Mth 410/510 inverse Problems & DA week 5

Generalized singular value decomposition (GSVD)

in the general form of the Tikhonov regularization, the solution $x_{\lambda,L}$ is obtained by minimizing

$$J_{\lambda,L}(x) = \|Ax-b\|_2^2 + \lambda^2 \|L(x-x_0)\|_2^2$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $L \in \mathbb{R}^{p\times n}$

$x_0 \in \mathbb{R}^n$ is a prior estimate / guess.

We assume $m \geq n \geq p$ and

$$\mathcal{N}(A) \cap \mathcal{N}(L) = \{0\}$$

Then there is a unique solution $x_{\lambda,L}$ obtained by solving the system

$$(A^TA + \lambda^2 L^TL)\,x = A^Tb + \lambda^2 L^TLx_0$$

For simplicity, we may assume $x_0 = 0$

The solution $x_{\lambda,L}$ may be expressed as a filtered solution and analyzed using GSVD.

<u>Definition</u> Let $A \in \mathbb{R}^{m \times n}$, $L \in \mathbb{R}^{p \times n}$ such that $m \geq n \geq p$. Assume that $\operatorname{rank}(L) = p$ and $\mathcal{N}(A) \cap \mathcal{N}(L) = \{0\}$.

The generalized singular value decomposition (GSVD) of (A, L) is a decomposition in the form

$$A = U \begin{pmatrix} S & 0 \\ 0 & I_{n-p} \end{pmatrix} X^{-1}$$

$$L = V \begin{pmatrix} M & 0 \end{pmatrix} X^{-1}$$

where $U \in \mathbb{R}^{m \times n}$ and $V \in \mathbb{R}^{p \times p}$ are orthonormal matrices

$$U^T U = I_n, \quad V^T V = I_p$$

$X \in \mathbb{R}^{n \times n}$ is a nonsingular matrix

$S \in \mathbb{R}^{p \times p}$ and $M \in \mathbb{R}^{p \times p}$ are diagonal matrices

$S = \mathrm{diag}(\sigma_1, \ldots, \sigma_p)$

$M = \mathrm{diag}(\mu_1, \ldots, \mu_p)$

where

$$0 \le \sigma_1 \le \ldots \le \sigma_p \le 1$$

$$1 \ge \mu_1 \ge \ldots \ge \mu_p > 0$$

satisfy $\sigma_i^2 + \mu_i^2 = 1, \quad i = 1:p$

The <u>generalized singular values</u> γ_i of the pair (A, L) are defined as

$$\gamma_i = \frac{\sigma_i}{\mu_i}, \quad i = 1:p$$

<u>Remark</u>: Notice that γ_i are ordered such that

$$0 \le \gamma_1 \le \ldots \le \gamma_p$$

Denoting
$$U = [u_1 \cdots\cdots u_n]$$
$$V = [v_1 \cdots\cdots v_p]$$
$$X = [x_1 \cdots\cdots x_n]$$
we have the following properties :

1) $A x_i = \sigma_i u_i \,, \quad L x_i = \mu_i v_i \,, \quad i = 1:p$

$A x_i = u_i \,, \quad L x_i = 0 \,, \quad i = p+1:n$

in compact form,
$$X^T A^T A X = \begin{pmatrix} S^2 & 0 \\ 0 & I_{n-p} \end{pmatrix}$$
$$X^T L^T L X = \begin{pmatrix} M^2 & 0 \\ 0 & 0 \end{pmatrix}$$

2) $$A^T A x_i = \gamma_i^2 L^T L x_i \,, \quad i = 1:p$$
such that (γ_i^2, x_i) are generalized eigenpairs to $(A^T A, L^T L)$.

in particular, $(\gamma_p, \frac{1}{\mu_p} x_p)$ satisfy

$$\gamma_p = \max_{\|Lx\|=1} \|Ax\| = \frac{1}{\mu_p} \|Ax_p\|$$

when L is the identity matrix, $L = I_n$

then U and V in the GSVD are identical with U and V in svd(A), whereas the generalized singular values of (A, I_n) are the singular values of A in reverse order.

For a general L matrix there is no connection between the generalized singular values/vectors and the ordinary singular values/vectors

<u>Property</u> Assuming $x_0 = 0$, the solution $x_{\lambda,L}$ to the general form of the Tikhonov regularization is expressed as

$$x_{\lambda,L} = \sum_{i=1}^{p} f_i(\lambda, L) \frac{u_i^T \cdot b}{\sigma_i} x_i + \sum_{i=p+1}^{n} (u_i^T \cdot b)\, x_i$$

where $f_i(\lambda, L)$ are the general-form Tikhonov filter factors given by

$$f_i(\lambda, L) = \frac{\sigma_i^2}{\sigma_i^2 + \lambda^2 \mu_i^2}, \quad i = 1:p$$

or equivalently, $f_i(\lambda, L) = \dfrac{\gamma_i^2}{\gamma_i^2 + \lambda^2}$

where γ_i are the generalized singular values.

Notice that $f_i(\lambda, L) \approx \begin{cases} 1, & \text{if } \gamma_i \gg \lambda \\ 0, & \text{if } \gamma_i \ll \lambda \end{cases}$

Proof: For $x_0 = 0$, the solution $x_{\lambda,L}$ is

$$x_{\lambda,L} = (A^T A + \lambda^2 L^T L)^{-1} A^T b$$

Using GSVD of (A, L) we have that

$$A^T A = X^{-T} \begin{bmatrix} S^2 & 0 \\ 0 & I_{n-p} \end{bmatrix} X^{-1} \quad (\text{since } U^T U = I)$$

$$L^T L = X^{-T} \begin{bmatrix} M^2 & 0 \\ 0 & 0 \end{bmatrix} X^{-1}$$

thus $$A^T A + \lambda^2 L^T L = X^{-T} \begin{bmatrix} S^2 + \lambda^2 M^2 & 0 \\ 0 & I_{n-p} \end{bmatrix} X^{-1}$$

such that

$$(A^T A + \lambda^2 L^T L)^{-1} = X \begin{bmatrix} (S^2 + \lambda^2 M^2)^{-1} & 0 \\ 0 & I_{n-p} \end{bmatrix} X^T$$

and

$$x_{\lambda,L} = X \begin{bmatrix} (S^2 + \lambda^2 M^2)^{-1} & 0 \\ 0 & I_{n-p} \end{bmatrix} X^T X^{-T} \begin{bmatrix} S & 0 \\ 0 & I_{n-p} \end{bmatrix} U^T b$$

$$x_{\lambda,L} = X \begin{bmatrix} (S^2 + \lambda^2 M^2)^{-1} S & 0 \\ 0 & I_{n-p} \end{bmatrix} U^T b$$

Notice that $(S^2+\lambda^2 M^2)^{-1} S \in \mathbb{R}^{p\times p}$ is a diagonal matrix with the diagonal entries given by

$$\frac{\sigma_i}{\sigma_i^2+\lambda^2\mu_i^2}$$

and $U^T b = \begin{bmatrix} u_1^T \cdot b \\ \\ u_n^T \cdot b \end{bmatrix} \in \mathbb{R}^n$

such that

$$\begin{bmatrix} (S^2+\lambda^2 M^2)^{-1} S & 0 \\ 0 & I_{n-p} \end{bmatrix} U^T b = \begin{bmatrix} \frac{\sigma_1}{\sigma_1^2+\lambda^2\mu_1^2} u_1^T \cdot b \\ \vdots \\ \frac{\sigma_p}{\sigma_p^2+\lambda^2\mu_p^2} u_p^T \cdot b \\ u_{p+1}^T \cdot b \\ \vdots \\ u_n^T \cdot b \end{bmatrix}$$

and we obtain

$$x_{\lambda,L} = \sum_{i=1}^{p} f_i(\lambda,L) \frac{u_i^T \cdot b}{\sigma_i} x_i + \sum_{i=p+1}^{n} (u_i^T \cdot b)\, x_i$$

where $f_i(\lambda, L) = \dfrac{\sigma_i^2}{\sigma_i^2+\lambda^2\mu_i^2} = \dfrac{\gamma_i^2}{\gamma_i^2+\lambda^2}$