HW #3 - due 11/16/2017

The goals of this exercise are to

- Illustrate that regularization techniques using a smoothing 2-norm are not best suited for problems where data has sharp gradients or discontinuities.
- Illustrate the potential benefits of the Total Variation (TV) regularization.

Mathematical Background and Setup

Let $\mathbf{A}$ denote the blurring operator defined in the context of HW #2 with $n = 220$ and consider the data $\mathbf{x}$ provided in the file TrueData.m (this is column 100 of the 220×220 image Datamatrix.png). The noisy burred data provided in the file BlurData.m was obtained as

$$\widehat{\mathbf{d}} = \mathbf{A}\mathbf{x} + \boldsymbol{\xi} \tag{1}$$

where $\boldsymbol{\xi}$ denotes a vector of random noise. The performance of the TSVD, Tikhonov, and TV methods in reconstructing the true vector $\mathbf{x}$ is tested as follows.

Consider the solution $\widehat{\mathbf{x}}_k$ provided by the TSVD

$$\widehat{\mathbf{x}}_k = \sum_{i=1}^{k} \frac{\mathbf{u}_i^{\mathrm{T}}\widehat{\mathbf{d}}}{\sigma_i}\mathbf{v}_i \tag{2}$$

and the solution $\widehat{\mathbf{x}}_{\lambda,L}$ provided by the Tikhonov regularization ($\mathbf{x}_0 = \mathbf{0}$)

$$\left(\mathbf{A}^{\mathrm{T}}\mathbf{A} + \lambda^2\mathbf{L}^{\mathrm{T}}\mathbf{L}\right)\widehat{\mathbf{x}}_{\lambda,L} = \mathbf{A}^{\mathrm{T}}\widehat{\mathbf{d}} \tag{3}$$

Consider the total variation solution $\widehat{\mathbf{x}}_{\alpha,\beta}$ obtained by solving the minimization problem

$$\min_{\widehat{\mathbf{x}}\in\mathbb{R}^n} J_{\alpha,\beta}(\widehat{\mathbf{x}}), \quad \mathit{where} \;\; J(\widehat{\mathbf{x}}) \overset{def}{=} \|\mathbf{A}\widehat{\mathbf{x}} - \widehat{\mathbf{d}}\|^2 + \alpha^2 T(\widehat{\mathbf{x}},\beta) \tag{4}$$

and $T(\widehat{\mathbf{x}},\beta)$ is a smooth approximation to the 1-norm $\|\mathbf{L}_1\widehat{\mathbf{x}}\|_1$ defined as

$$T(\widehat{\mathbf{x}},\beta) = \sum_{i=1}^{n-1} \sqrt{\beta^2 + |\hat{x}_{i+1} - \hat{x}_i|^2} \tag{5}$$

Homework requirements:

(30/10 pts) [1] Implement the TSVD (2) and the Tikhonov regularization (3) with $\mathbf{L} = \mathbf{I}$ to reconstruct the true data $\mathbf{x}$. Provide the graphs of the reconstructed data $\widehat{\mathbf{x}}$ and the error in the approximation, $\widehat{\mathbf{x}} - \mathbf{x}$.

(40 pts) Further experiment with the $\mathbf{L}$ operator taken as $\mathbf{L}_1$ and $\mathbf{L}_2$. Provide the graphs of the reconstructed data $\widehat{\mathbf{x}}$ and the error in the approximation, $\widehat{\mathbf{x}} - \mathbf{x}$.

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(20 pts) Implement the TV method (4)-(5). Find appropriate values for the parameters α, β and provide the graphs of the reconstructed data $\widehat{\mathbf{x}}_{\alpha,\beta}$ and the error in the approximation, $\widehat{\mathbf{x}}_{\alpha,\beta} - \mathbf{x}$.

[1] Mth 510 students get 10 points for this question

[2] Mth 410 students get bonus points for this question