Mth 410/510 Inverse Problems & DA — Week 6

TSVD and Tikhonov rely on the 2-norm to perform regularization of ill-posed problems. In practice these methods work well for smooth and continuous data. However, reconstruction of images with sharp gradients (edges) or discontinuities is more difficult (see barcode example).

Total variation regularization relies on the 1-norm of the first derivative,

$$\|f'\|_1 = \int_0^1 |f'(t)|\,dt .$$

In the discrete version, TV is

$$\min_{x \in \mathbb{R}^n} \left\{ \|Ax-b\|_2^2 + \alpha^2 \|L_1 x\|_1 \right\}$$

where

$$L_1 = \begin{pmatrix} -1 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & -1 & 1 \end{pmatrix} \in \mathbb{R}^{(n-1)\times n}$$

Notice that $L_1 x = \begin{bmatrix} x_2 - x_1 \\ \vdots \\ x_n - x_{n-1} \end{bmatrix} \in \mathbb{R}^{n-1}$

is the same operator as L_1 defined previously, but

$$\|L_1 x\|_1 = \sum_{i=1}^{n-1} |x_{i+1} - x_i|$$

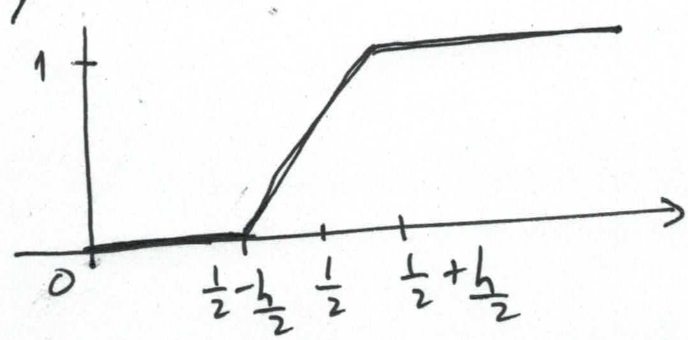
Notice that the functional

$$J(x) = \|Ax - b\|_2^2 + \alpha^2 \|L_1 x\|_1$$

is not quadratic and not differentiable.

To illustrate the difference between the smoothing norms $\|L_1 x\|_2^2$ and $\|L_1 x\|_1$ consider the function $f : [0,1] \to \mathbb{R}$

$f(t)$ piecewise linear



$f(t) = 0$ in $(0, \frac{1}{2} - \frac{h}{2})$

$f(t) = 1$ in $(\frac{1}{2} + \frac{h}{2}, 1)$

$f(t) = \frac{t}{h} - \frac{1-h}{2h}$ in $(\frac{1-h}{2}, \frac{1+h}{2})$

Then $f'(t) = \begin{cases} 0, & \text{in } (0, \frac{1}{2}-h) \\ \frac{1}{h}, & \text{in } (\frac{1-h}{2}, \frac{1+h}{2}) \\ 0, & \text{in } (\frac{1+h}{2}, 1) \end{cases}$

$$\|f'\|_1 = \int_0^1 |f'(t)|\,dt = 1$$

$$\|f'\|_2^2 = \int_0^1 |f'(t)|^2\,dt = \frac{1}{h^2}\cdot h = \frac{1}{h}$$

→ regularization based on the 2-norm penalizes steep gradients (small h) whereas the 1-norm is independent of the slope h.

Notice that $J(x)$ is a convex functional, however it is not differentiable everywhere w.r.t. x. Therefore, an explicit form of the solution is not available, iterative procedures are required.

To avoid the issue of nondifferentiability, a small positive constant β^2 may be added to the regularization term and the TV problem is formulated as

$$\min_{x \in \mathbb{R}^n} J_{\alpha,\beta}(x)$$

where $J_{\alpha,\beta}(x) = \|Ax-b\|_2^2 + \alpha^2 \sum_{i=1}^{n-1} \sqrt{\beta^2 + (x_{i+1}-x_i)^2}$

$\alpha > 0$ and $\beta > 0$ are parameters

The $J_{\alpha,\beta}(x)$ is continuously differentiable and the first order optimality system is $\nabla_x J_{\alpha,\beta}(x) = 0 \quad (*)$

Notice that the gradient of

$$T(x) = \sum_{i=1}^{n-1} \sqrt{\beta^2 + (x_{i+1}-x_i)^2}$$

is evaluated as : for $i = 2 : n-1$

$$\frac{\partial T}{\partial x_i} = \frac{x_i - x_{i-1}}{\sqrt{\beta^2 + (x_i - x_{i-1})^2}} + \frac{x_i - x_{i+1}}{\sqrt{\beta^2 + (x_{i+1} - x_i)^2}}$$

and

$$\frac{\partial T}{\partial x_1} = \frac{x_1 - x_2}{\sqrt{\beta^2 + (x_2 - x_1)^2}}$$

$$\frac{\partial T}{\partial x_n} = \frac{x_n - x_{n-1}}{\sqrt{\beta^2 + (x_n - x_{n-1})^2}}$$

such that (*) is written

$$\boxed{2A^T(Ax - b) + \alpha^2 \nabla_x T(x) = 0}$$

which is a non-linear system of equations.