MTH 410/510 Inverse Problems & Data Assimilation

Lecture 1 Introductory concepts & examples

Inverse Problems (IP) : Infer actual values of the parameters in a physical system (model) based on information provided by observational data.

Forward modeling (direct problem): specification of the mapping (operator) F from the parameter space X into the observation space Y,

$$F: X \to Y, \qquad y = F(x)$$

Inverse modeling (inverse problem)
Find the actual values of the parameters x using information provided by data y

Find x such that $y = F(x)$

or find x such that $\|y - F(x)\|$ is minimized.

in practice, we only have access to noisy data, $y = y_{true} + \varepsilon$

where ε represents an unknown measurement error (may include error in instrument measurement and/or error in the representation of data)

Definition (Hadamard) We say that a problem is well-posed if it satisfies all of the following:

1) Existence : there is a solution
2) Uniqueness : there is only one solution
3) Stability : the solution depends continuously on the data

if at least one of these properties is not satisfied, the problem is said to be ill-posed. if small variations in the input produce large variations in the solution, the problem is ill-conditioned

Example 1

Direct problem: Evaluation of a polynomial

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n$$

inverse problem: Polynomial interpolation

Find the polynomial of degree n given the values $P(x_0) = y_0$, $P(x_1) = y_1$, ----, $P(x_n) = y_n$

Requires solution to the $(n+1)$-dimensional linear system

$$A a = y$$

where

$$A = \begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ & & & \\ 1 & x_i & \cdots & x_i^n \\ & & & \\ 1 & x_n & \cdots & x_n^n \end{bmatrix} \qquad a = \begin{bmatrix} a_0 \\ \vdots \\ \vdots \\ a_n \end{bmatrix}$$

$$y = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}$$

Solution may be expressed in terms of Lagrange polynomials of degree n,

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$L_i(x_i) = 1, \quad L_i(x_j) = 0, \quad i \neq j$$

$$P(x) = y_0 L_0(x) + \cdots + y_n L_n(x)$$

For linear interpolation,

$$P_1(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}$$

such that $a_0 = \dfrac{y_1 x_0 - x_1 y_0}{x_0 - x_1}$

$$a_1 = \frac{y_0 - y_1}{x_0 - x_1}$$

in the presence of errors in data,

$$\bar{y}_0 = y_0 + \delta y_0 \,, \quad \bar{y}_1 = y_1 + \delta y_1$$

we have
$$\delta a_0 = \frac{x_0}{x_0 - x_1}\delta y_1 - \frac{x_1}{x_0 - x_1}\delta y_0$$

$$\delta a_1 = \frac{\delta y_0}{x_0 - x_1} - \frac{\delta y_1}{x_0 - x_1}$$

The problem becomes ill-conditioned if the interpolation nodes $x_0, x_1, \dots, x_n$ are close together (clustered).

The <u>sensitivity</u> of the solution with respect to data is given by the Jacobian matrix

$$\frac{\partial [a_0, a_1]}{\partial [y_0, y_1]} = \frac{1}{x_0 - x_1}\begin{bmatrix} -x_1 & x_0 \\ 1 & -1 \end{bmatrix}$$

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<u>Example 2</u> integration vs. differentiation

<u>Direct problem</u> find the function

$f : [0,1] \to \mathbb{R}$ such that

$$\begin{cases} f'(t) = 3t^2 \\ f(0) = 0 \end{cases}$$

<u>Inverse problem</u> find the derivative of

$f : [0,1] \to \mathbb{R}$, $f(t) = t^3$

Difficulties arise if instead of f

we are given $f(t) + \delta f(t)$

for example, $\delta f_n(t) = \frac{1}{n} \sin nt$

$$\lim_{n \to \infty} \delta f_n(t) = 0 \,, \quad \forall t$$

however, $\delta f_n'(t) = \cos nt$

is relatively large!

Example 3 Population growth model

$$\begin{cases} \dfrac{dP}{dt} = KP \\ P(0) = P_0 \end{cases}$$

initial-value problem

K = rate growth parameter

P_0 = initial condition.

Forward problem : Given K, P_0 evaluate $P(t)$

Solution : $P(t) = P_0 e^{Kt}$

inverse problem : Given data points (t_1, P_1), (t_2, P_2) find P_0 and K.

$$\begin{cases} P_0 e^{Kt_1} = P_1 \\ P_0 e^{Kt_2} = P_2 \end{cases}$$

Assume $t_1 \neq t_2$

Notice that a solution exists only if $\text{sign}(P_1) = \text{sign}(P_2)$

Q: Find the solution & sensitivity to data

Example 4 inverse boundary-value problem

Direct problem find $u : [0, \pi] \to \mathbb{R}$

such that
$$\begin{cases} -u''(x) = f(x), & x \in (0, \pi) \\ u(0) = u(\pi) = 0 \end{cases}$$

Solution may be obtained using eigenvalues and eigenfunction expansion (Fourier series)
$$\begin{cases} -\phi''(x) = \lambda \phi(x) \\ \phi(0) = \phi(\pi) = 0 \end{cases}$$

Eigenvalues : $\lambda_n = n^2$

Eigenfunctions : $\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$

such that $\displaystyle\int_0^\pi \phi_n(x)\, \phi_m(x) = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$

Solution: $\displaystyle u(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$

where $\displaystyle a_n = \frac{1}{\lambda_n} \int_0^\pi f(x)\, \phi_n(x)\, dx$

The solution may be expressed as

$$u(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \int_0^{\pi} f(\bar{x})\, \phi_n(\bar{x})\, d\bar{x}\ \phi_n(x)$$

$$= \int_0^{\pi} \underbrace{\left[\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \phi_n(\bar{x})\, \phi_n(x) \right]}_{G(x,\bar{x})} f(\bar{x})\, d\bar{x}$$

$G(x, \bar{x})$ = Green's function

$$u(x) = \int_0^{\pi} G(x, \bar{x})\, f(\bar{x})\, d\bar{x}$$

The inverse problem is : find f given u

Given $u(x)$ find the solution to

$$u(x) = \int_0^{\pi} G(x, \bar{x})\, f(\bar{x})\, d\bar{x} \qquad \text{(iBVP)}$$

→ Fredholm integral equation of first kind for $f(x)$

We show that the (iBVP) is ill-posed.

Consider the perturbation in data

$$\delta u_n(x) = \frac{1}{\lambda_n} \phi_n(x) \xrightarrow[n\to\infty]{} 0 \quad , \quad \forall x \in [0,\pi].$$

Then $-[u + \delta u_n]'' = f + \delta f_n$

such that $\delta f_n = -[\delta u_n]'' = \phi_n$

and $\int_0^\pi [\delta f_n]^2 dx = 1$ since $\int_0^\pi \phi_n^2 dx = 1$

In general, the Fredholm integral equation of first kind is

$$\int_0^1 K(s,t) f(t)\, dt = g(s) \,, \quad 0 \le s \le 1$$

$K : [0,1] \times [0,1] \to \mathbb{R}$ is the <u>Kernel operator</u>

Both K and g are known functions, the ivp is to solve for f.

Examples Backward heat equation

Forward problem models the evolution of the temperature $u(x,t)$ as a partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad 0 < x < \pi$$

with boundary conditions

$$u(0,t) = 0, \quad u(\pi,t) = 0, \quad t \geq 0$$

and initial condition

$$u(x,0) = f(x), \quad 0 \leq x \leq 1$$

Solution to the forward problem is expressed as a fourier series

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \phi_n(x)$$

where the coefficients are obtained

from the initial condition as

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \Rightarrow a_n = \int_0^{\pi} \phi_n(x) f(x)\,dx$$

The inverse problem is the backward heat equation: Given $u(x,T)$ at time $T > 0$ find the initial condition $f(x)$.

We have $u(x,T) = \sum_{n=1}^{\infty} a_n e^{-n^2 T} \phi_n(x)$

thus $$a_n = e^{n^2 T} \int_0^{\pi} u(x,T)\,\phi_n(x)\,dx$$

and $$u(x,0) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

Consider a perturbation in data

$$\delta u_n(x,T) = \frac{1}{n}\phi_n(x) \xrightarrow[n\to\infty]{} 0$$

the corresponding perturbation in the retrieved initial condition is

$$\delta u_n(x,0) = \delta a_n \phi_n(x) = \frac{1}{n} e^{n^2 T} \phi_n(x)$$

and the amplification factor is

$$\frac{1}{n} e^{n^2 T} \xrightarrow[n\to\infty]{} \infty$$

which shows that the backward heat equation is an ill-posed problem.