Chapter10: Fourier Transform Solutions of PDEs

In this chapter we show how the method of separation of variables may be extended to solve PDEs defined on an infinite or semi-infinite spatial domain. Several new concepts such as the "Fourier integral representation" and "Fourier transform" of a function are introduced as an extension of the Fourier series representation to an infinite domain.

We consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \ \ -\infty < x < \infty \tag{1}$$

with the initial condition

$$u(x,0) = f(x), \ \ -\infty < x < \infty \tag{2}$$

As *physical conditions* at $\pm\infty$ we impose

$$u(-\infty,t) = 0, \ u(\infty,t) = 0, \ t > 0 \tag{3}$$

Using separation of variables,

$$u(x,t) = \Phi(x)h(t) \tag{4}$$

we obtain the differential equations

$$\frac{dh}{dt} = -\lambda kh \Rightarrow h(t) = ce^{-\lambda kt} \tag{5}$$

$$\frac{d^2\Phi}{dx^2} = -\lambda\Phi \tag{6}$$

First problem we face is to impose boundary conditions at $\pm\infty$ for $\Phi(x)$.

Q: Show that equation (6) with the boundary conditions $\Phi(-\infty) = \Phi(\infty) = 0$ has no solutions.

Therefore, the correct boundary condition for $\Phi(x)$ at $x = \pm\infty$ is different from the boundary condition for $u(x,t)$ at $x = \pm\infty$. We require that $\Phi(x)$ is bounded at $x = \pm\infty$

$$|\Phi(\pm\infty)| < \infty \tag{7}$$

Q: Show that problem (6)-(7) has no solutions if $\lambda < 0$.

If $\lambda > 0$ the solution of the problem (6),(7) is

$$\Phi(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x \tag{8}$$

which remains bounded at $x = \pm\infty$ for all $\lambda > 0$. *The major difference from the finite domain problem is that in this case all $\lambda > 0$ are eigenvalues!*

Q: Show that $\lambda = 0$ is an eigenvalue of the problem (6),(7) and the corresponding eigenfunction is $\Phi(x) \equiv c$.

The set of eigenvalues for a problem is usually referred to as the *spectrum*. For the heat equation on a finite domain we have a *discrete spectrum* $\lambda_n = (n\pi/L)^2$, whereas for the heat equation defined on $-\infty < x < \infty$ we have a *continuous spectrum* $\lambda \geq 0$.

Superposition principle. From (5) and (8) we obtain the product solutions $u(x,t) = \sin\sqrt{\lambda}xe^{-\lambda kt}$ and $u(x,t) = \cos\sqrt{\lambda}xe^{-\lambda kt}$ for all $\lambda \geq 0$. The general solution is obtained using the *principle of superposition*:

$$u(x,t) = \int_0^\infty [c_1(\lambda)\cos\sqrt{\lambda}xe^{-\lambda kt} + c_2(\lambda)\sin\sqrt{\lambda}xe^{-\lambda kt}]d\lambda \tag{9}$$

where $c_1(\lambda)$ and $c_2(\lambda)$ are arbitrary functions of λ.

Q: Show that (9) is a solution of the equation (1) for any $c_1(\lambda)$ and $c_2(\lambda)$.

If we let $\lambda = \omega^2$ then (9) becomes

$$u(x,t) = \int_0^\infty [A(\omega)\cos\omega x\, e^{-k\omega^2 t} + B(\omega)\sin\omega x\, e^{-k\omega^2 t}]d\omega \tag{10}$$

where $A(\omega) = 2\omega c_1(\omega^2), B(\omega) = 2\omega c_2(\omega^2)$ are arbitrary functions.

To satisfy the initial condition (2) we must have

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega \tag{11}$$

The problem is now to determine the functions $A(\omega)$ and $B(\omega)$ such that (11) is satisfied.

Complex exponentials

Complex exponentials may be used to express the sin and cos functions (Euler's formulas):

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},\ \ \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{12}$$

With these formulas, we may write (10) as

$$\begin{aligned}
u(x,t) &= \int_0^\infty [A(\omega)\frac{e^{i\omega x} + e^{-i\omega x}}{2} + B(\omega)\frac{e^{i\omega x} - e^{-i\omega x}}{2i}]e^{-k\omega^2 t}d\omega \\
&= \int_0^\infty [A(\omega)\frac{e^{i\omega x} + e^{-i\omega x}}{2} - iB(\omega)\frac{e^{i\omega x} - e^{-i\omega x}}{2}]e^{-k\omega^2 t}d\omega \\
&= \int_0^\infty \frac{A(\omega) - iB(\omega)}{2}e^{i\omega x}e^{-k\omega^2 t}d\omega + \int_0^\infty \frac{A(\omega) + iB(\omega)}{2}e^{-i\omega x}e^{-k\omega^2 t}d\omega \\
&= \int_{-\infty}^0 \frac{A(-\omega) - iB(-\omega)}{2}e^{-i\omega x}e^{-k\omega^2 t}d\omega + \int_0^\infty \frac{A(\omega) + iB(\omega)}{2}e^{-i\omega x}e^{-k\omega^2 t}d\omega
\end{aligned}$$

such that we obtain

$$u(x,t) = \int_{-\infty}^\infty c(\omega)e^{-i\omega x}e^{-k\omega^2 t}d\omega \tag{13}$$

where the function $c(\omega) : R \to C$ (takes a real argument and returns a complex value) is defined as

$$c(\omega) = \begin{cases} \frac{A(\omega)+iB(\omega)}{2}, & \omega > 0 \\ \frac{A(-\omega)-iB(-\omega)}{2}, & \omega < 0 \end{cases} \tag{14}$$

Remark: Notice that from (14) it follows that $c(-\omega) = \overline{c}(\omega)$.
At $t = 0$ we obtain from (13)

$$f(x) = \int_{-\infty}^\infty c(\omega)e^{-i\omega x}d\omega \tag{15}$$

Q: Show that (15) with $c(\omega)$ given by (14) is equivalent to (11).
To determine the coefficients $c(\omega)$ from (15) we need to introduce a couple of new concepts: *Fourier transform and Fourier integral representation* of a function.

Fourier transform pair

Given a piecewise smooth function $f(x)$ defined on $-L \leq x \leq L$, the Fourier series representation if f is

$$\frac{f(x+)+f(x-)}{2} = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \tag{16}$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

The complex form of (16) is (see Section 3.6 for a proof):

$$\frac{f(x+)+f(x-)}{2} = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L} \tag{17}$$

where the the coefficients c_n are complex numbers given by

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}dx \tag{18}$$

By replacing (18) in (17) we obtain the *Fourier series identity*

$$\frac{f(x+)+f(x-)}{2} = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2L}\int_{-L}^{L} f(\overline{x})e^{in\pi\overline{x}/L}d\overline{x}\right]e^{-in\pi x/L} \tag{19}$$

For functions defined for $-\infty < x < \infty$ the following *Fourier integral identity* is valid

$$\frac{f(x+)+f(x-)}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\overline{x})e^{i\omega\overline{x}}d\overline{x}\right]e^{-i\omega x}d\omega \tag{20}$$

You may think about (20) as a limit of (19) as $L \to \infty$.

Definition: Given a function $f(x), -\infty < x < \infty$, we define the **Fourier transform of f(x)** as the function

$$F(\omega) = \frac{\gamma}{2\pi}\int_{-\infty}^{\infty} f(\overline{x})e^{i\omega\overline{x}}d\overline{x} \tag{21}$$

where γ is an arbitrary constant ($\neq 0$).
From (20) and (21) it follows that

$$\frac{f(x+)+f(x-)}{2} = \frac{1}{\gamma}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}d\omega \tag{22}$$

Relation (22) is called the **Fourier integral representation of** $f(x)$ and $f(x)$ determined by (22) is called the **inverse Fourier transform of** $F(\omega)$. If f is continuous we simply have

$$f(x) = \frac{1}{\gamma}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}d\omega \tag{23}$$

The relations (21) and (22) are valid for any function $f(x)$ that satisfies $\int_{-\infty}^{\infty} |f(x)|dx < \infty$ and are also known as the **Fourier transform pair**. In our applications we will let $\gamma = 1$.

Next we mention several properties of the *Fourier transform*.

1. The Fourier transform is a linear operator:

$$\mathcal{F}[c_1 f(x) + c_2 g(x)] = c_1 \mathcal{F}[f(x)] + c_2 \mathcal{F}[g(x)] \tag{24}$$

where $\mathcal{F}[f(x)] = F(\omega)$ denotes the Fourier transform of $f(x)$.

2. Given a real valued function $f(x)$ we have

$$F(-\omega) = F^*(\omega) \tag{25}$$

where $F^*(\omega)$ denotes the complex conjugate of $F(\omega)$.

3. If $f(x)$ is an *odd function* then $F(\omega)$ is an odd function. In addition, in this case we have

$$f(x) = \frac{1}{\gamma} \int_{-\infty}^{\infty} F(\omega)(\cos \omega x - i \sin \omega x) dx = \frac{-2i}{\gamma} \int_{0}^{\infty} F(\omega) \sin \omega x \, d\omega$$

$$\begin{aligned} F(\omega) &= \frac{\gamma}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx = \frac{\gamma}{2\pi} \int_{-\infty}^{\infty} f(x)(\cos \omega x + i \sin \omega x) \, dx \\ &= \frac{i\gamma}{2\pi} \int_{-\infty}^{\infty} f(x) \sin \omega x \, dx = \frac{i\gamma}{\pi} \int_{0}^{\infty} f(x) \sin \omega x \, dx \end{aligned}$$

If we choose, for convenience, $\gamma = -2i$, then we obtain

$$f(x) = \int_{0}^{\infty} F(\omega) \sin \omega x \, d\omega \tag{26}$$

$$F(\omega) = \frac{2}{\pi} \int_{0}^{\infty} f(x) \sin \omega x \, dx \tag{27}$$

Relations (26) and (27) are called the *Fourier sine transform pair* and are valid if $f(x)$ is an *odd* function. $F(\omega) \equiv \mathcal{S}[f(x)]$ is called the *Fourier sine transform* of $f(x)$ and $f(x) \equiv \mathcal{S}^{-1}[F(\omega)]$ is called the *inverse Fourier sine transform* of $F(\omega)$.

4. Similarly, if $f(x)$ is an *even function* then $F(\omega)$ is an even function and we obtain the *Fourier cosine transform pair*

$$f(x) = \int_{0}^{\infty} F(\omega) \cos \omega x d\omega \tag{28}$$

$$F(\omega) = \frac{2}{\pi} \int_{0}^{\infty} f(x) \cos \omega x \, dx \tag{29}$$

In this case $F(\omega) \equiv \mathcal{C}[f(x)]$ is called the *Fourier cosine transform* of $f(x)$ and $f(x) \equiv \mathcal{C}^{-1}[F(\omega)]$ is called the *inverse Fourier cosine transform* of $F(\omega)$.

5. The Fourier transform of a Gaussian is a Gaussian and the inverse Fourier transform of a Gaussian is a Gaussian

$$f(x) = e^{-\beta x^2} \Leftrightarrow F(\omega) = \frac{1}{\sqrt{4\pi\beta}} e^{-\frac{\omega^2}{4\beta}} \tag{30}$$

$$f(x) = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{x^2}{4\alpha}} \Leftrightarrow F(\omega) = e^{-\alpha\omega^2} \tag{31}$$

6. *Convolution Theorem*: If $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(x)$ and $g(x)$, respectively, then

$$h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\overline{x}) f(x - \overline{x}) d\overline{x} \tag{32}$$

is the inverse Fourier transform of the product $F(\omega)G(\omega)$. The function $h(x)$ defined in (32) is called the convolution of the functions f and g and is denoted $h = f * g$. Notice that $f * g = g * f$.

Fourier transform and the heat equation

We return now to the solution of the heat equation on an infinite interval and show how to use Fourier transforms to obtain $u(x,t)$. From (15) it follows that $c(\omega)$ is the Fourier transform of the initial temperature distribution $f(x)$:

$$c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \tag{33}$$

such that by replacing (33) in (13) we may express the solution $u(x,t)$ as

$$u(x,t) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} f(\overline{x}) e^{i\omega\overline{x}} d\overline{x} \right] e^{-i\omega x} e^{-k\omega^2 t} d\omega \tag{34}$$

which may be written in the equivalent form

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\overline{x}) \left[\int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega(x-\overline{x})} d\omega \right] d\overline{x} \tag{35}$$

Notice that

$$g(x) = \int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega x} d\omega \tag{36}$$

is the inverse Fourier transform of $e^{-k\omega^2 t}$. With this notation, the solution (35) becomes

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\overline{x}) g(x - \overline{x}) d\overline{x} \tag{37}$$

The only problem now is to obtain an explicit formula for $g(x)$ defined by (36).

Inverse Fourier Transform of a Gaussian

Functions of the form

$$G(\omega) = e^{-\alpha\omega^2}$$

where $\alpha > 0$ is a constant are usually referred to as *Gaussian* functions. The function $g(x)$ whose Fourier transform is $G(\omega)$ is given by the inverse Fourier transform formula

$$g(x) = \int_{-\infty}^{\infty} G(\omega) e^{-i\omega x} d\omega = \int_{-\infty}^{\infty} e^{-\alpha\omega^2} e^{-i\omega x} d\omega \tag{38}$$

The last integral in (38) may be evaluated (the proof is not trivial, see the Appendix to 10.3 for details) to obtain

$$g(x) = \sqrt{\frac{\pi}{\alpha}} e^{\frac{-x^2}{4\alpha}} \tag{39}$$

showing that the *inverse Fourier transform of a Gaussian is itself a Gaussian.*

If we let $\alpha = kt$ in (38), (39) we obtain

$$\int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega x} d\omega = \sqrt{\frac{\pi}{kt}} e^{\frac{-x^2}{4kt}}$$

such that replacing in (36) it follows that

$$g(x) = \sqrt{\frac{\pi}{kt}} e^{\frac{-x^2}{4kt}} \tag{40}$$

The solution $u(x,t)$ is obtained by replacing (40) in (37)

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\overline{x}) \sqrt{\frac{\pi}{kt}} e^{\frac{-(x-\overline{x})^2}{4kt}} d\overline{x} \tag{41}$$

which may be written as

$$u(x,t) = \int_{-\infty}^{\infty} f(\overline{x}) \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-\overline{x})^2}{4kt}} d\overline{x} \tag{42}$$

Remark: Notice that if the initial condition is specified as a Dirac impulse concentrated at the origin, $f(x) = \delta(x)$, then from (42) we get

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{\frac{-x^2}{4kt}} \tag{43}$$

which is called **the fundamental solution of the heat equation**.

Transform of the derivatives: the following properties hold:

1. The Fourier transform of a time derivative equals the time derivative of the Fourier transform

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{\partial}{\partial t} U(\omega,t) \tag{44}$$

2. In general, the Fourier transform of the n^{th} derivative of a function $u(x,t)$ with respect to x equals $(-i\omega)^n$ times the Fourier transform of $u(x,t)$, if $u(x,t) \to 0$, sufficiently fast as $x \to \pm\infty$. In particular,

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = (-i\omega)^2 U(\omega,t) \tag{45}$$

Q: Prove (44) and (45).

Fourier transforming the heat equation

Next we show how the Fourier transform may be used to solve directly the heat equation on an infinite interval

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty \tag{46}$$

with the initial condition

$$u(x,0) = f(x), \quad -\infty < x < \infty \tag{47}$$

$$u(-\infty,t) = 0, \ u(\infty,t) = 0, \ t > 0 \tag{48}$$

Since the Fourier transform is a linear operator, by applying the Fourier transform to the equation (46) we obtain

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = k\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] \tag{49}$$

The Fourier transform of $u(x,t)$ is

$$\mathcal{F}[u] \equiv U(\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{i\omega x}\,dx \tag{50}$$

If we are able to find $\mathcal{F}[u]$ then the solution $u(x,t)$ is given by the inverse Fourier transform formula

$$u(x,t) = \int_{-\infty}^{\infty} U(\omega,t)e^{-i\omega x}\,d\omega \tag{51}$$

After replacing (44) and (45) in (49) we obtain *an ordinary differential equation* for the Fourier transform $U(\omega,t)$

$$\frac{\partial}{\partial t}U(\omega,t) = -k\omega^2 U(\omega,t) \tag{52}$$

Therefore, *the Fourier transform operation converts a linear PDE with constant coefficients into an ODE!*

The general solution of (52) is

$$U(\omega,t) = c(\omega)e^{-k\omega^2 t} \tag{53}$$

To determine $c(\omega)$ we use the initial condition (47) and require that $U(\omega,0)$ is the Fourier transform of $f(x)$. Then

$$c(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx \tag{54}$$

After replacing (53) in (51) we obtain the solution $u(x,t)$

$$u(x,t) = \int_{-\infty}^{\infty} c(\omega)e^{-i\omega x}e^{-k\omega^2 t}d\omega \tag{55}$$

which may be written after replacing (54) and rearranging the terms

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\overline{x})\left[\int_{-\infty}^{\infty} e^{-k\omega^2 t}e^{-i\omega(x-\overline{x})}d\omega\right]d\overline{x} \tag{56}$$

This is the same formula we obtained by using the separation of variables.

Heat equation on semi-infinite intervals

We consider the heat equation on a semi-infinite interval $x > 0$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},\ \ 0 < x < \infty \tag{57}$$

with the initial condition

$$u(x,0) = f(x),\ \ 0 < x < \infty \tag{58}$$

$$u(0,t) = 0,\ \ u(\infty,t) = 0,\ \ t > 0 \tag{59}$$

Using separation of variables,

$$u(x,t) = \Phi(x)h(t)$$

we obtain

$$\frac{dh}{dt} = -\lambda k h$$

and the boundary value problem

$$\frac{d^2\Phi}{dx^2} = -\lambda\Phi$$
$$\Phi(0) = 0$$
$$|\Phi(\infty)| < \infty$$

whose eigenvalues are all positive λ and the eigenfunctions are

$$\Phi(x) = c \sin\sqrt{\lambda}x = c\sin\omega x$$

where $\omega \equiv \sqrt{\lambda} > 0$. Therefore, the product solutions are

$$u(x,t) = A\sin\omega x e^{-k\omega^2 t}$$

and using the principle of superposition we obtain

$$u(x,t) = \int_0^\infty A(\omega)\sin\omega x e^{-k\omega^2 t}\, d\omega \tag{60}$$

To satisfy the initial condition we require

$$f(x) = \int_0^\infty A(\omega)\sin\omega x\, d\omega$$

If we consider the odd extension $\overline{f}(x)$ of $f(x)$ to $-\infty < x < \infty$ then

$$\overline{f}(x) = f(x) = \int_0^\infty F(\omega)\sin\omega x\, d\omega, \;\; x > 0$$

where $F(\omega)$ is the *Fourier sine transform* of $\overline{f}(x)$

$$F(\omega) = \frac{2}{\pi}\int_0^\infty \overline{f}(x)\sin\omega x\, dx = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\, dx$$

Therefore, $A(\omega)$ is given by the *Fourier sine transform* of $f(x)$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\, dx \tag{61}$$

Fourier sine and cosine transforms of derivatives

We defined the Fourier sine and cosine transforms of a function $f(x)$ as

$$\mathcal{S}[f(x)] = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx \tag{62}$$

$$\mathcal{C}[f(x)] = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx \tag{63}$$

For the Fourier sine and cosine transforms of derivatives, assuming that $\lim_{x\to\infty} f(x) = 0$, we have the following formulas:

$$\mathcal{C}\left[\frac{df}{dx}\right] = -\frac{2}{\pi}f(0) + \omega\mathcal{S}[f] \tag{64}$$

$$\mathcal{S}\left[\frac{df}{dx}\right] = -\omega\mathcal{C}[f] \tag{65}$$

$$\mathcal{C}\left[\frac{d^2f}{dx^2}\right] = -\frac{2}{\pi}\frac{df}{dx}(0) + \omega\mathcal{S}\left[\frac{df}{dx}\right] = -\frac{2}{\pi}\frac{df}{dx}(0) - \omega^2\mathcal{C}[f] \tag{66}$$

$$\mathcal{S}\left[\frac{d^2f}{dx^2}\right] = -\omega\mathcal{C}\left[\frac{df}{dx}\right] = \frac{2}{\pi}\omega f(0) - \omega^2\mathcal{S}[f] \tag{67}$$

Q: Prove formulas (64-67).

Formulas (66) and (67) suggest the appropriate choice of the Fourier sine/cosine transform when solving PDEs involving second order derivatives on a semi-infinite interval $x \geq 0$. Replace $f(x)$ by $u(x,t)$ in the formulas above: if $u(0,t)$ is given we will use a Fourier sine transform; if $\partial u/\partial x(0,t)$ is given we will use a Fourier cosine transform.

Example: Consider the heat equation on $0 < x < \infty$ with nonhomogeneous boundary condition at $x = 0$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},\ \ 0 < x < \infty \tag{68}$$

$$u(x,0) = f(x),\ \ 0 < x < \infty \tag{69}$$

$$u(0,t) = g(t),\ \ u(\infty,t) = 0,\ \ t > 0 \tag{70}$$

We introduce the Fourier sine transform of $u(x,t)$

$$U(\omega,t) = \frac{2}{\pi}\int_0^\infty u(x,t)\sin\omega x\,dx \tag{71}$$

By applying the Fourier sine transform to equation (68) and using the property (67) we obtain a first order linear ODE

$$\frac{\partial U}{\partial t} = k\left(\frac{2}{\pi}\omega g(t) - \omega^2 U\right) \tag{72}$$

We require that $U(\omega,0)$ must be the Fourier sine transform of $u(x,0) = f(x)$ such that

$$U(\omega,0) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx \tag{73}$$

By solving (72-73) we obtain $U(\omega,t)$. The solution $u(x,t)$ is given then by the inverse Fourier sine transform

$$u(x,t) = \int_0^\infty U(\omega,t)\sin\omega x\,d\omega \tag{74}$$

Worked examples using transforms

One-dimensional wave equation on an infinite interval

Consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty \tag{75}$$

with the initial conditions

$$u(x, 0) = f(x) \tag{76}$$

$$\frac{\partial u}{\partial t}(x, 0) = 0 \tag{77}$$

To solve this problem we consider the Fourier transform

$$U(\omega, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x, t) e^{i\omega x} \, dx \tag{78}$$

$$u(x, t) = \int_{-\infty}^{\infty} U(\omega, t) e^{-i\omega x} \, d\omega \tag{79}$$

By applying the Fourier transform to equation (75) and using the properties of the transforms of the derivatives we obtain the ODE

$$\frac{\partial^2 U}{\partial t^2} = -c^2 \omega^2 U \tag{80}$$

with the initial conditions

$$U(\omega, 0) = \mathcal{F}[u(x, 0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx \tag{81}$$

$$\frac{\partial}{\partial t} U(\omega, 0) = 0 \tag{82}$$

The general solution of equation (80) is

$$U(\omega, t) = A(\omega) \cos c\omega t + B(\omega) \sin c\omega t \tag{83}$$

and from the initial conditions (81-82) we obtain the coefficients

$$B(\omega) = 0 \tag{84}$$

$$A(\omega) = U(\omega, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx \tag{85}$$

Therefore,

$$U(\omega, t) = U(\omega, 0) \cos c\omega t$$

and from (79) we obtain the solution

$$u(x, t) = \int_{-\infty}^{\infty} U(\omega, 0) \cos c\omega t e^{-i\omega x} \, d\omega \tag{86}$$

To simplify this expression we use Euler's formula

$$\cos c\omega t = \frac{e^{ic\omega t} + e^{-ic\omega t}}{2}$$

such that (86) becomes

$$u(x, t) = \frac{1}{2} \int_{-\infty}^{\infty} U(\omega, 0) \left[e^{-i\omega(x-ct)} + e^{-i\omega(x+ct)} \right] \, d\omega \tag{87}$$

From (81) we have (inverse Fourier transform)

$$f(x) = \int_{-\infty}^{\infty} U(\omega, 0) e^{-i\omega x} \, d\omega$$

such that (87) becomes

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] \tag{88}$$

which is the same expression we obtained using the method of characteristics.

Laplace's equation in a semi-infinite strip

Consider the Laplace's equation in a semi-infinite strip

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \;\; 0 < x < L, \, y > 0 \tag{89}$$

with the boundary conditions

$$u(0,y) = g_1(y), \;\; y > 0 \tag{90}$$

$$u(L,y) = g_2(y), \;\; y > 0 \tag{91}$$

$$u(x,0) = f(x), \;\; 0 < x < L \tag{92}$$

In addition we assume that

$$\lim_{y \to \infty} u(x,y) = 0, \;\; 0 < x < L \tag{93}$$

We split this problem into two subproblems that are easier to solve:

$$u(x,y) = u_1(x,y) + u_2(x,y) \tag{94}$$

where u_1 and u_2 satisfy the Laplace's equation (89) and respectively, the boundary conditions

$$u_1(0,y) = g_1(y), \;\; u_1(L,y) = g_2(y), \;\; u_1(x,0) = 0, \;\; \lim_{y \to \infty} u_1(x,y) = 0 \tag{95}$$

$$u_2(0,y) = 0, \;\; u_2(L,y) = 0, \;\; u_2(x,0) = f(x), \;\; \lim_{y \to \infty} u_2(x,y) = 0 \tag{96}$$

Zero-temperature sides

Using separation of variables, we search for a solution

$$u_2(x,y) = \Phi(x) h(y) \tag{97}$$

After replacing (97) in (89) and using the boundary conditions (95) we obtain

$$\frac{d^2\Phi}{dx^2} = -\lambda \Phi, \;\; \Phi(0) = 0, \;\; \Phi(L) = 0 \tag{98}$$

$$\frac{d^2 h}{dy^2} = \lambda h \tag{99}$$

From (98) we obtain the eigenvalues $\lambda_n = (n\pi x/L)^2$ and the corresponding eigenfunctions $\Phi_n(x) = \sin \frac{n\pi x}{L}$. The general solution of (99) is

$$h(y) = a_n e^{-\sqrt{\lambda_n} y} + b_n e^{\sqrt{\lambda_n} y}$$

such that using the principle of superposition

$$u_2(x,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} e^{-n\pi y/L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} e^{n\pi y/L} \tag{100}$$

The boundary condition

$$\lim_{y\to\infty} u_2(x,y) = 0$$

implies $b_n = 0$, $n = 1, 2, \ldots$. The nonhomogeneous boundary condition $u_2(x,0) = f(x)$ implies then

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}$$

such that a_n are the Fourier sine coefficients of $f(x)$

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx \tag{101}$$

Therefore, the solution $u_2(x,y)$ is

$$u_2(x,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} e^{-n\pi y/L} \tag{102}$$

where the coefficients a_n are evaluated according to (101).

Zero-temperature bottom

To find the solution $u_1(x,y)$ we introduce the Fourier sine transform in y

$$u_1(x,y) = \int_0^{\infty} U_1(x,\omega) \sin \omega y\, d\omega \tag{103}$$

$$U_1(x,\omega) = \frac{2}{\pi} \int_0^{\infty} u_1(x,y) \sin \omega y\, dy \tag{104}$$

Taking the Fourier sine transform with respect to y of the Laplace's equation and using the properties of the transforms of the derivatives, we obtain

$$\frac{\partial^2}{\partial x^2} U_1(x,\omega) + \frac{2}{\pi}\omega U_1(x,0) - \omega^2 U_1(x,\omega) = 0 \tag{105}$$

Now, $U_1(x,0) = \mathcal{S}[u_1(x,0)] = \mathcal{S}[0] = 0$ such that (105) becomes

$$\frac{\partial^2}{\partial x^2} U_1(x,\omega) - \omega^2 U_1(x,\omega) = 0 \tag{106}$$

The solution of (106) may be expressed in terms of sinh as

$$U_1(x,\omega) = a(\omega) \sinh \omega x + b(\omega) \sinh \omega (L - x) \tag{107}$$

The coefficients $a(\omega)$ and $b(\omega)$ are obtained from the boundary conditions

$$U_1(0,\omega) = \mathcal{S}[g_1(y)] = \frac{2}{\pi} \int_0^{\infty} g_1(y) \sin \omega y\, dy = b(\omega) \sinh \omega L \Rightarrow b(\omega) = \frac{2}{\pi} \frac{1}{\sinh \omega L} \int_0^{\infty} g_1(y) \sin \omega y\, dy$$

$$U_1(L,\omega) = \mathcal{S}[g_2(y)] = \frac{2}{\pi} \int_0^{\infty} g_2(y) \sin \omega y\, dy = a(\omega) \sinh \omega L \Rightarrow a(\omega) = \frac{2}{\pi} \frac{1}{\sinh \omega L} \int_0^{\infty} g_2(y) \sin \omega y\, dy$$

This completes the evaluation of the Fourier sine transform $U_1(x,\omega)$ and the solution $u_1(x,y)$ is given by (103).

Remark: The solution $u(x,y) = u_1(x,y) + u_2(x,y)$ of the Laplace's equation in a semi-infinite strip is the sum of a solution $u_1(x,y)$ obtained using the Fourier sine transform and a solution $u_2(x,y)$ obtained using a Fourier sine series.

An alternative approach to obtain the solution $u(x,y)$ is to directly apply the Fourier sine transform in y to the equation (89) with nonhomogeneous boundary conditions

$$u(x,y) = \int_0^\infty U(x,\omega)\sin\omega y\, d\omega$$

The differential equation for $U(x,\omega)$ is then nonhomogeneous

$$\frac{\partial^2 U}{\partial x^2} - \omega^2 U = -\frac{2}{\pi}\omega f(x)$$

and must be solved with two nonhomogeneous boundary conditions at $x = 0$ and $x = L$.

<u>*Laplace's equation in a half-plane*</u>

We consider the Laplace's equation in a half-plane

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty,\ y > 0 \tag{108}$$

with the boundary conditions

$$u(x,0) = f(x) \tag{109}$$

$$\lim_{x\to\infty} u(x,y) = 0 \tag{110}$$

$$\lim_{x\to-\infty} u(x,y) = 0 \tag{111}$$

$$\lim_{y\to\infty} u(x,y) = 0 \tag{112}$$

To solve this problem we consider the Fourier transform in x

$$u(x,y) = \int_{-\infty}^{\infty} U(\omega,y)e^{-i\omega x}\, d\omega \tag{113}$$

$$U(\omega,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,y)e^{i\omega x}\, dx \tag{114}$$

Taking the Fourier transform in x of the Laplace's equation (108) we obtain the ODE

$$\frac{\partial^2 U}{\partial y^2} - \omega^2 U = 0 \tag{115}$$

whose general solution is

$$U(\omega,y) = a(\omega)e^{\omega y} + b(\omega)e^{-\omega y} \tag{116}$$

Next we use the boundary conditions to determine $a(\omega)$ and $b(\omega)$.

$$\lim_{y\to\infty} u(x,y) = 0 \Rightarrow \lim_{y\to\infty} U(\omega,y) = 0$$

such that $a(\omega) = 0$ if $\omega > 0$ and $b(\omega) = 0$ if $\omega < 0$. Therefore, (116) becomes

$$U(\omega, y) = \begin{cases} a(\omega)e^{\omega y}, & \omega < 0 \\ b(\omega)e^{-\omega y}, & \omega > 0 \end{cases} \tag{117}$$

which may be written in a compact form

$$U(\omega, y) = c(\omega)e^{-|\omega|y} \tag{118}$$

where $c(\omega) = a(\omega), \omega < 0$ and $c(\omega) = b(\omega), \omega > 0$.
From the boundary condition (109) we have

$$U(\omega, 0) = \mathcal{F}[f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\, dx$$

such that $c(\omega)$ in (118) must be the Fourier transform of $f(x)$:

$$c(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\, dx \tag{119}$$

The solution $u(x, y)$ is then given by (113) where $U(\omega, y)$ is evaluated according to (118)-(119).

Q: Using the convolution theorem, show that the solution $u(x, y)$ may be expressed as

$$u(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\overline{x})\frac{2y}{(x-\overline{x})^2 + y^2}\, d\overline{x} \tag{120}$$

Laplace's equation in a quarter-plane

To solve the Laplace's equation in a quarter-plane $x > 0, y > 0$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \ \ x > 0, y > 0 \tag{121}$$

with the boundary conditions

$$u(0, y) = g(y), \ \ y > 0 \tag{122}$$

$$\frac{\partial u}{\partial y}(x, 0) = f(x), \ \ x > 0 \tag{123}$$

$$\lim_{x\to\infty} u(x, y) = \lim_{y\to\infty} u(x, y) = 0 \tag{124}$$

we decompose the problem into two subproblems that are easier to solve:

$$u(x, y) = u_1(x, y) + u_2(x, y)$$

where

$$\begin{aligned} \nabla^2 u_1 &= 0 & \qquad \nabla^2 u_2 &= 0 \\ u_1(0, y) &= g(y) & u_2(0, y) &= 0 \\ \frac{\partial}{\partial y}u_1(x, 0) &= 0 & \tfrac{\partial}{\partial y}u_2(x, 0) &= f(x) \\ \lim_{x\to\infty} u_1(x, y) &= 0 & \lim\nolimits_{x\to\infty} u_2(x, y) &= 0 \\ \lim_{y\to\infty} u_1(x, y) &= 0 & \lim\nolimits_{y\to\infty} u_2(x, y) &= 0 \end{aligned}$$

The techniques for solving these problems are very similar, and we only indicate how to solve the problem for $u_1(x, y)$. We consider the Fourier cosine transform in y since we have a homogeneous boundary condition $\partial u_1/\partial y(x, 0) = 0$.

$$u_1(x, y) = \int_0^\infty U_1(x, \omega) \cos \omega y \, d\omega \tag{125}$$

$$U_1(x, \omega) = \frac{2}{\pi} \int_0^\infty u_1(x, y) \cos \omega y \, dy \tag{126}$$

Taking the Fourier cosine transform in y of the Laplace's equation for u_1 and using the homogeneous boundary condition $\partial u_1/\partial y(x, 0) = 0$ we obtain the ordinary differential equation

$$\frac{\partial^2 U_1}{\partial x^2} - \omega^2 U_1 = 0$$

whose general solution is

$$U_1(x, \omega) = a(\omega)e^{-\omega x} + b(\omega)e^{\omega x}, \;\; x > 0, \omega > 0$$

To determine $a(\omega)$ and $b(\omega)$ we use the boundary conditions.

$$\lim_{x\to\infty} u_1(x, y) = 0 \Rightarrow \lim_{x\to\infty} U_1(x, \omega) = 0 \Rightarrow b(\omega) = 0$$

Therefore,

$$U_1(x, \omega) = a(\omega)e^{-\omega x} \tag{127}$$

$$U_1(0, \omega) = \mathcal{C}[u_1(0, y)] = \mathcal{C}[g(y)] \Rightarrow a(\omega) = \frac{2}{\pi} \int_0^\infty g(y) \cos \omega y \, dy \tag{128}$$

The solution $u_1(x, y)$ is obtained from (125) with $U_1(x, \omega)$ given by (127)-(128).