G326 midterm review

1 preliminaries

1. "model"
   (a) data visualization, conceptual, physical (bench-top), mathematical
   (b) mathematical: differential equations
      i. physical laws
      ii. empirical coefficients
      iii. boundary/initial conditions
      iv. analytic or numerical solution
   (c) hypothesis testing, infer quantities not observable directly, coupled processes, prediction/retrodiction

2. building a model
   (a) statement of problem
   (b) math representation: observations, physics, simplifications (calculus & assumptions about physical system), boundary conditions
   $\frac{\mathrm{d}(\textit{dependent variable})}{\mathrm{d}(\textit{independent variable})} = (\textit{some terms involving dependent variable})$
   (c) solution scheme: analytical or numerical?
   computation
   (d) analysis of model result

3. numerical models
   (a) no analytic solution available
   (b) direct or iterative
   (c) forward or inverse

4. analytical solutions: integrate exactly
   (a) exact
   (b) approximate (a simplification is made in order to arrive at an analytic solution, such as laminar flow assumption in our buoyancy problem)

5. definitions
   (a) ordinary vs. partial differential equation
   (b) linear vs. nonlinear equation
   (c) order of differentiation (higher order ODEs can be reduced)
   (d) model domain
   (e) initial value vs. boundary value problems
   (f) stability; explicit vs. implicit schemes

2 approximation of continuous functions

1. differential equations must be continuous over range of the independent variable
2. continuous function can be represented as a polynomial
3. Taylor series
   (a)
   $$f(x) = \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k \right) \tag{1}$$
   (b) exact but impossible to compute
   (c) approximation
      i. truncate: keep small number of terms in Taylor series
      ii. everything else is error; magnitude of error goes with order of first truncated term
4. apply approximation over discrete intervals of dependent variable

3 initial value problems

1. the basic set-up is:
   $$y_{j+1} = y_j + \int_{x_j}^{x_{j+1}} g(x)\, dx \tag{2}$$
   where $g(x)$ is our function over the interval $\{x_j : x_{j+1}\}$
2. single-step & predictor-corrector approaches to dealing with $g(x)$
3. Euler single-step

(a) assume $g(x)$ is constant and equal to $f(x_j)$ so that

$$y_{j+1} \;=\; y_{j_j} \;+\; h\; f(x_j) \tag{3}$$

where h represents the interval size

(b) error: local (truncation) & global (integrated effect)

4. midpoint predictor-corrector method: supposes that mean of $f(x_j)$ and $f(x_{j+1})$ is better representation of f over the interval

   (a) predictor: Euler step $f(x_j)$

   (b) corrector: compute $f(x_{j+1})$ using result of predictor step

$$y_{j+1} \;=\; y_{j_j} \;+\; \frac{h}{2}\;\{f(x_j) + f(x_{j+1})\} \tag{4}$$

5. higher-order single step methods: weighted sum of multiple evaluations of f, all based on $f(x_j)$

$$y_{j+1} \;=\; y_j \;+\; \sum\nolimits_{l=1}^{m} \gamma_l k_l \tag{5}$$

   where γ_l are the weights, and k_l are the function evaluations, none of which involve $f(x_{j+1}, y_{j+1})$.

6. coupled ODEs

4 homework problems