\) are two linearly independent solutions of the homogeneous problem \(L(u) = 0\), the general solution of the homogeneous problem is

\[ u = c_1 u_1 + c_2 u_2 \]

where \(c_1\) and \(c_2\) are arbitrary constants. To solve the nonhomogeneous problem, we use the method of variation of parameters and search for a particular solution of (11) of the form

\[ u(x) = v_1(x)u_1(x) + v_2(x)u_2(x) \]  \tag{12} \]

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where \( v_1(x) \) and \( v_2(x) \) are functions to be determined such that (12) satisfies (11). Since we have only one equation to be satisfied and two unknown functions, we impose an additional constraint

\[
\frac{dv_1}{dx}u_1 + \frac{dv_2}{dx}u_2 = 0
\]  

(13)

Using (13), we obtain from (12)

\[
\frac{du}{dx} = v_1 \frac{du_1}{dx} + v_2 \frac{du_2}{dx}
\]

and the equation (11) is satisfied if

\[
\frac{dv_1}{dx}u_1 + \frac{dv_2}{dx}u_2 = f(x)
\]  

(14)

From (13) and (14) we obtain

\[
\frac{dv_1}{dx} = -\frac{fu_2}{c} \quad \text{and} \quad \frac{dv_2}{dx} = \frac{fu_1}{c}
\]

(15)

(16)

where

\[
c = p \left( u_1 \frac{du_2}{dx} - u_2 \frac{du_1}{dx} \right)
\]

(17)

Remark: The quantity

\[
W = u_1 \frac{du_2}{dx} - u_2 \frac{du_1}{dx} = \left| \begin{array}{cc} u_1 & u_2 \\ \frac{du_1}{dx} & \frac{du_2}{dx} \end{array} \right|
\]

is called the Wronskian of \( u_1 \) and \( u_2 \) and satisfies the differential equation

\[
\frac{dW}{dx} = -\frac{1}{p} \frac{dp}{dx} W
\]

Q: Using the Wronskian, show that expression (17) is constant.

Then, integrating (15) and (16) we obtain

\[
v_1(x) = -\frac{1}{c} \int_a^x f(x_0)u_2(x_0)dx_0 + c_1
\]

(18)

\[
v_2(x) = \frac{1}{c} \int_a^x f(x_0)u_1(x_0)dx_0 + c_2
\]

(19)

such that the general solution of the nonhomogeneous problem (11) is

\[
u(x) = v_1(x)u_1(x) + v_2(x)u_2(x)
\]

\[
= c_1 u_1(x) + c_2 u_2(x) - \frac{u_1(x)}{c} \int_a^x f(x_0)u_2(x_0)dx_0 + \frac{u_2(x)}{c} \int_a^x f(x_0)u_1(x_0)dx_0
\]

(20)

(21)

The constants \( c_1 \) and \( c_2 \) are determined by the boundary conditions.

**A simple example.** Consider the following problem

\[
\frac{d^2u}{dx^2} = f(x)
\]

with homogeneous boundary conditions

\[
u(0) = 0, \quad u(L) = 0
\]

Two linearly independent solutions of the homogeneous differential equation are

\[
u_1(x) = x
\]

(22)
\[ u_2(x) = L - x \]  

Then, \( W = u_1 \frac{du_2}{dx} - u_2 \frac{du_1}{dx} = -L \) such that from (18) and (19) we obtain

\[ v_1(x) = \frac{1}{L} \int_0^x f(x_0)(L - x_0)dx_0 + c_1 \]  

\[ v_2(x) = -\frac{1}{L} \int_0^x f(x_0)x_0dx_0 + c_2 \]  

Using the boundary conditions,

\[ u(0) = 0 \Rightarrow c_2 = 0 \]

\[ u(L) = 0 \Rightarrow c_1 = -\frac{1}{L} \int_0^L f(x_0)(L - x_0)dx_0 \]

and replacing in (24), (25) it follows that

\[ v_1(x) = \frac{1}{L} \int_0^x f(x_0)(L - x_0)dx_0 - \frac{1}{L} \int_0^L f(x_0)(L - x_0)dx_0 = -\frac{1}{L} \int_x^L f(x_0)(L - x_0)dx_0 \]  

\[ v_2(x) = -\frac{1}{L} \int_0^x f(x_0)x_0dx_0 \]

and the solution of the nonhomogeneous problem is

\[ u(x) = v_1(x)u_1(x) + v_2(x)u_2(x) \]

\[ = -\frac{x}{L} \int_x^L f(x_0)(L - x_0)dx_0 - \frac{L - x}{L} \int_0^x f(x_0)x_0dx_0 \]

This solution may be written as

\[ u(x) = \int_0^L f(x_0)G(x, x_0)dx_0 \]

where the Green's function \( G(x, x_0) \) is given by

\[ G(x, x_0) = \begin{cases}  
-\frac{x(L - x_0)}{L}, & x < x_0 \\ -\frac{x_0(L - x)}{L}, & x > x_0 
\end{cases} \]

The method of eigenfunction expansion for Green's functions

Consider a general Sturm-Liouville nonhomogeneous ODE

\[ L(u) = f(x), \ a < x < b \]

subject to two homogeneous boundary conditions. We know that the eigenfunctions \( \Phi_n(x) \) of the related eigenvalue problem

\[ L(\Phi) = -\lambda \sigma \Phi \]

(for an arbitrary weight function \( \sigma \)) subject to the same homogeneous boundary conditions form a ”complete set”, such that \( u(x) \) may be expressed as a generalized Fourier series of eigenfunctions

\[ u(x) = \sum_{n=1}^{\infty} a_n \Phi_n(x) \]


\[ L(u) = L \left( \sum_{n=1}^{\infty} a_n \Phi_n(x) \right) = \sum_{n=1}^{\infty} a_n L(\Phi_n) = -\sum_{n=1}^{\infty} a_n \lambda_n \sigma \Phi_n \]

In the last relation above we used \( L(\Phi_n) = -\lambda_n \sigma \Phi_n \). Therefore,

\[ L(u) = f(x) \Rightarrow -\sum_{n=1}^{\infty} a_n \lambda_n \sigma \Phi_n = f(x) \]

and using the orthogonality of the eigenfunctions (with weight \( \sigma \)) we obtain the coefficients

\[ a_n = \frac{\int_a^b f(x) \Phi_n dx}{-\lambda_n \int_a^b \phi_n^2 \sigma dx} \quad (33) \]

Replacing (33) in (32) we obtain (after interchanging \( \int \) and \( \sum \))

\[ u(x) = \sum_{n=1}^{\infty} \frac{\int_a^b f(x) \Phi_n(x) dx}{-\lambda_n \int_a^b \phi_n^2 \sigma dx} \Phi_n(x) = \int_a^b f(x_0) \sum_{n=1}^{\infty} \frac{\Phi_n(x) \Phi_n(x_0)}{-\lambda_n \int_a^b \phi_n^2 \sigma dx} dx_0 \quad (34) \]

For this problem,

\[ u(x) = \int_a^b f(x_0) G(x,x_0) dx_0 \]

where the Green’s function \( G(x,x_0) \) is

\[ G(x,x_0) = \sum_{n=1}^{\infty} \frac{\Phi_n(x_0) \Phi_n(x)}{-\lambda_n \int_a^b \phi_n^2 \sigma dx} \quad (35) \]

**Remark:** Formula (35) implies that the Green’s function is symmetric:

\[ G(x,x_0) = G(x_0,x) \]

### The Dirac Delta Function and its relationship to Green’s function

In the previous section we proved that the solution of the nonhomogeneous problem

\[ L(u) = f(x) \]

subject to homogeneous boundary conditions is

\[ u(x) = \int_a^b f(x_0) G(x,x_0) dx_0 \]

In this section we want to give an interpretation of the Green’s function. Let \( x_s, a < x_s < b \) represent an arbitrary fixed point. We consider a perturbation of the source term

\[ f(x) + \delta_\epsilon(x) \]

where \( \delta_\epsilon(x) \) is a continuous function with the following properties:

1. \( \int_a^b \delta_\epsilon(x) dx = 1 \)
2. \( \delta_\epsilon(x) = 0 \) if \( x \geq x_s + \epsilon \) or \( x \leq x_s - \epsilon \)
A possible choice for \( \delta_\epsilon(x) \) is the hat function

\[
\delta_\epsilon(x) = \begin{cases} 
0, & a < x \leq x_s - \epsilon \\
\frac{1}{\epsilon^2}(x - x_s + \epsilon), & x_s - \epsilon < x < x_s \\
\frac{1}{\epsilon^2}(x - x_s - \epsilon), & x_s < x < x_s + \epsilon \\
0, & x_s + \epsilon \leq x < b
\end{cases}
\] (36)

Denote \( u_\epsilon(x) \) the solution of the perturbed problem

\[ L(u_\epsilon) = f(x) + \delta_\epsilon(x) \]

subject to the same homogeneous boundary conditions as \( u \). The variation \( \Delta u = u_\epsilon - u \) in the solution \( u \) due to the perturbation of the source term satisfy the following problem:

\[ L(\Delta u) = \delta_\epsilon(x) \]

with homogeneous boundary conditions, such that we have

\[ \Delta u(x) = \int_a^b \delta_\epsilon(x_0)G(x, x_0)dx_0 = \int_{x_s - \epsilon}^{x_s + \epsilon} \delta_\epsilon(x_0)G(x, x_0)dx_0 = G(x, x_s) \int_{x_s - \epsilon}^{x_s + \epsilon} \delta_\epsilon(x_0)dx_0 \]

where \( x_s \) is a point in the interval \((x_s - \epsilon, x_s + \epsilon)\). The existence of such point is a consequence of the mean value theorem, since \( G(x, x_0) \) is continuous. Notice that

\[ \int_{x_s - \epsilon}^{x_s + \epsilon} \delta_\epsilon(x_0)dx_0 = \int_a^b \delta_\epsilon(x_0)dx_0 = 1 \]

If we let \( \epsilon \to 0 \), we obtain

\[ \Delta u(x) = G(x, x_s) \]

The limit

\[ \lim_{\epsilon \to 0} \delta_\epsilon = \delta(x - x_s) \] (37)

represents an infinitely concentrated pulse at \( x_s \), that is zero everywhere, except at \( x = x_s \)

\[
\delta(x - x_s) = \begin{cases} 
0, & x \neq x_s \\
\infty, & x = x_s
\end{cases}
\] (38)

We define the **Dirac delta function** \( \delta(x - x_s) \) as an operator with the property (38) such that for every continuous function \( f(x) \) we have

\[ f(x) = \int_{-\infty}^{\infty} f(x_s)\delta(x - x_s)dx_s \]

The Dirac delta function has unit area

\[ \int_{-\infty}^{\infty} \delta(x - x_s)dx_s = 1 \]

and is even in the argument \( x - x_s \)

\[ \delta(x - x_s) = \delta(x_s - x) \]

Then \( G(x, x_s) \) satisfies

\[ L[G(x, x_s)] = \delta(x - x_s) \]

and the homogeneous boundary conditions at \( x = a \) and \( x = b \), such that we obtain the following interpretation: the **Green’s function** \( G(x, x_s) \) is the response at \( x \) due to a concentrated source at \( x_s \). The symmetry of the Green’s function

\[ G(x, x_s) = G(x_s, x) \]
implies that the response at \( x \) due to a concentrated source at \( x_s \) is the same as the response at \( x_s \) due to a concentrated source at \( x \). This property is known as Maxwell’s reciprocity.

**Jump conditions**

Next we show how the Green’s function may be obtained by directly solving

\[
L[G(x, x_0)] = \delta(x - x_0)
\]

subject to homogeneous boundary conditions.

**Example** consider the steady-state problem

\[
u''(x) = f(x)
\]

\[u(0) = 0, u(L) = 0\]

The Green’s function to this problem is then a solution to

\[
\frac{d^2 G(x, x_0)}{dx^2} = \delta(x - x_0)
\]

(40)

\[G(0, x_0) = 0, \ G(L, x_0) = 0\] (41)

It follows then that \( \frac{d^2 G(x, x_0)}{dx^2} = 0, x \neq x_0 \), therefore \( G(x, x_0) \) must be a linear function on each interval \( x < x_0 \) and \( x > x_0 \)

\[
G(x, x_0) = \begin{cases} 
  a + bx, & x < x_0 \\
  c + dx, & x > x_0 
\end{cases}
\]

(42)

The constants \( a, b, c, d \) need to be determined. From the boundary conditions we have

\[
G(0, x_0) = 0 \Rightarrow a = 0
\]

\[
G(L, x_0) = 0 \Rightarrow c = -dL
\]

such that we obtain

\[
G(x, x_0) = \begin{cases} 
  bx, & x < x_0 \\
  d(x - L), & x > x_0 
\end{cases}
\]

(43)

To find \( b \) and \( d \), first we require that \( G(x, x_0) \) must be continuous at \( x_0 \)

\[
G(x_0^-, x_0) = G(x_0^+, x_0) \Rightarrow bx_0 = d(x_0 - L)
\]

(44)

Next, for any \( \epsilon > 0 \), integrating (40) in \((x_0 - \epsilon, x_0 + \epsilon)\)

\[
\frac{d}{dx} G(x_0 + \epsilon, x_0) - \frac{d}{dx} G(x_0 - \epsilon, x_0) = 1
\]

then passing to the limit as \( \epsilon \to 0 \), we get the jump condition

\[
\frac{d}{dx} G(x_0^-, x_0) - \frac{d}{dx} G(x_0^+, x_0) = 1 \Rightarrow d - b = 1
\]

(45)

Solving (44-45) we get

\[
d = \frac{x_0}{L}, \quad b = \frac{x_0 - L}{L}
\]

such that the Green’s function (43) is

\[
G(x, x_0) = \begin{cases} 
  -\frac{x}{L}(L - x_0), & x < x_0 \\
  -\frac{x_0}{L}(L - x), & x > x_0 
\end{cases}
\]

(46)

which is the same expression we obtained using the variation of parameters.
Nonhomogeneous boundary conditions

The Green’s function may be used to solve problems with nonhomogeneous boundary conditions e.g.,

\[ u'' = f(x) \] (47)

\[ u(0) = \alpha, \ u(L) = \beta \] (48)

The solution \( u(x) \) may be written as the sum

\[ u = u_1 + u_2 \]

where \( u_1(x) \) solves the nonhomogenous ODE with homogeneous boundary conditions

\[ u_1'' = f(x), \quad u_1(0) = u_1(L) = 0 \]

and \( u_2(x) \) solves the homogenous ODE with nonhomogeneous boundary conditions

\[ u_2'' = 0, \quad u_2(0) = \alpha, u_2(L) = \beta \]

Then,

\[ u_1(x) = \int_0^L f(x_0)G(x, x_0) \, dx_0 \]

\[ u_2(x) = \alpha + \frac{\beta - \alpha}{L} x \]

such that

\[ u(x) = u_1(x) + u_2(x) = \int_0^L f(x_0)G(x, x_0) \, dx_0 + \alpha + \frac{\beta - \alpha}{L} x \]

is the solution to the nonhomogeneous problem (47-48).