HW #4 - due 11/21/2017

The goals of this exercise are to

- Illustrate the sensitivity of the solutions obtained through Tikhonov regularization to small variations in data
- Illustrate how the equation of the observation sensitivity may be used to characterize the uncertainty in the regularized solution.

Mathematical Background and Setup

Consider the linear system of equations

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{1}$$

where $\mathbf{b} \in \mathbb{R}^n$ is a given vector (observed/received data), $\mathbf{x} \in \mathbb{R}^n$ is an unknown vector (transmitted data), and $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a given nonsingular matrix (transformation of data). The SVD of the matrix $\mathbf{A}$ is

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}} \in \mathbb{R}^{n\times n} \tag{2}$$

Consider the noisy data vector

$$\widehat{\mathbf{b}} = \mathbf{b} + \boldsymbol{\xi} \tag{3}$$

where $\boldsymbol{\xi}$ denotes a noise vector. The Tikhonov regularization solution $\widehat{\mathbf{x}}_\lambda$ obtained in the standard formulation is expressed as

$$\widehat{\mathbf{x}}_\lambda = \sum_{i=1}^{n} f_i(\lambda) \frac{\mathbf{u}_i^{\mathrm{T}}\widehat{\mathbf{b}}}{\sigma_i}\mathbf{v}_i \tag{4}$$

where the filter factors $f_i(\lambda)$ are defined as

$$f_i(\lambda) = \frac{\sigma_i^2}{\sigma_i^2 + \lambda^2},\ \ i = 1, 2, \ldots n \tag{5}$$

An appropriate value of the regularization parameter λ is obtained by means of the L-curve method.

Henceforth the value of the parameter λ is fixed, as determined above. We are further interested to investigate an ensemble (a series) of problems with different specifications of the noise $\boldsymbol{\xi}$ added to the exact data $\mathbf{b}$. The size of the ensemble is denoted M (number of problems/samples considered).

Assume that the components of the noise vector are independent random variables drawn from a Gaussian (normal) distribution with mean 0 and standard deviation σ_o, $\xi_i \sim N(0, \sigma_o^2), i = 1 : n$. For each noise realization (sample) $\boldsymbol{\xi}^{(j)}, j = 1 : M$, a data sample is obtained as

$$\widehat{\mathbf{b}}^{(j)} = \mathbf{d} + \boldsymbol{\xi}^{(j)}, \quad j = 1 : M$$

and regularized solutions $\widehat{\mathbf{x}}_\lambda^{(j)}$ are obtained from (4)-(5) using the same (fixed) regularization parameter λ.

The ensemble mean is obtained as

$$\bar{\mathbf{x}}_\lambda = \frac{1}{M}\sum_{j=1}^{M}\widehat{\mathbf{x}}_\lambda^{(j)} \tag{6}$$

and the ensemble variance is expressed componentwise as

$$\bar{\Sigma}^2(i) = \frac{1}{M-1}\sum_{j=1}^{M}\left(\widehat{\mathbf{x}}_\lambda^{(j)}(i) - \bar{\mathbf{x}}_\lambda(i)\right)^2, \quad i = 1:n \tag{7}$$

Homework requirements: Consider the *shaw* problem in the Regu package, with $n = 100$,

$$[A, b, x] = shaw(n)$$

and generate a noisy version of data using random samples from a Gaussian distribution with mean 0 and standard deviation $\sigma_o = 10^{-3}$,

$$\hat{b} = b + \sigma_o * randn(n, 1)$$

(10 pts) Find an appropriate value of the regularization parameter λ using the L-curve guidance.

(20 pts) For an ensemble of size $M = 100$, evaluate the ensemble mean (6) and variance (7).

(10 pts) Provide a plot of the true solution $\mathbf{x}$ and the ensemble mean $\bar{\mathbf{x}}_\lambda$.

(10 pts) Provide a plot of the ensemble of solutions $\{\widehat{\mathbf{x}}_\lambda^{(j)}\}_{j=1:M}$, the ensemble mean $\bar{\mathbf{x}}_\lambda$, together with plots of $\bar{\mathbf{x}}_\lambda - 3\bar{\Sigma}$ and $\bar{\mathbf{x}}_\lambda + 3\bar{\Sigma}$ (99.7% probability interval).

(20/10 pts) Consider the vector $\Sigma \in \mathbb{R}^n$ with entries defined as

$$\Sigma(j) = \sigma_o\left(\sum_{i=1}^{n}\left(\frac{f_i}{\sigma_i}\right)^2 \mathbf{v}_i^2(j)\right)^{\frac{1}{2}}, \quad j = 1:n \tag{8}$$

Provide a plot of the ensemble of solutions $\{\widehat{\mathbf{x}}_\lambda^{(j)}\}_{j=1:M}$, the ensemble mean $\bar{\mathbf{x}}_\lambda$, together with plots of $\bar{\mathbf{x}}_\lambda - 3\Sigma$ and $\bar{\mathbf{x}}_\lambda + 3\Sigma$

MTH 510 students only[1]

(10 pts) Use (4)-(5) to derive the equation of the sensitivity of the regularized solution to observations

$$\nabla_{\widehat{\mathbf{b}}}\widehat{\mathbf{x}}_\lambda = \left(\frac{\partial \widehat{\mathbf{x}}_\lambda}{\partial \widehat{\mathbf{b}}}\right)^{\mathrm{T}} \in \mathbb{R}^{n\times n}$$

and explain its relationship to (8).

[1] Mth 410 students get bonus points for this question