MTH 410/510 Inverse Problems & Data Assimilation: Final Project

Due by noon on 12/05/2017

Image deblurring with missing pixel data

The goal of this project is to apply regularization techniques for reconstructing an image from an incomplete blurred version of it (with missing pixel data). The setup is as follows.

Consider an image represented as a matrix $\mathbf{X} \in \mathbb{R}^{n\times m}$ and a columnwise (1-dimensional) blurring/transmission process represented by the nonsingular matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ such that the true image is the solution to the matrix equation

$$\mathbf{AX} = \mathbf{D}, \quad \mathbf{AX}(:,j) = \mathbf{D}(:,j)\ ,\ \text{for } j = 1:m \tag{1}$$

In practice, it is often the case that we only have available an incomplete data set corrupted by noise (measurement and/or representation errors),

$$\mathbf{M} \circ \widehat{\mathbf{D}} = \mathbf{M} \circ (\mathbf{D} + \boldsymbol{\xi}) \tag{2}$$

In equation (2), $\boldsymbol{\xi} \in \mathbb{R}^{n\times m}$ denotes a matrix of random noise, $\circ$ denotes the elementwise (Hadamard) matrix product and $\mathbf{M} \in \mathbb{R}^{n\times m}$ is a "mask" matrix whose entries are 0 or 1 and are used to indicate data availability:

$M_{i,j} = 1$ if data $\widehat{\mathbf{D}}_{i,j}$ is available; $M_{i,j} = 0$ if data $\widehat{\mathbf{D}}_{i,j}$ is *not available.*

Essentially, if $J \in \mathbb{R}^{n_j}, n_j \le n$, denotes the vector of indices of all nonzero entries in the column j of the mask matrix,

$$J = find(M(:,j))$$

then the column j of the reconstructed image $\widehat{\mathbf{X}}$ is obtained as a regularized solution to the underdetermined linear system of n_j equations for n unknowns

$$\mathbf{A}(J,:)\widehat{\mathbf{X}}(:,j) = \widehat{\mathbf{D}}(J,j) \tag{3}$$

Columnwise Reconstruction Algorithm

for $j = 1:m$

$J = find(M(:,j))$

$\widehat{\mathbf{X}}(:,j) = regusol(\mathbf{A}(J,:), \widehat{\mathbf{D}}(J,j)$

end

where *regusol* represents the regularization method used to solve (3).

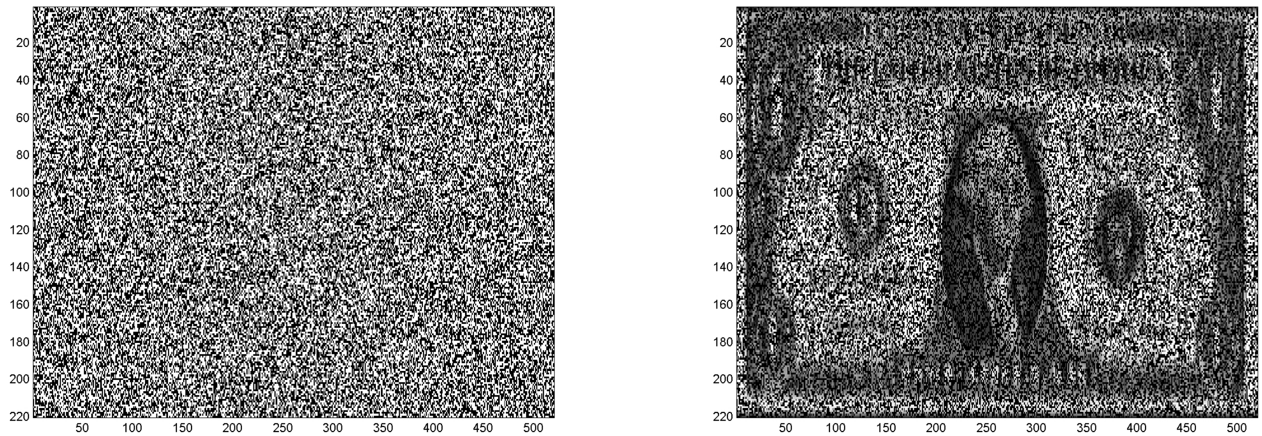

Figure 1: The mask matrix $\mathbf{M}$ (left figure) and the masked noisy data $\mathbf{M} \circ \widehat{\mathbf{D}}$ (right figure). Roughly, 50% of data is masked (missing).

Project requirements

In this project $n = 220$, $m = 520$, and X is a 220×520 matrix representing the image of a dollar bill. The blurring operation is represented as follows: Consider a $n \times n$ symmetric tridiagonal matrix B with entries [1]

$$B(i,i) = 1 - 2s, \ \ i = 1, 2, \dots n$$

$$B(i, i+1) = s, \ \ i = 1, 2, \dots n-1; \quad B(i+1, i) = s, \ \ i = 1, 2, \dots n-1$$

where $s = 0.45$. Then the blurring operator is

$$\mathbf{A} = \mathbf{B}^{10}$$

The file ”mask.m” contains the 220×520 mask matrix $\mathbf{M}$ and the file ”prdata.m” contains the 220×520 masked noisy data matrix $\mathbf{M} \circ \widehat{\mathbf{D}}$ in (2). In MATLAB you may execute:

load mask.m; imagesc(mask); colormap(gray);

load prdata.m; D = prdata; imagesc(D); colormap(gray)

to visualize $\mathbf{M}$ and $\mathbf{M} \circ \widehat{\mathbf{D}}$ as shown in Fig. 1.

Your job: Implement a regularization procedure of your choice to provide an approximation $\widehat{\mathbf{X}}$ of the unknown image $\mathbf{X}$ such that the serial number on the dollar bill can be identified.

Provide the serial number, an image of the reconstructed bill and a listing of the code you used to obtained the solution $\widehat{\mathbf{X}}$, clearly indicating the regularization procedure implemented and the specification of the regularization parameters (e.g., TSVD, standard/generalized Tikhonov, TV).

[1]such matrix results from discretization of the 1-D heat equation $u_t - ku_{xx} = 0$ with $s = k\Delta t/(\Delta x)^2$.