

HW #2 - due 10/31/2017

The goal of this homework is to illustrate applications of the singular value decomposition (SVD) and Tikhonov regularization to image deblurring.

Mathematical Background:

Consider a linear system of equations

$$\mathbf{A}\mathbf{x} = \mathbf{d} \quad (1)$$

where $\mathbf{d} \in \mathbb{R}^n$ is a given vector (observed/received data), $\mathbf{x} \in \mathbb{R}^n$ is an unknown vector (transmitted data), and $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a given nonsingular matrix (transformation of data).

In many applications the observed/received data is corrupted by unknown noise or measurement errors, such that we only have access to

$$\hat{\mathbf{d}} = \mathbf{d} + \boldsymbol{\xi}$$

where $\boldsymbol{\xi} \in \mathbb{R}^n$ is an unknown noise vector. The problem is then formulated as follows:

Given the matrix \mathbf{A} and a vector $\hat{\mathbf{d}}$, provide an approximation of the unknown vector \mathbf{x} .

Practical difficulties arise when \mathbf{A} is ill-conditioned. In such case, an attempt to provide an approximation $\hat{\mathbf{x}} \approx \mathbf{x}$ by solving

$$\mathbf{A}\hat{\mathbf{x}} = \hat{\mathbf{d}} \quad (2)$$

will not work since $\hat{\mathbf{x}}$ will be drastically corrupted by noise. Regularization techniques are required to address this issue.

Truncated SVD. Consider the singular value decomposition of the matrix A

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T \in \mathbb{R}^{n \times n} \quad (3)$$

where $\mathbf{U} = [\mathbf{u}_1 \mathbf{u}_2 \dots \mathbf{u}_n]$, $\mathbf{S} = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$, $\mathbf{V} = [\mathbf{v}_1 \mathbf{v}_2 \dots \mathbf{v}_n]$. The solution to the system (2) is expressed as

$$\hat{\mathbf{x}} = [\mathbf{U}\mathbf{S}\mathbf{V}^T]^{-1}\hat{\mathbf{d}} = \mathbf{V}\mathbf{S}^{-1}\mathbf{U}^T\hat{\mathbf{d}} = \sum_{i=1}^n \frac{\mathbf{u}_i^T \hat{\mathbf{d}}}{\sigma_i} \mathbf{v}_i \quad (4)$$

For very small σ_i , the terms $(\mathbf{u}_i^T \hat{\mathbf{d}})/\sigma_i$ will result in noise amplification.

In the *truncated SVD* regularization, an approximation $\hat{\mathbf{x}}_k$ to the solution is obtained by truncating the sum in (4) at a selected index k ,

$$\hat{\mathbf{x}}_k = \sum_{i=1}^k \frac{\mathbf{u}_i^T \hat{\mathbf{d}}}{\sigma_i} \mathbf{v}_i \quad (5)$$

The main difficulty is to find an appropriate truncation index k such that $\hat{\mathbf{x}}_k$ is a good approximation to the true value \mathbf{x} .

Tikhonov Regularization. In this approach, an approximation to \mathbf{x} is obtained as the solution to the least squares minimization problem:

$$\min_{\hat{\mathbf{x}} \in \mathbb{R}^n} J(\hat{\mathbf{x}}), \quad \text{where } J(\hat{\mathbf{x}}) \stackrel{\text{def}}{=} \|\mathbf{A}\hat{\mathbf{x}} - \hat{\mathbf{d}}\|^2 + \lambda^2 \|\hat{\mathbf{x}}\|^2 \quad (6)$$

In (6), $\|\cdot\|$ denotes the Euclidean vector norm and λ is a scalar *regularization parameter* that controls the *smoothness* of the solution. If $\lambda = 0$ then no regularization is applied and the solution is (4). If λ is large, then $\lambda^2 \|\hat{\mathbf{x}}\|^2$ has a significant contribution to the cost (6) and the solution $\hat{\mathbf{x}}$ can not be a good approximation of \mathbf{x} . We view the solution to (6) as a function of the parameter λ and denote it $\hat{\mathbf{x}}_\lambda$. The main difficulty is to find an appropriate value of the regularization parameter λ such that $\hat{\mathbf{x}}_\lambda$ is a good approximation to the true value \mathbf{x} .

The solution $\hat{\mathbf{x}}_\lambda$ to the minimization problem (6) is obtained by solving the linear system

$$(\mathbf{A}^T \mathbf{A} + \lambda^2 \mathbf{I}) \hat{\mathbf{x}}_\lambda = \mathbf{A}^T \hat{\mathbf{d}} \quad (7)$$

and may be expressed using the SVD (3) of \mathbf{A} as

$$\hat{\mathbf{x}}_\lambda = \sum_{i=1}^n f_i(\lambda) \frac{\mathbf{u}_i^T \hat{\mathbf{d}}}{\sigma_i} \mathbf{v}_i \quad (8)$$

where the filter factors $f_i(\lambda)$ are defined as

$$f_i(\lambda) = \frac{\sigma_i^2}{\sigma_i^2 + \lambda^2}, \quad i = 1, 2, \dots, n \quad (9)$$

The main difficulty in this approach is to select an appropriate value of the *regularization parameter* λ .

Homework content

Consider an image represented as a matrix $\mathbf{X} \in \mathbb{R}^{n \times m}$ and a blurring process represented by the matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$. Whereas the true image is the solution to the matrix equation

$$\mathbf{A}\mathbf{X} = \mathbf{D}$$

we want to reconstruct an approximation $\hat{\mathbf{X}}$ to the image \mathbf{X} given the blurring matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ and a matrix of received noisy data

$$\hat{\mathbf{D}} = \mathbf{A}\mathbf{X} + \boldsymbol{\xi}$$

The regularization procedure (5) or (8-9) is used to find an approximate solution $\hat{\mathbf{X}}$, one column at a time:

ALGORITHM

$$[U, S, V] = \text{svd}(A)$$

FOR $i = 1 : m$

$$\hat{d} = \hat{D}(:, i)$$

evaluate \hat{x}_i from (5) - TSVD solution *or* from (8-9) - Tikhonov solution

$$\hat{X} = [\hat{x}_1 \hat{x}_2 \dots \hat{x}_m]$$

For this assignment $n = m = 128$, and X is a 128×128 matrix representing the image of a coin. The blurring operation is represented as follows: Consider a 128×128 symmetric tridiagonal matrix B with entries ¹

$$B(i, i) = 1 - 2L, \quad i = 1, 2, \dots, n$$

$$B(i, i + 1) = L, \quad i = 1, 2, \dots, n - 1; \quad B(i + 1, i) = L, \quad i = 1, 2, \dots, n - 1$$

where $L = 0.45$. Then the blurring operator is

$$\mathbf{A} = \mathbf{B}^{10}$$

The file "hw2data.m" represents the 128×128 noisy data matrix $\hat{\mathbf{D}}$, obtained as

$$\hat{\mathbf{D}} = \mathbf{A}\mathbf{X} + \boldsymbol{\xi}$$

where $\boldsymbol{\xi}$ is an unknown 128×128 noise matrix. To visualize $\hat{\mathbf{D}}$, in MATLAB you may execute:

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load hw2data.m; D = hw2data; imagesc(D); colormap(gray)
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Jour job: Implement the TSVD and the Tikhonov regularization procedures to reconstruct an approximation $\hat{\mathbf{X}}$ of the unknown image \mathbf{X} such that the year and the inscriptions on the coin can be read.

- (50 points) Implement a function $\hat{X} = \text{regularize}(A, \hat{D}, \text{method}, p)$ that takes as input the noisy data matrix \hat{D} , the blurring matrix A and the regularization parameter p , and returns \hat{X} , the regularization approximation to X . The input value *method* is used to distinguish between the TSVD solution (5) in which case p is the truncation index and the Tikhonov solution (8-9) in which case p is the regularization parameter λ . Alternatively, you may write a function for each method *tsvd* and *tikhonov*, respectively.
- (20 points) Find values of k and λ such that from $\hat{\mathbf{X}}$ the year and the inscriptions on the coin can be read. Plot the *l - curve*.

Things to hand in: listing of the *tsvd* and *tikhonov* functions, value of index k (tsvd) and parameter λ (Tikhonov), reconstructed image: year, inscriptions; a plot of the singular values σ_i on a \log_{10} scale and the filter factors f_i ; a plot of the *l - curve*.

¹such matrix results from discretization of the 1-D heat equation $u_t - ku_{xx} = 0$ with $L = k\Delta t/(\Delta x)^2$.