Method of separation of variables

Linear operators

Definition: An operator L is linear if

$$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2) \tag{1}$$

for any functions (arguments) u_1 and u_2 and for any constants c_1 and c_2.
Example: The *heat operator*

$$\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2} \tag{2}$$

is linear.

PDE's may be of various types: linear, nonlinear, homogeneous, or nonhomogeneous. See Lecture 1 notes for a definition.
In general, a linear equation is an equation of the form

$$L(u) = f \tag{3}$$

where L is a linear operator. A *linear homogeneous* equation is an equation of the form

$$L(u) = 0 \tag{4}$$

The trivial function $u \equiv 0$ is always a solution of the homogeneous equation (4). Why?

Principle of Superposition: if u_1 and u_2 are two solutions of equation (4), then any linear combination $u = c_1u_1 + c_2u_2$ is also a solution of (4).

Definition: A scalar (real or complex number) λ is called an *eigenvalue* of the linear operator L if there exists a non-trivial (not identically zero) function v such that $Lv = \lambda v$. If such function exists, then v is called the *eigenfunction* corresponding to the eigenvalue λ.

Heat equation with zero temperatures at finite ends

Consider the 1-D heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},\ 0 < x < L,\ t > 0 \tag{5}$$

with prescribed zero boundary conditions

$$u(0,t) = 0 \tag{6}$$

$$u(L,t) = 0 \tag{7}$$

and the initial condition

$$u(x,0) = f(x) \tag{8}$$

Separation of variables

In the method of *separation of variables* we seek a solution of the form

$$u(x,t) = X(x)T(t) \tag{9}$$

If we are able to find such a solution, the problem is solved since from the previous lecture (see also Lecture 1 notes) we know that problem (5-8) has at most one solution.

Replacing (9) in (5) it results

$$X(x)\frac{dT}{dt} = k\frac{d^2X}{dx^2}T(t) \tag{10}$$

Therefore,

$$\underbrace{\frac{1}{kT}\frac{dT}{dt}}_{\text{function of } t \text{ only}} \quad = \quad \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x \text{ only}} \quad = -\lambda \tag{11}$$

where λ is an arbitrary constant. Thus, we have obtained two ordinary differential equations

$$\frac{d^2X}{dx^2} = -\lambda X \tag{12}$$

$$\frac{dT}{dt} = -\lambda k T \tag{13}$$

In addition, we require that the homogeneous boundary conditions (6) and (7) must be satisfied by $u(x,t) = X(x)T(t)$. We have

$$u(0,t) = 0 \Rightarrow X(0)T(t) = 0 \tag{14}$$

$$u(L,t) = 0 \Rightarrow X(L)T(t) = 0 \tag{15}$$

Since we are looking for nontrivial solutions, $T(t)$ can not be identically zero. It follows then from (14) and (15) that we must have

$$\begin{aligned} X(0) &= 0 && (16) \\ X(L) &= 0 && (17) \end{aligned}$$

The general solution of equation (13) is

$$T(t) = ce^{-\lambda kt} \tag{18}$$

where c is an arbitrary constant.

Q: Justify why we must have $\lambda > 0$.

Next we consider the *boundary value problem*

$$\begin{aligned} \frac{d^2X}{dx^2} &= -\lambda X && (19) \\ X(0) &= 0 && (20) \\ X(L) &= 0 && (21) \end{aligned}$$

Eigenvalues and eigenfunctions ($\lambda > 0$)

When $\lambda > 0$, the general solution of equation (19) is (why?)

$$X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x) \tag{22}$$

Using (22) and the boundary conditions (20)-(21) we obtain

$$X(0) = 0 \Rightarrow c_1 = 0 \tag{23}$$

$$X(L) = 0 \Rightarrow c_2 \sin(\sqrt{\lambda}L) = 0 \tag{24}$$

Since we seek a nontrivial solution, the eigenvalues λ must satisfy the equation

$$\sin(\sqrt{\lambda}L) = 0 \tag{25}$$

which has the solutions

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots \tag{26}$$

The corresponding eigenfunctions are

$$X(x) = c_2 \sin(\sqrt{\lambda}L) = c_2 \sin\left(\frac{n\pi x}{L}\right) \tag{27}$$

Q: Show that there are no eigenvalues $\lambda \leq 0$.

From (9), (18), and (27) we obtain that each of

$$u(x,t) = B \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}, \; n = 1, 2, \ldots \tag{28}$$

is a solution of the homogeneous problem (5),(6), (7). Next we want to impose the initial condition (8).

Principle of Superposition

We know that for linear homogeneous problems, any linear combination of two solutions is also a solution.

Q: Show that any finite linear combination of solutions is also a solution. That is, for any fixed number M, if $u_1, u_2, \ldots, u_M$ are solutions, then for any constants $c_1, c_2, \ldots, c_M$

$$c_1 u_1 + c_2 u_2 + \ldots + c_M u_M \tag{29}$$

is also a solution.

Therefore, for any finite M

$$u(x,t) = \sum_{n=1}^{M} B_n \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t} \tag{30}$$

is a solution of the heat equation with homogeneous boundary conditions.

We also claim that the infinite series

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t} \tag{31}$$

is a solution of the heat equation with homogeneous boundary conditions.

Fourier sine series of f

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \tag{32}$$

Next we use the initial condition (8) and relations (31) and (32) to determine the coefficients B_n, $n = 1, 2, \ldots$.

Q: Show that if m and n are arbitrary positive integers, then

$$\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} dx = 0 \;\; \text{if } m \neq n \tag{33}$$

$$\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} dx = L/2 \;\; \text{if } m = n \tag{34}$$

Q: Using (33)-(34), show that the *Fourier series coefficients* of f are given by

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L} dx \tag{35}$$

The solution of the heat equation (5) with homogeneous boundary values (6), (7) and initial condition (8) is then given by expression (31) where the coefficients B_n are evaluated using (35).

Heat conduction in a rod with insulated ends

We consider the 1-D heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \;\; 0 < x < L, \; t > 0 \tag{36}$$

with insulated boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0, \;\; t > 0 \tag{37}$$

$$\frac{\partial u}{\partial x}(L,t) = 0, \;\; t > 0 \tag{38}$$

and the initial condition

$$u(x,0) = f(x), \;\; 0 \le x \le L \tag{39}$$

We apply the method of separation of variables and seek a solution of the product form

$$u(x,t) = \Phi(x)G(t) \tag{40}$$

After replacing (40) in (36) we obtain two ODEs

$$\frac{dG}{dt} = -\lambda k G \tag{41}$$

$$\frac{d^2\Phi}{dx^2} = -\lambda\Phi \tag{42}$$

where λ is the separation constant. The general solution of the equation (41) is

$$G(t) = ce^{-\lambda k t} \tag{43}$$

where c is an arbitrary constant. Next we use the boundary conditions (37), (38) to obtain

$$\frac{d\Phi}{dx}(0) = 0 \tag{44}$$

$$\frac{d\Phi}{dx}(L) = 0 \tag{45}$$

Therefore, Φ must satisfy the boundary value problem (42), (44), (45). Since we have a linear homogeneous problem, the trivial function $\Phi \equiv 0$ is a solution of this problem. But we are interested in solutions that are not identically zero. To write the general solution of the equation (42) we must consider three cases: $\lambda > 0, \lambda = 0, \lambda < 0$. From (43), intuition tells us that $\lambda < 0$ is not good. But this is just intuition, not a proof. Rigorously,

Q: Show that there are no negative eigenvalues for (12) subject to (44), (45). See also problem 2.4.4. in the book.

First we consider the case when $\lambda > 0$. In this case the general solution of equation (42) is

$$\Phi(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x \tag{46}$$

To impose the boundary conditions, we evaluate

$$\frac{d\Phi}{dx} = \sqrt{\lambda}(-c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x) \tag{47}$$

at $x = 0$ and $x = L$. We obtain

$$\frac{d\Phi}{dx}(0) = 0 \Rightarrow \sqrt{\lambda}c_2 = 0 \Rightarrow c_2 = 0 \ \text{(since } \lambda > 0) \tag{48}$$

$$\frac{d\Phi}{dx}(L) = 0 \Rightarrow c_1\sqrt{\lambda}\sin\sqrt{\lambda}L = 0 \tag{49}$$

Since we are looking for nontrivial solutions, we must have $c_1 \neq 0$, such that from (49) we obtain

$$\sin\sqrt{\lambda}L = 0 \tag{50}$$

whose general solution is

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \;\; n = 1, 2, 3, \dots \tag{51}$$

These are the eigenvalues for $\lambda > 0$. The corresponding eigenfunctions are

$$\Phi(x) = c_1 \cos\frac{n\pi x}{L}, \;\; n = 1, 2, 3, \dots \tag{52}$$

Replacing (51), (52), and (43) in (40) we obtain the product solutions of the PDE with homogeneous boundary conditions

$$u(x,t) = A\cos\frac{n\pi x}{L}e^{-(n\pi/L)^2kt}, \;\; n = 1, 2, 3, \dots \tag{53}$$

Second, we consider the case when $\lambda = 0$. In this case, from (42) we obtain

$$\Phi = c_1 + c_2 x \tag{54}$$

and using the boundary conditions (44), (45) we obtain

$$\Phi(x) = c_1 \tag{55}$$

To satisfy the initial condition, we use (55), (53) and the principle of superposition, to obtain

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}e^{-(n\pi/L)^2kt} \tag{56}$$

The initial condition $u(x,0) = f(x)$ is satisfied if

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \tag{57}$$

To determine the coefficients A_i we use the *orthogonality relation*

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \neq 0 \\ L & n = m = 0 \end{cases} \tag{58}$$

which is true for any nonnegative integers n, m.

Q: Prove that property (58) is true for any nonnegative integers n, m.

Using property (58) we obtain from (57)

$$A_0 = \frac{1}{L}\int_0^L f(x)dx \tag{59}$$

$$A_m = \frac{2}{L}\int_0^L f(x)\cos\frac{m\pi x}{L}dx, \;\; m = 1, 2, 3, \dots \tag{60}$$

Therefore, the solution of the problem (36-39) is (56) where the coefficients are evaluated using (59) and (60).