G326 final review

1 preliminaries

1. "model"
   (a) data visualization, conceptual, physical (bench-top), mathematical
   (b) mathematical: differential equations
      i. conservation
      ii. empirical coefficients
      iii. boundary/initial conditions
      iv. analytical or numerical solution
   (c) hypothesis testing (prediction), infer quantities not observable directly, coupled processes, projection
2. building a model
   (a) statement of problem
   (b) math representation: observations, physics, simplifications (calculus & assumptions about physical system), boundary conditions
   $\frac{\mathrm{d}(dependent\ variable)}{\mathrm{d}(independent\ variable)} = (some\ terms\ involving\ dependent\ variable)$
   (c) solution scheme: analytical or numerical?
   computation
   (d) analysis of model result
3. numerical solutions
   (a) analytical solution not available/desirable
   (b) always have error
   (c) stability issues
4. analytical solutions: integrate exactly
   (a) exact
   (b) approximate (a simplification is made in order to arrive at an analytic solution, such as laminar flow assumption in our buoyancy problem)

5. conservation equations

   (a) conservation of ϕ in a volume V enclosed by a surface S with an outward-pointing normal $\hat{n}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \phi \mathrm{d}V \;=\; -\int_S \mathbf{F}\cdot\hat{n}\;\mathrm{d}S \;-\; \int_S \phi\mathbf{V}\cdot\hat{n}\;\mathrm{d}S \;+\; \int_V H\mathrm{d}V \qquad (1)$$

   flux due to diffusion, flux due to advection, source

   (b) using the divergence theorem and assuming continuous fields at scales important to the problem,

$$\frac{\partial\phi}{\partial t} \;+\; \nabla\cdot\left(\mathbf{F} \;+\; \phi\mathbf{V}\right) \;-\; H \;=\; 0 \qquad (2)$$

6. definitions

   (a) ordinary vs. partial differential equation
   (b) linear vs. nonlinear equation
   (c) order of differentiation (higher order ODEs can be reduced)
   (d) model domain
   (e) initial value vs. boundary value problems
   (f) stability

2 approximation of continuous functions and finite differences

1. differential equations must be continuous over domain of the independent variable

2. continuous function can be represented as a polynomial

3. Taylor series

   (a)

$$f\left(x\right) \;=\; \lim_{n\to\infty}\left(\sum_{k=0}^{n}\frac{f^{(k)}\left(a\right)}{k!}\left(x \;-\; a\right)^{k}\right) \qquad (3)$$

   (b) exact when $n \to \infty$ but impossible to compute
   (c) approximation
      i. truncate: keep small number of terms in Taylor series

ii. everything else is error; magnitude of error goes with order of first truncated term

4. apply approximation over discrete intervals of dependent variable: forward, backward, centered

5. finite differences

   (a) forward difference
   $$f(x_{j+1}) \; - \; f(x_j)$$
   (b) backward difference
   $$f(x_j) \; - \; f(x_{j-1})$$
   (c) single-interval central difference
   $$f(x_{j+\frac{1}{2}}) \; - \; f(x_{j-\frac{1}{2}})$$
   (d) double-interval central difference
   $$f(x_{j+1}) \; - \; f(x_{j-1})$$
   (e) approximation of derivative: divide difference by step size

   (f) higer-order derivatives: repeated application of difference formula

3 initial value problems

1. the basic set-up (using a forward difference):
   $$y_{j+1} \; = \; y_j \; + \; \int_{x_j}^{x_{j+1}} g(x) \; dx \tag{4}$$
   where $g(x)$ is our function over the interval $\{x_j : x_{j+1}\}$

2. Euler single-step (forward difference)

   (a) assume $g(x)$ is constant and equal to $f(x_j)$ so that
   $$y_{j+1} \; = \; y_{j_j} \; + \; h \; f(x_j) \tag{5}$$
   where h represents the interval size

   (b) error: local (truncation) & global (integrated effect)

3. midpoint predictor-corrector method: supposes that mean of $f(x_j)$ and $f(x_{j+1})$ is better representation of f over the interval

$$y_{j+1} \;=\; y_{j_j} \;+\; \frac{h}{2}\left\{f(x_j) + f(x_{j+1})\right\} \tag{6}$$

4. higher-order single step methods: weighted sum of multiple evaluations of f, all based on $f(x_j)$

$$y_{j+1} \;=\; y_j \;+\; \sum\nolimits_{l=1}^{m} \gamma_l k_l \tag{7}$$

   where γ_l are the weights, and k_l are the function evaluations, none of which involve $f(x_{j+1}, y_{j+1})$.

5. coupled ODEs

4 partial differential equations

1. finite differences
   (a) Taylor Series again
   (b) forward, backward, centered differences
2. types of boundary conditions
   (a) Dirichlet

$$y^n_{N+1} = \theta^n_{N+1} \tag{8}$$

   (b) Neumann

$$\left.\frac{\partial y}{\partial x}\right|_{x_{N+1}, t_n} = \phi^n_{N+1} \tag{9}$$

3. selection of numerical scheme
   (a) choices about time derivative: different schemes suited to different types of equations
   (b) stability criteria for different schemes associated with truncation error
   (c) direct calculation or simultaneous equations
4. simultaneous equations $\mathbf{A}y = \mathbf{b}$
5. multiple dimensions in space: same ideas as 1D, more terms

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu + g = 0 \tag{10}$$