Further applications of Bessel’s functions

1. Vibrations of a circularly symmetric membrane

Consider the vibrations of a circular membrane

\[ u_{tt} = c^2 \left( \frac{1}{r} ru_r + \frac{1}{r^2} u_{\theta\theta} \right), \quad 0 < r < a, \quad -\pi < \theta < \pi \]  

(1)

with zero boundary conditions

\[ u(a, \theta) = 0, \quad -\pi < \theta < \pi \]  

(2)

and radially symmetric initial conditions

\[ u(r, \theta, 0) = u_0(r); \quad u_t(r, \theta, 0) = v_0(r) \]  

(3)

The solution to the problem (1-3) is then radially symmetric, \( u(r, \theta, t) = u(r, t) \) such that we are looking for product solutions

\[ u(r, t) = h(t) \phi(r) \]

Replacing in (1) it results,

\[ \frac{1}{c^2} \frac{h''}{h} = \left( \frac{r\phi'}{r\phi} \right)' = -\lambda \]

for some constant \( \lambda > 0 \) (why?) such that we have

\[ h'' + \lambda c^2 h = 0 \Rightarrow h(t) = A \cos(\sqrt{\lambda} t) + B \sin(\sqrt{\lambda} t) \]

with \( A, B \) arbitrary constants, and

\[ (r\phi')' + \lambda r \phi = 0 \]  

(4)

\[ \phi(a) = 0 \]  

(5)

\[ |\phi(0)| < \infty \]  

(6)

Using the change of variable

\[ z = \sqrt{\lambda} r, \quad \Phi(z) = \phi(r) \]

the corresponding problem for \( \Phi(z) \) is written

\[ z^2 \Phi'' + z \Phi' + z^2 \Phi = 0 \]  

(7)

\[ \Phi(\sqrt{\lambda} a) = 0 \]  

(8)

\[ |\Phi(0)| < \infty \]  

(9)

Equation (7) is a Bessel’s equation of order zero, such that its general solution is expressed in terms of the Bessel’s functions of order zero,

\[ \Phi(z) = c_1 J_0(z) + c_2 Y_0(z) \]

Since we require \( \Phi \) to be bounded at the origin, \( c_2 = 0 \). Then

\[ \Phi(\sqrt{\lambda} a) = 0 \Rightarrow J_0(\sqrt{\lambda} a) = 0 \Rightarrow \lambda_n = \left( \frac{\mu_n^{(0)}}{a} \right)^2, \quad n = 1, 2, \ldots \]

where \( \mu_n^{(0)}, n = 1, 2, \ldots \) denote the zeros of the regular Bessel’s function \( J_0(z) \). Then \( \phi_n(r) = J_0(\sqrt{\lambda}_n r) \) and product solutions \( u(r, t) = h(t) \phi(r) \) are of the form

\[ u_n(r, t) = A_n \cos(\sqrt{\lambda}_n t) J_0(\sqrt{\lambda}_n r) + B_n \sin(\sqrt{\lambda}_n t) J_0(\sqrt{\lambda}_n r) \]
We seek for the solution to (1-3) as an infinite series

\[ u(r, t) = \sum_{n=1}^{\infty} A_n \cos(c\sqrt{\lambda_n}t)J_0(\sqrt{\lambda_n}r) + B_n \sin(c\sqrt{\lambda_n}t)J_0(\sqrt{\lambda_n}r) \]  

(10)

The coefficients \( a_n, b_n \) are obtained by imposing the boundary conditions (3):

\[ u(r, \theta, 0) = u_0(r) \Rightarrow \sum_{n=1}^{\infty} A_n J_0(\sqrt{\lambda_n}r) = u_0(r) \]

and using the orthogonality property

\[ \int_0^a r J_0(\sqrt{\lambda_n}r)J_0(\sqrt{\lambda_m}r) \, dr = 0 \]  

(11)

we get

\[ A_n = \frac{\int_0^a r u_0(r)J_0(\sqrt{\lambda_n}r) \, dr}{\int_0^a r J_0^2(\sqrt{\lambda_n}r) \, dr}, \quad n = 1, 2, \ldots \]  

(12)

The second initial condition is used to find the coefficients \( B_n \):

\[ u_t(r, \theta, 0) = v_0(r) \Rightarrow \sum_{n=1}^{\infty} B_n c\sqrt{\lambda_n}J_0(\sqrt{\lambda_n}r) = v_0(r) \Rightarrow B_n = \frac{1}{c\sqrt{\lambda_n}} \frac{\int_0^a r v_0(r)J_0(\sqrt{\lambda_n}r) \, dr}{\int_0^a r J_0^2(\sqrt{\lambda_n}r) \, dr}, \quad n = 1, 2, \ldots \]  

(13)

The solution to the problem (1-3) is thus expressed as the infinite series (10) with the coefficients given by (12-13).

2. Laplace’s equation in a cylinder

We consider the Laplace equation

\[ \nabla^2 u = 0 \]

in a cylinder of height \( H \) and radius \( a \). Introducing the cylindrical coordinates

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]
\[ z = z \]

the Laplace’s equation is written

\[ \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial u}{\partial r} \right] + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \]  

(14)

We assume that Dirichlet boundary conditions are prescribed on the top, bottom, and lateral surface of the cylinder:

\[ u(r, \theta, H) = \beta(r, \theta) \quad (\text{top}) \]  

(15)
\[ u(r, \theta, 0) = \alpha(r, \theta) \quad (\text{bottom}) \]  

(16)
\[ u(a, \theta, z) = \gamma(\theta, z) \quad (\text{lateral boundary}) \]  

(17)

To find the solution, we split the problem (14-17) into three subproblems,

\[ u = u_1 + u_2 + u_3 \]
where each of $u_1, u_2, u_3$ satisfies only one nonhomogeneous boundary condition

\[
\begin{align*}
    u_1(r, \theta, H) &= \beta(r, \theta) \\
    u_2(r, \theta, 0) &= \alpha(r, \theta) \\
    u_3(a, \theta, z) &= \gamma(\theta, z)
\end{align*}
\]

and takes zero values on the rest of the boundary. We used this approach before for the Laplace’s equation in a rectangle. We search for product solutions of the form

\[
    u(r, \theta, z) = f(r)g(\theta)h(z) \tag{18}
\]

such that, after replacing (18) into (14),

\[
    \left[ \frac{rf'}{f} \right]' + \frac{1}{r^2} \frac{g''}{g} + \frac{h''}{h} = 0 \tag{19}
\]

The $z$-variable may be first separated,

\[
    \frac{h''}{h} = -\left( \frac{rf'}{rf} + \frac{1}{r^2} \frac{g''}{g} \right) = \lambda
\]

then,

\[
    \frac{rf'}{r} + \lambda r^2 = -\frac{g''}{g} = \mu
\]

Corresponding to $g(\theta)$ function we impose periodic boundary conditions,

\[
    g(-\pi) = g(\pi), \quad g'(-\pi) = g'(+\pi)
\]

such that the eigenpairs $(\mu, g)$ are

\[
    \mu_m = m^2, \quad g_m(\theta) = c_1 \cos(m\theta) + c_2 \sin(m\theta), \quad m = 0, 1, \ldots
\]

For each of $u_1, u_2, u_3$ the differential equations

\[
\begin{align*}
    h'' &= \lambda h \tag{20} \\
    rf' + (\lambda r^2 - m^2)f &= 0 \tag{21}
\end{align*}
\]

must be solved with appropriate boundary conditions.

### Subproblems 1 and 2

Notice that from the mathematical point of view the subproblems for $u_1$ and $u_2$ are quite similar, just flip the cylinder upside down. Then is enough to study the problem for $u_1$ which involves the Bessel’s equation of order $m$:

\[
    rf' + (\lambda r^2 - m^2)f = 0 \tag{22}
\]

with boundary conditions

\[
\begin{align*}
    f(a) &= 0 \\
    |f(0)| &< \infty
\end{align*}
\]

From a previous analysis we know that $\lambda > 0$ and the eigenpairs $(\lambda, f)$ are

\[
    \lambda_{mn} = \left( \frac{\mu_m}{a} \right)^2, \quad f(r) = J_m(\sqrt{\lambda_{mn}} r), \quad n = 1, 2, \ldots \tag{23}
\]
where \( \mu_n^{(m)} \), \( n = 1, 2, \ldots \) denote the zeros of the Bessel’s function \( J_m \).

The differential equation for \( h \) is
\[
h'' = \lambda h
\]  
and since \( u_1 \) is zero on the bottom boundary \( z = 0 \), we impose \( h(0) = 0 \) such that, up to a multiplicative constant,
\[
h(z) = \sinh(\sqrt{\lambda} z)
\]
The solution \( u_1(r, \theta, z) \) is then expressed as a series
\[
u_1(r, \theta, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sinh(\sqrt{\lambda_{mn}} z) J_m(\sqrt{\lambda_{mn}} r) \cos(m\theta)
\]
where the coefficients \( A_{mn} \) are determined from the boundary condition
\[
u_1(r, \theta, H) = \beta(r, \theta)
\]

**Subproblem 3**

For the \( u_3 \)-subproblem the differential equation for \( h \) is
\[
h'' = \lambda h
\]  
with boundary conditions
\[
h(0) = 0, \quad h(H) = 0
\]
since \( u_3 \) takes zero values on the top and bottom boundaries of the cylinder. The eigenpairs \((\lambda, h)\) are then
\[
\lambda_n = -(n\pi/H)^2, \quad h_n(z) = \sin(n\pi z/H), \quad n = 1, 2, \ldots
\]  
such that solutions corresponding to both \( \theta \) and \( z \) variables have an oscillatory behavior (sine and cosine functions). With \( \lambda_n \) above, the differential equation for the \( r \)-dependent solution becomes
\[
r r' f'' + \left[ -(n\pi/H)^2 r^2 - m^2 \right] f = 0
\]  
to which we must impose
\[
|f(0)| < \infty
\]
but there is no homogeneous condition at \( r = a \). The change of variable
\[
w = \frac{n\pi}{H} r, \quad F(w) = f(r)
\]
may be used to transform (28) into a modified Bessel’s equation of order \( m \)
\[
w^2 F'' + w F' \left( -w^2 - m^2 \right) F = 0
\]  
which has a solution that is well defined at \( w = 0 \), the modified Bessel’s function of order \( m \) of first kind, \( I_m(w) \), and a solution that is singular at \( w = 0 \), the modified Bessel’s function of order \( m \) of second kind, \( K_m(w) \). Then
\[
f(r) = c_1 K_m \left( \frac{n\pi}{H} r \right) + c_2 I_m \left( \frac{n\pi}{H} r \right)
\]  
and \( |f(0)| < \infty \) implies \( c_1 = 0 \).

In conclusion, the solution \( u_3 \) is expressed as a double series
\[
\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} E_{mn} I_m \left( \frac{n\pi}{H} r \right) \sin \left( \frac{n\pi z}{H} \right) \cos(m\theta) + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} F_{mn} I_m \left( \frac{n\pi}{H} r \right) \sin \left( \frac{n\pi z}{H} \right) \sin(m\theta)
\]  
where the coefficients \( E_{mn}, F_{mn} \) are determined by imposing the boundary condition \( u_3(a, \theta, z) = \gamma(\theta, z) \).