Chapter 10: Fourier Transform Solutions of PDEs

In this chapter we show how the method of separation of variables may be extended to solve PDEs defined on an infinite or semi-infinite spatial domain. Several new concepts such as the "Fourier integral representation" and "Fourier transform" of a function are introduced as an extension of the Fourier series representation to an infinite domain.

We consider the heat equation
\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty
\] (1)
with the initial condition
\[
u(x, 0) = f(x), \quad -\infty < x < \infty
\] (2)
As physical conditions at ±∞ we impose
\[
u(-\infty, t) = 0, \quad \nu(\infty, t) = 0, \quad t > 0
\] (3)
Using separation of variables,
\[
u(x, t) = \Phi(x) h(t)
\] (4)
we obtain the differential equations
\[
\frac{dh}{dt} = -\lambda k h \Rightarrow h(t) = c e^{-\lambda kt}
\] (5)
\[
\frac{d^2 \Phi}{dx^2} = -\lambda \Phi
\] (6)
First problem we face is to impose boundary conditions at ±∞ for \(\Phi(x)\).

Q: Show that equation (6) with the boundary conditions \(\Phi(-\infty) = \Phi(\infty) = 0\) has no solutions.

Therefore, the correct boundary condition for \(\Phi(x)\) at \(x = \pm \infty\) is different from the boundary condition for \(\nu(x, t)\) at \(x = \pm \infty\). We require that \(\Phi(x)\) is bounded at \(x = \pm \infty\)
\[
|\Phi(\pm \infty)| < \infty
\] (7)
Q: Show that problem (6)-(7) has no solutions if \(\lambda < 0\).

If \(\lambda > 0\) the solution of the problem (6),(7) is
\[
\Phi(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x
\] (8)
which remains bounded at \(x = \pm \infty\) for all \(\lambda > 0\). The major difference from the finite domain problem is that in this case all \(\lambda > 0\) are eigenvalues!

Q: Show that \(\lambda = 0\) is an eigenvalue of the problem (6),(7) and the corresponding eigenfunction is \(\Phi(x) \equiv c\).

The set of eigenvalues for a problem is usually referred to as the spectrum. For the heat equation on a finite domain we have a discrete spectrum \(\lambda_n = (n\pi/L)^2\), whereas for the heat equation defined on \(-\infty < x < \infty\) we have a continuous spectrum \(\lambda \geq 0\).

Superposition principle. From (5) and (8) we obtain the product solutions \(\nu(x, t) = \sin \sqrt{\lambda} x e^{-\lambda kt}\) and \(\nu(x, t) = \cos \sqrt{\lambda} x e^{-\lambda kt}\) for all \(\lambda \geq 0\). The general solution is obtained using the principle of superposition:
\[
u(x, t) = \int_0^\infty [c_1(\lambda) \cos \sqrt{\lambda} x e^{-\lambda kt} + c_2(\lambda) \sin \sqrt{\lambda} x e^{-\lambda kt}] d\lambda
\] (9)
where \( c_1(\lambda) \) and \( c_2(\lambda) \) are arbitrary functions of \( \lambda \).

**Q:** Show that (9) is a solution of the equation (1) for any \( c_1(\lambda) \) and \( c_2(\lambda) \).

If we let \( \lambda = \omega^2 \) then (9) becomes

\[
u(x, t) = \int_0^\infty [A(\omega) \cos \omega x e^{-k\omega^2 t} + B(\omega) \sin \omega x e^{-k\omega^2 t}] d\omega
\]

where \( A(\omega) = 2\omega c_1(\omega^2) \), \( B(\omega) = 2\omega c_2(\omega^2) \) are arbitrary functions.

To satisfy the initial condition (2) we must have

\[ f(x) = \int_0^\infty [A(\omega) \cos \omega x + B(\omega) \sin \omega x] d\omega \tag{11} \]

The problem is now to determine the functions \( A(\omega) \) and \( B(\omega) \) such that (11) is satisfied.

**Complex exponentials**

Complex exponentials may be used to express the sin and cos functions (Euler’s formulas):

\[
\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}
\]

With these formulas, we may write (10) as

\[
u(x, t) = \int_0^\infty \left[ A(\omega) \frac{e^{i\omega x} + e^{-i\omega x}}{2} + B(\omega) \frac{e^{i\omega x} - e^{-i\omega x}}{2i} \right] e^{-k\omega^2 t} d\omega
\]

such that we obtain

\[
u(x, t) = \int_{-\infty}^\infty c(\omega) e^{-i\omega x} e^{-k\omega^2 t} d\omega \tag{13}
\]

where the function \( c(\omega) : R \rightarrow C \) (takes a real argument and returns a complex value) is defined as

\[
c(\omega) = \begin{cases} 
\frac{A(\omega) + iB(\omega)}{2}, & \omega > 0 \\
\frac{A(\omega) - iB(\omega)}{2}, & \omega < 0 
\end{cases}
\]

**Remark:** Notice that from (14) it follows that \( c(-\omega) = \bar{c}(\omega) \).

At \( t = 0 \) we obtain from (13)

\[
f(x) = \int_{-\infty}^\infty c(\omega) e^{-i\omega x} d\omega \tag{15}
\]

**Q:** Show that (15) with \( c(\omega) \) given by (14) is equivalent to (11).

To determine the coefficients \( c(\omega) \) from (15) we need to introduce a couple of new concepts: **Fourier transform and Fourier integral representation** of a function.
Fourier transform pair

Given a piecewise smooth function \( f(x) \) defined on \(-L \leq x \leq L\), the Fourier series representation if \( f \) is

\[
\frac{f(x+) + f(x-)}{2} = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}
\]

where

\[
a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx
\]

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx
\]

The complex form of (16) is (see Section 3.6 for a proof):

\[
\frac{f(x+) + f(x-)}{2} = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}
\]

where the coefficients \( c_n \) are complex numbers given by

\[
c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{in\pi x/L} dx
\]

By replacing (18) in (17) we obtain the Fourier series identity

\[
\frac{f(x+) + f(x-)}{2} = \sum_{n=-\infty}^{\infty} \left[ \frac{1}{2L} \int_{-L}^{L} f(x) e^{in\pi x/L} dx \right] e^{-in\pi x/L}
\]

For functions defined for \(-\infty < x < \infty\) the following Fourier integral identity is valid

\[
\frac{f(x+) + f(x-)}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \right] e^{-i\omega x} d\omega
\]

You may think about (20) as a limit of (19) as \( L \to \infty \).

Definition: Given a function \( f(x), -\infty < x < \infty \), we define the Fourier transform of \( f(x) \) as the function

\[
F(\omega) = \frac{\gamma}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx
\]

where \( \gamma \) is an arbitrary constant \((\neq 0)\).

From (20) and (21) it follows that

\[
\frac{f(x+) + f(x-)}{2} = \frac{1}{\gamma} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega
\]

Relation (22) is called the Fourier integral representation of \( f(x) \) and \( f(x) \) determined by (22) is called the inverse Fourier transform of \( F(\omega) \). If \( f \) is continuous we simply have

\[
f(x) = \frac{1}{\gamma} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega
\]
The relations (21) and (22) are valid for any function \( f(x) \) that satisfies \( \int_{-\infty}^{\infty} |f(x)|dx < \infty \) and are also known as the **Fourier transform pair**. In our applications we will let \( \gamma = 1 \).

Next we mention several properties of the **Fourier transform**.

1. The Fourier transform is a linear operator:

   \[
   \mathcal{F}[c_1 f(x) + c_2 g(x)] = c_1 \mathcal{F}[f(x)] + c_2 \mathcal{F}[g(x)]
   \]  
   (24)

   where \( \mathcal{F}[f(x)] = F(\omega) \) denotes the Fourier transform of \( f(x) \).

2. Given a real valued function \( f(x) \) we have

   \[
   F(-\omega) = F^*(\omega)
   \]  
   (25)

   where \( F^*(\omega) \) denotes the complex conjugate of \( F(\omega) \).

3. If \( f(x) \) is an odd function then \( F(\omega) \) is an odd function. In addition, in this case we have

   \[
   f(x) = \frac{1}{\gamma} \int_{-\infty}^{\infty} F(\omega)(\cos \omega x - i \sin \omega x)d\omega = \frac{-2i}{\gamma} \int_{0}^{\infty} F(\omega) \sin \omega x d\omega
   \]

   \[
   F(\omega) = \frac{\gamma}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx = \frac{\gamma}{2\pi} \int_{-\infty}^{\infty} f(x)(\cos \omega x + i \sin \omega x) dx
   \]

   \[
   = \frac{i\gamma}{2\pi} \int_{-\infty}^{\infty} f(x) \sin \omega x dx = \frac{i\gamma}{\pi} \int_{0}^{\infty} f(x) \sin \omega x d\omega
   \]

   If we choose, for convenience, \( \gamma = -2i \), then we obtain

   \[
   f(x) = \int_{0}^{\infty} F(\omega) \sin \omega x d\omega
   \]  
   (26)

   \[
   F(\omega) = \frac{2}{\pi} \int_{0}^{\infty} f(x) \sin \omega x dx
   \]  
   (27)

   Relations (26) and (27) are called the **Fourier sine transform pair** and are valid if \( f(x) \) is an odd function. \( F(\omega) \equiv \mathcal{S}[f(x)] \) is called the Fourier sine transform of \( f(x) \) and \( f(x) \equiv \mathcal{S}^{-1}[F(\omega)] \) is called the inverse Fourier sine transform of \( F(\omega) \).

4. Similarly, if \( f(x) \) is an even function then \( F(\omega) \) is an even function and we obtain the **Fourier cosine transform pair**

   \[
   f(x) = \int_{0}^{\infty} F(\omega) \cos \omega x d\omega
   \]  
   (28)

   \[
   F(\omega) = \frac{2}{\pi} \int_{0}^{\infty} f(x) \cos \omega x dx
   \]  
   (29)

   In this case \( F(\omega) \equiv \mathcal{C}[f(x)] \) is called the Fourier cosine transform of \( f(x) \) and \( f(x) \equiv \mathcal{C}^{-1}[F(\omega)] \) is called the inverse Fourier cosine transform of \( F(\omega) \).

5. The Fourier transform of a Gaussian is a Gaussian and the inverse Fourier transform of a Gaussian is a Gaussian

   \[
   f(x) = e^{-\beta x^2} \Leftrightarrow F(\omega) = \frac{1}{\sqrt{4\pi\beta}} e^{-\frac{\omega^2}{4\beta}}
   \]  
   (30)
\[ f(x) = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{x^2}{4\alpha}} \Leftrightarrow F(\omega) = e^{-\alpha \omega^2} \]  

6. **Convolution Theorem**: If \( F(\omega) \) and \( G(\omega) \) are the Fourier transforms of \( f(x) \) and \( g(x) \), respectively, then

\[ h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\tau) f(x - \tau) d\tau \]  

is the inverse Fourier transform of the product \( F(\omega)G(\omega) \). The function \( h(x) \) defined in (32) is called the convolution of the functions \( f \) and \( g \) and is denoted \( h = f * g \). Notice that \( f * g = g * f \).

**Fourier transform and the heat equation**

We return now to the solution of the heat equation on an infinite interval and show how to use Fourier transforms to obtain \( u(x, t) \). From (15) it follows that \( c(\omega) \) is the Fourier transform of the initial temperature distribution \( f(x) \):

\[ c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \]  

such that by replacing (33) in (13) we may express the solution \( u(x, t) \) as

\[ u(x, t) = \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\tau) e^{i\omega \tau} d\tau \right] e^{-i\omega x} e^{-k\omega^2 t} d\omega \]  

which may be written in the equivalent form

\[ u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\tau) \left[ \int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega (x - \tau)} d\omega \right] d\tau \]  

(35)

Notice that

\[ g(x) = \int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega x} d\omega \]  

is the inverse Fourier transform of \( e^{-k\omega^2 t} \). With this notation, the solution (35) becomes

\[ u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\tau) g(x - \tau) d\tau \]  

(37)

The only problem now is to obtain an explicit formula for \( g(x) \) defined by (36).

**Inverse Fourier Transform of a Gaussian**

Functions of the form

\[ G(\omega) = e^{-\alpha \omega^2} \]

where \( \alpha > 0 \) is a constant are usually referred to as Gaussian functions. The function \( g(x) \) whose Fourier transform is \( G(\omega) \) is given by the inverse Fourier transform formula

\[ g(x) = \int_{-\infty}^{\infty} G(\omega) e^{-i\omega x} d\omega = \int_{-\infty}^{\infty} e^{-\alpha \omega^2} e^{-i\omega x} d\omega \]  

(38)

The last integral in (38) may be evaluated (the proof is not trivial, see the Appendix to 10.3 for details) to obtain

\[ g(x) = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{x^2}{4\alpha}} \]  

(39)
showing that the inverse Fourier transform of a Gaussian is itself a Gaussian.

If we let $\alpha = kt$ in (38), (39) we obtain

$$
\int_{-\infty}^{\infty} e^{-k\omega^2} e^{-i\omega x} d\omega = \sqrt{\frac{\pi}{kt}} e^{-\frac{x^2}{4kt}}
$$

such that replacing in (36) it follows that

$$
g(x) = \sqrt{\frac{\pi}{kt}} e^{-\frac{x^2}{4kt}}
$$

The solution $u(x, t)$ is obtained by replacing (40) in (37)

$$
u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) \sqrt{\frac{\pi}{kt}} e^{-\frac{(\bar{x}-x)^2}{4kt}} d\bar{x}
$$

which may be written as

$$
u(x, t) = \int_{-\infty}^{\infty} f(\bar{x}) \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(\bar{x}-x)^2}{4kt}} d\bar{x}
$$

**Remark:** Notice that if the initial condition is specified as a Dirac impulse concentrated at the origin, $f(x) = \delta(x)$, then from (42) we get

$$
u(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}
$$

which is called the fundamental solution of the heat equation.

**Transform of the derivatives:** the following properties hold:

1. The Fourier transform of a time derivative equals the time derivative of the Fourier transform

$$
\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{\partial}{\partial t} U(\omega, t)
$$

2. In general, the Fourier transform of the $n^{th}$ derivative of a function $u(x, t)$ with respect to $x$ equals $(-i\omega)^n$ times the Fourier transform of $u(x, t)$, if $u(x, t) \to 0$, sufficiently fast as $x \to \pm \infty$. In particular,

$$
\mathcal{F}\left[\frac{\partial^n u}{\partial x^n}\right] = (-i\omega)^n U(\omega, t)
$$

Q: Prove (44) and (45).

**Fourier transforming the heat equation**

Next we show how the Fourier transform may be used to solve directly the heat equation on an infinite interval

$$
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty
$$

with the initial condition

$$
u(x, 0) = f(x), \quad -\infty < x < \infty \quad \text{(47)}
$$

$$
u(-\infty, t) = 0, \quad u(\infty, t) = 0, \quad t > 0 \quad \text{(48)}
$$
Since the Fourier transform is a linear operator, by applying the Fourier transform to the equation (46) we obtain

\[ \mathcal{F}\left[ \frac{\partial u}{\partial t} \right] = k \mathcal{F}\left[ \frac{\partial^2 u}{\partial x^2} \right] \]  

(49)

The Fourier transform of \( u(x, t) \) is

\[ \mathcal{F}[u] \equiv U(\omega, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x, t) e^{i\omega x} \, dx \]  

(50)

If we are able to find \( \mathcal{F}[u] \) then the solution \( u(x, t) \) is given by the inverse Fourier transform formula

\[ u(x, t) = \int_{-\infty}^{\infty} U(\omega, t) e^{-i\omega x} \, d\omega \]  

(51)

After replacing (44) and (45) in (49) we obtain an ordinary differential equation for the Fourier transform \( U(\omega, t) \)

\[ \frac{\partial}{\partial t} U(\omega, t) = -k\omega^2 U(\omega, t) \]  

(52)

Therefore, the Fourier transform operation converts a linear PDE with constant coefficients into an ODE!

The general solution of (52) is

\[ U(\omega, t) = c(\omega) e^{-k\omega^2 t} \]  

(53)

To determine \( c(\omega) \) we use the initial condition (47) and require that \( U(\omega, 0) \) is the Fourier transform of \( f(x) \). Then

\[ c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx \]  

(54)

After replacing (53) in (51) we obtain the solution \( u(x, t) \)

\[ u(x, t) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} e^{-k\omega^2 t} \, d\omega \]  

(55)

which may be written after replacing (54) and rearranging the terms

\[ u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\overline{x}) \left[ \int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega x} \, d\omega \right] d\overline{x} \]  

(56)

This is the same formula we obtained by using the separation of variables.

**Heat equation on semi-infinite intervals**

We consider the heat equation on a semi-infinite interval \( x > 0 \)

\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty \]  

(57)

with the initial condition

\[ u(x, 0) = f(x), \quad 0 < x < \infty \]  

(58)

\[ u(0, t) = 0, \quad u(\infty, t) = 0, \quad t > 0 \]  

(59)

Using separation of variables,

\[ u(x, t) = \Phi(x) h(t) \]

we obtain

\[ \frac{dh}{dt} = -\lambda h \]
and the boundary value problem
\[ \frac{d^2 \Phi}{dx^2} = -\lambda \Phi \]
\[ \Phi(0) = 0 \]
\[ |\Phi(\infty)| < \infty \]
whose eigenvalues are all positive \( \lambda \) and the eigenfunctions are
\[ \Phi(x) = c \sin \sqrt{\lambda} x = c \sin \omega x \]
where \( \omega \equiv \sqrt{\lambda} > 0 \). Therefore, the product solutions are
\[ u(x, t) = A \sin \omega x e^{-k \omega^2 t} \]
and using the principle of superposition we obtain
\[ u(x, t) = \int_0^\infty A(\omega) \sin \omega x e^{-k \omega^2 t} d\omega \]
(60)
To satisfy the initial condition we require
\[ f(x) = \int_0^\infty A(\omega) \sin \omega x d\omega \]
If we consider the odd extension \( \overline{f}(x) \) of \( f(x) \) to \( -\infty < x < \infty \) then
\[ \overline{f}(x) = f(x) = \int_0^\infty F(\omega) \sin \omega x d\omega, \quad x > 0 \]
where \( F(\omega) \) is the Fourier sine transform of \( \overline{f}(x) \)
\[ F(\omega) = \frac{2}{\pi} \int_0^\infty \overline{f}(x) \sin \omega x dx = \frac{2}{\pi} \int_0^\infty f(x) \sin \omega x dx \]
Therefore, \( A(\omega) \) is given by the Fourier sine transform of \( f(x) \)
\[ A(\omega) = \frac{2}{\pi} \int_0^\infty f(x) \sin \omega x dx \]
(61)
Fourier sine and cosine transforms of derivatives

We defined the Fourier sine and cosine transforms of a function \( f(x) \) as

\[
S[f(x)] = \frac{2}{\pi} \int_0^\infty f(x) \sin \omega x \, dx \quad (62)
\]

\[
C[f(x)] = \frac{2}{\pi} \int_0^\infty f(x) \cos \omega x \, dx \quad (63)
\]

For the Fourier sine and cosine transforms of derivatives, assuming that \( \lim_{x \to \infty} f(x) = 0 \), we have the following formulas:

\[
C \left[ \frac{df}{dx} \right] = -\frac{2}{\pi} f(0) + \omega S[f] \quad (64)
\]

\[
S \left[ \frac{df}{dx} \right] = -\omega C[f] \quad (65)
\]

\[
C \left[ \frac{d^2f}{dx^2} \right] = -\frac{2}{\pi} \frac{df}{dx} (0) + \omega S \left[ \frac{df}{dx} \right] = -\frac{2}{\pi} \frac{df}{dx} (0) - \omega^2 C[f] \quad (66)
\]

\[
S \left[ \frac{d^2f}{dx^2} \right] = -\omega C \left[ \frac{df}{dx} \right] = \frac{2}{\pi} \omega f(0) - \omega^2 S[f] \quad (67)
\]

Q: Prove formulas (64-67).

Formulas (66) and (67) suggest the appropriate choice of the Fourier sine/cosine transform when solving PDEs involving second order derivatives on a semi-infinite interval \( x \geq 0 \). Replace \( f(x) \) by \( u(x, t) \) in the formulas above: if \( u(0, t) \) is given we will use a Fourier sine transform; if \( \partial u/\partial x(0, t) \) is given we will use a Fourier cosine transform.

Example: Consider the heat equation on \( 0 < x < \infty \) with nonhomogeneous boundary condition at \( x = 0 \)

\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty \quad (68)
\]

\[
u(x, 0) = f(x), \quad 0 < x < \infty \quad (69)
\]

\[
u(0, t) = g(t), \quad u(\infty, t) = 0, \quad t > 0 \quad (70)
\]

We introduce the Fourier sine transform of \( u(x, t) \)

\[
U(\omega, t) = \frac{2}{\pi} \int_0^\infty u(x, t) \sin \omega x \, dx \quad (71)
\]

By applying the Fourier sine transform to equation (68) and using the property (67) we obtain a first order linear ODE

\[
\frac{\partial U}{\partial t} = k \left( \frac{2}{\pi} \omega g(t) - \omega^2 U \right) \quad (72)
\]

We require that \( U(\omega, 0) \) must be the Fourier sine transform of \( u(x, 0) = f(x) \) such that

\[
U(\omega, 0) = \frac{2}{\pi} \int_0^\infty f(x) \sin \omega x \, dx \quad (73)
\]

By solving (72-73) we obtain \( U(\omega, t) \). The solution \( u(x, t) \) is given then by the inverse Fourier sine transform

\[
u(x, t) = \int_0^\infty U(\omega, t) \sin \omega x \, d\omega \quad (74)
\]

9
Worked examples using transforms

One-dimensional wave equation on an infinite interval

Consider the one-dimensional wave equation

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty
\]  
(75)

with the initial conditions

\[
u(x, 0) = f(x)\]
(76)
\[
\frac{\partial u}{\partial t}(x, 0) = 0
\]  
(77)

To solve this problem we consider the Fourier transform

\[
U(\omega, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x, t) e^{i\omega x} \, dx
\]  
(78)

\[
u(x, t) = \int_{-\infty}^{\infty} U(\omega, t) e^{-i\omega x} \, d\omega
\]  
(79)

By applying the Fourier transform to equation (75) and using the properties of the transforms of the derivatives we obtain the ODE

\[
\frac{\partial^2 U}{\partial t^2} = -c^2 \omega^2 U
\]  
(80)

with the initial conditions

\[
U(\omega, 0) = \mathcal{F}[u(x, 0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx
\]  
(81)
\[
\frac{\partial}{\partial t} U(\omega, 0) = 0
\]  
(82)

The general solution of equation (80) is

\[
U(\omega, t) = A(\omega) \cos c\omega t + B(\omega) \sin c\omega t
\]  
(83)

and from the initial conditions (81-82) we obtain the coefficients

\[
B(\omega) = 0
\]  
(84)
\[
A(\omega) = U(\omega, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx
\]  
(85)

Therefore,

\[
U(\omega, t) = U(\omega, 0) \cos c\omega t
\]

and from (79) we obtain the solution

\[
u(x, t) = \int_{-\infty}^{\infty} U(\omega, 0) \cos c\omega t e^{-i\omega x} \, d\omega
\]  
(86)

To simplify this expression we use Euler’s formula

\[
\cos c\omega t = \frac{e^{ic\omega t} + e^{-ic\omega t}}{2}
\]

such that (86) becomes

\[
u(x, t) = \frac{1}{2} \int_{-\infty}^{\infty} U(\omega, 0) \left[ e^{-i\omega(x-ct)} + e^{-i\omega(x+ct)} \right] \, d\omega
\]  
(87)
From (81) we have (inverse Fourier transform)

\[ f(x) = \int_{-\infty}^{\infty} U(\omega, 0)e^{-i\omega x} d\omega \]

such that (87) becomes

\[ u(x, t) = \frac{1}{2} \left[ f(x - ct) + f(x + ct) \right] \] (88)

which is the same expression we obtained using the method of characteristics.

Laplace’s equation in a semi-infinite strip

Consider the Laplace’s equation in a semi-infinite strip

\[ \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < L, \ y > 0 \] (89)

with the boundary conditions

\[ u(0, y) = g_1(y), \ y > 0 \] (90)
\[ u(L, y) = g_2(y), \ y > 0 \] (91)
\[ u(x, 0) = f(x), \ 0 < x < L \] (92)

In addition we assume that

\[ \lim_{y \to \infty} u(x, y) = 0, \ 0 < x < L \] (93)

We split this problem into two subproblems that are easier to solve:

\[ u(x, y) = u_1(x, y) + u_2(x, y) \] (94)

where \( u_1 \) and \( u_2 \) satisfy the Laplace’s equation (89) and respectively, the boundary conditions

\[ u_1(0, y) = g_1(y), \ u_1(L, y) = g_2(y), \ u_1(x, 0) = 0, \ \lim_{y \to \infty} u_1(x, y) = 0 \] (95)
\[ u_2(0, y) = 0, \ u_2(L, y) = 0, \ u_2(x, 0) = f(x), \ \lim_{y \to \infty} u_2(x, y) = 0 \] (96)

Zero-temperature sides

Using separation of variables, we search for a solution

\[ u_2(x, y) = \Phi(x)h(y) \] (97)

After replacing (97) in (89) and using the boundary conditions (95) we obtain

\[ \frac{d^2 \Phi}{dx^2} = -\lambda \Phi, \ \Phi(0) = 0, \ \Phi(L) = 0 \] (98)
\[ \frac{d^2 h}{dy^2} = \lambda h \] (99)

From (98) we obtain the eigenvalues \( \lambda_n = \left( n\pi x / L \right)^2 \) and the corresponding eigenfunctions \( \Phi_n(x) = \sin \frac{n\pi x}{L} \).

The general solution of (99) is

\[ h(y) = a_n e^{-\sqrt{\lambda_n} y} + b_n e^{\sqrt{\lambda_n} y} \]
such that using the principle of superposition

\[
    u_2(x, y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} e^{-n\pi y/L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} e^{n\pi y/L} \tag{100}
\]

The boundary condition

\[
    \lim_{y \to \infty} u_2(x, y) = 0
\]

implies \( b_n = 0 \), \( n = 1, 2, \ldots \). The nonhomogeneous boundary condition \( u_2(x, 0) = f(x) \) implies then

\[
    f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}
\]

such that \( a_n \) are the Fourier sine coefficients of \( f(x) \)

\[
    a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \tag{101}
\]

Therefore, the solution \( u_2(x, y) \) is

\[
    u_2(x, y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} e^{-n\pi y/L} \tag{102}
\]

where the coefficients \( a_n \) are evaluated according to (101).

**Zero-temperature bottom**

To find the solution \( u_1(x, y) \) we introduce the Fourier sine transform in \( y \)

\[
    u_1(x, y) = \int_0^\infty U_1(x, \omega) \sin \omega y \, d\omega \tag{103}
\]

\[
    U_1(x, \omega) = \frac{2}{\pi} \int_0^\infty u_1(x, y) \sin \omega y \, dy \tag{104}
\]

Taking the Fourier sine transform with respect to \( y \) of the Laplace’s equation and using the properties of the transforms of the derivatives, we obtain

\[
    \frac{\partial^2}{\partial x^2} U_1(x, \omega) + \frac{2}{\pi} \omega U_1(x, 0) - \omega^2 U_1(x, \omega) = 0 \tag{105}
\]

Now, \( U_1(x, 0) = S[u_1(x, 0)] = S[0] = 0 \) such that (105) becomes

\[
    \frac{\partial^2}{\partial x^2} U_1(x, \omega) - \omega^2 U_1(x, \omega) = 0 \tag{106}
\]

The solution of (106) may be expressed in terms of sinh as

\[
    U_1(x, \omega) = a(\omega) \sinh \omega x + b(\omega) \sinh \omega (L - x) \tag{107}
\]

The coefficients \( a(\omega) \) and \( b(\omega) \) are obtained from the boundary conditions

\[
    U_1(0, \omega) = S[g_1(y)] = \frac{2}{\pi} \int_0^\infty g_1(y) \sin \omega y \, dy = b(\omega) \sinh \omega L \Rightarrow b(\omega) = \frac{1}{\sinh \omega L} \int_0^\infty g_1(y) \sin \omega y \, dy
\]

\[
    U_1(L, \omega) = S[g_2(y)] = \frac{2}{\pi} \int_0^\infty g_2(y) \sin \omega y \, dy = a(\omega) \sinh \omega L \Rightarrow a(\omega) = \frac{1}{\sinh \omega L} \int_0^\infty g_2(y) \sin \omega y \, dy
\]
This completes the evaluation of the Fourier sine transform $U_1(x, \omega)$ and the solution $u_1(x, y)$ is given by (103).

**Remark:** The solution $u(x, y) = u_1(x, y) + u_2(x, y)$ of the Laplace’s equation in a semi-infinite strip is the sum of a solution $u_1(x, y)$ obtained using the Fourier sine transform and a solution $u_2(x, y)$ obtained using a Fourier sine series.

An alternative approach to obtain the solution $u(x, y)$ is to directly apply the Fourier sine transform in $y$ to the equation (89) with nonhomogeneous boundary conditions

$$u(x, y) = \int_0^\infty U(x, \omega) \sin \omega y \, d\omega$$

The differential equation for $U(x, \omega)$ is then nonhomogeneous

$$\frac{\partial^2 U}{\partial x^2} - \omega^2 U = -\frac{2}{\pi} \omega f(x)$$

and must be solved with two nonhomogeneous boundary conditions at $x = 0$ and $x = L$.

**Laplace’s equation in a half-plane**

We consider the Laplace’s equation in a half-plane

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad y > 0$$

(108)

with the boundary conditions

$$u(x, 0) = f(x)$$

(109)

$$\lim_{x \to \infty} u(x, y) = 0$$

(110)

$$\lim_{x \to -\infty} u(x, y) = 0$$

(111)

$$\lim_{y \to \infty} u(x, y) = 0$$

(112)

To solve this problem we consider the Fourier transform in $x$

$$u(x, y) = \int_{-\infty}^{\infty} U(\omega, y) e^{-i\omega x} \, d\omega$$

(113)

$$U(\omega, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x, y) e^{i\omega x} \, dx$$

(114)

Taking the Fourier transform in $x$ of the Laplace’s equation (108) we obtain the ODE

$$\frac{\partial^2 U}{\partial y^2} - \omega^2 U = 0$$

(115)

whose general solution is

$$U(\omega, y) = a(\omega) e^{\omega y} + b(\omega) e^{-\omega y}$$

(116)

Next we use the boundary conditions to determine $a(\omega)$ and $b(\omega)$.

$$\lim_{y \to \infty} u(x, y) = 0 \Rightarrow \lim_{y \to \infty} U(\omega, y) = 0$$
such that \( a(\omega) = 0 \) if \( \omega > 0 \) and \( b(\omega) = 0 \) if \( \omega < 0 \). Therefore, (116) becomes

\[
U(\omega, y) = \begin{cases} 
  a(\omega)e^{\omega y}, & \omega < 0 \\
  b(\omega)e^{-\omega y}, & \omega > 0 
\end{cases}
\] (117)

which may be written in a compact form

\[
U(\omega, y) = c(\omega)e^{-|\omega|y}
\] (118)

where \( c(\omega) = a(\omega), \omega < 0 \) and \( c(\omega) = b(\omega), \omega > 0 \).

From the boundary condition (109) we have

\[
U(\omega, 0) = \mathcal{F}[f(x)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{i\omega x} dx
\]

such that \( c(\omega) \) in (118) must be the Fourier transform of \( f(x) \):

\[
c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{i\omega x} dx
\] (119)

The solution \( u(x, y) \) is then given by (113) where \( U(\omega, y) \) is evaluated according to (118)-(119).

Q: Using the convolution theorem, show that the solution \( u(x, y) \) may be expressed as

\[
u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\tau) \frac{2y}{(x-\tau)^2 + y^2} d\tau
\] (120)

Laplace’s equation in a quarter-plane

To solve the Laplace’s equation in a quarter-plane \( x > 0, y > 0 \)

\[
\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad x > 0, y > 0
\]

(121)

with the boundary conditions

\[
u(0, y) = g(y), \quad y > 0
\]

(122)

\[
\frac{\partial u}{\partial y}(x, 0) = f(x), \quad x > 0
\]

(123)

\[
\lim_{x \to \infty} u(x, y) = \lim_{y \to \infty} u(x, y) = 0
\]

(124)

we decompose the problem into two subproblems that are easier to solve:

\[
u(x, y) = u_1(x, y) + u_2(x, y)
\]

where

\[
\nabla^2 u_1 = 0 \quad \nabla^2 u_2 = 0
\]

\[
u_1(0, y) = g(y) \quad u_2(0, y) = 0
\]

\[
\frac{\partial}{\partial y} u_1(x, 0) = 0 \quad \frac{\partial}{\partial y} u_2(x, 0) = f(x)
\]

\[
\lim_{x \to \infty} u_1(x, y) = 0 \quad \lim_{x \to \infty} u_2(x, y) = 0
\]

\[
\lim_{y \to \infty} u_1(x, y) = 0 \quad \lim_{y \to \infty} u_2(x, y) = 0
\]

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The techniques for solving these problems are very similar, and we only indicate how to solve the problem for \( u_1(x, y) \). We consider the Fourier cosine transform in \( y \) since we have a homogeneous boundary condition \( \partial u_1 / \partial y(x, 0) = 0 \).

\[
\begin{align*}
  u_1(x, y) &= \int_0^\infty U_1(x, \omega) \cos \omega y \, d\omega \\
  U_1(x, \omega) &= \frac{2}{\pi} \int_0^\infty u_1(x, y) \cos \omega y \, dy
\end{align*}
\]

Taking the Fourier cosine transform in \( y \) of the Laplace’s equation for \( u_1 \) and using the homogeneous boundary condition \( \partial u_1 / \partial y(x, 0) = 0 \) we obtain the ordinary differential equation

\[
\frac{\partial^2 U_1}{\partial x^2} - \omega^2 U_1 = 0
\]

whose general solution is

\[
U_1(x, \omega) = a(\omega)e^{-\omega x} + b(\omega)e^{\omega x}, \quad x > 0, \omega > 0
\]

To determine \( a(\omega) \) and \( b(\omega) \) we use the boundary conditions.

\[
\lim_{x \to \infty} u_1(x, y) = 0 \Rightarrow \lim_{x \to \infty} U_1(x, \omega) = 0 \Rightarrow b(\omega) = 0
\]

Therefore,

\[
U_1(x, \omega) = a(\omega)e^{-\omega x}
\]

\[
U_1(0, \omega) = C[u_1(0, y)] = C[g(y)] \Rightarrow a(\omega) = \frac{2}{\pi} \int_0^\infty g(y) \cos \omega y \, dy
\]

The solution \( u_1(x, y) \) is obtained from (125) with \( U_1(x, \omega) \) given by (127)-(128).