

# Appendix A

## Conservation equations

### A.1 introduction

The mathematical descriptions of physical systems we encounter in Earth science problems have at their hearts statements requiring the conservation of energy, momentum, or mass. We see this fundamental principle written in many different forms for many different quantities but strictly, they all derive from one place. In this chapter, the origin of conservation equations is described.

### A.2 conservation of ...

Conservation involves the change in a quantity  $\phi$  (with dimension of *something*/L<sup>3</sup>) in an arbitrary inertial volume  $V$  enclosed by a surface  $S$ . A unit vector  $\hat{n}$  normal to  $S$ , defined to be positive outward, is used to identify the regions of space within and outside of  $S$ .

The value of  $\phi$  within  $V$  may change over time  $t$  if there is a flux through  $S$  or creation of  $\phi$  within  $V$ . The flux may have two parts, one due to diffusion and another due to advection. The change in  $\phi$  within  $V$  is written:

$$\frac{d}{dt} \int_V \phi dV = - \int_S \mathbf{F} \cdot \hat{n} dS - \int_S \phi \mathbf{V} \cdot \hat{n} dS + \int_V H dV \quad (\text{A.1})$$

where  $\mathbf{F}$  represents the flux due to diffusion,  $\phi \mathbf{V}$  represents advection of  $\phi$  in a fluid flow with velocity  $\mathbf{V}$ ,  $H$  represents a source (or sink) of  $\phi$  within  $V$  and boldface indicates a vector-valued quantity. The negative signs in front of the first two terms on the right-hand side of (A.1) indicate that an outward flux results in a decrease of  $\phi$  in the volume enclosed by  $S$ .

Equation (A.1) is a statement of conservation of  $\phi$  for the unit volume  $V$ . This statement is always true, independent of the size of  $V$  and even if the fields enclosed by  $S$  are not continuous. This is the case because we integrate over the whole volume. It is important to note, however, that the integration also means that information on spatial scales smaller than  $V$  is not available to us.

One way to visualize the importance of the integration in equation (A.1) is to consider rice being poured from a bag. The rice is a discontinuous mixture of grains and air. If we focus in on a small volume of the flowing rice, in any instant of time we might see rice, we might see air, or we might see a mixture of the two. If we zoom out, and look again, we see the flow. If we integrate over that larger volume, the pouring rice looks like a fluid flow but we can't say anything about what is happening to individual grains.

### A.2.1 reduced dimensionality

Reduced spatial detail is often beneficial, as was the case in the zero-dimension energy balance model in chapter 4. By considering average (or “integrated”) properties of Earth’s surface and atmosphere, we were able to avoid complicated equations for radiative transfer and build an efficient model that let us explore some fundamental aspects of Earth’s mean climate. Another example where reduced dimensionality is beneficial are linear reservoir problems such as the ocean chemistry “box” model in chapter 3.

The zero-dimension approximation dictates that we consider only mean quantities. This is expressed:

$$\bar{f} = \frac{1}{V} \int_V f \, dV \quad (\text{A.2})$$

where  $f$  is any function ( $\phi$  or  $H$  in equation (A.1)) and the overbar indicates a mean. In our polluted lake example, we assume that the reservoir is well-mixed and  $f$  corresponds to the contaminant in the lake and  $V$  represents the lake volume. We would use a similar statement to address any sources or sinks within the lake.

The flux terms are grouped into two terms, one each for the rates at which the conserved quantity is going in and coming out through  $S$ :

$$\int_S (\mathbf{F} + \phi \mathbf{V}) \cdot d\mathbf{S} = \dot{\phi}_{out} - \dot{\phi}_{in} \quad (\text{A.3})$$

where the dot indicates a rate. The  $\phi$  rate is equivalent to the product of water flow rate and contaminant concentration in our lake pollution problem.

Using these definitions, equation (A.1) is simplified:

$$\frac{d}{dt} \frac{1}{V} \int_V \phi \, dV = \dot{\phi}_{in} - \dot{\phi}_{out} + \frac{1}{V} \int_V H \, dV \quad (\text{A.4})$$

and becomes:

$$\frac{d\phi}{dt} = \dot{\Phi}_{in} - \dot{\Phi}_{out} + \bar{H} \quad (\text{A.5})$$

in which  $\Phi$  represents the volumetric rate of  $\phi$ ,

$$\dot{\Phi} = \frac{\dot{\phi}}{V} \quad (\text{A.6})$$

Equation (A.5) is an ordinary differential equation, which may be solved exactly, by integration, or numerically, as an initial value problem. Equations derived in this manner are often linked together to form what are called *box models*. As we saw in the planetary energy balance problem, such models are quite powerful for conceptual studies but interpretation of the results derived from such models must be done with care. Additionally, it is often the case that the fluxes depend on the concentrations (as was the case with the temperature-dependent albedo in the energy balance model), in which case the equations are nonlinear and must be solved iteratively.

The zero dimension conservation equation is elegant but its use is restricted. Spatial, as well as temporal, variations are important to many problems in Earth science. Thus, we are also interested in applying equation (A.1) to cases in which  $\phi$  is not constant within  $V$ .

### A.2.2 (locally) continuous fields

In order to make progress for situations in which  $\phi$  varies in space, we must be able to write equation (A.1) as a partial differential equation. This requires the derivatives of  $\phi$  to exist within  $V$ . That is, our spatial field  $\phi$  must be nearly constant within smaller, subdivided regions of  $V$  and vary smoothly from little region to little region. It is important to recognize that there is still some lower limit at which our assumption of a locally continuous field breaks down (recall the example of rice pouring from a bag). That limit varies from system to system and among quantities within a given system. When we discuss the variation of  $\phi$  in space, we are really talking about the average  $\phi$ 's in small regions of space. We can make no assertions about anything that happens on a smaller scale and we assume that no smaller scale processes are important to the problem we wish to address. (The assumption about “sub-grid” processes is sometimes a poor one, in which case we might attempt to parameterize them, as we did for radiative transfer in the energy balance model.)

With the assumption of a locally continuous field, we can rewrite equation (A.1) as a (local) partial differential equation. We begin by applying the (very handy) divergence theorem which, for a vector-valued quantity, states:

$$\int_S \mathbf{F} \cdot \mathbf{n} \, dS = \int_V \nabla \cdot \mathbf{F} \, dV \quad (\text{A.7})$$

Using index notation, equation (A.7) is written:

$$\int_S \mathbf{F}_j n_j \, dS = \int_V \frac{\partial \mathbf{F}_j}{\partial x_j} \, dV \quad (\text{A.8})$$

Using equation (A.7), the surface integrals in equation (A.1) may be replaced:

$$-\int_S \mathbf{F} \cdot dS - \int_S \phi \mathbf{V} \cdot dS = -\int_V \nabla \cdot (\mathbf{F} + \phi \mathbf{V}) dV \quad (\text{A.9})$$

In our Eulerian reference frame it must be the case that

$$\frac{d}{dt} \int_V \phi dV = \int_V \frac{\partial \phi}{\partial t} dV \quad (\text{A.10})$$

Substituting (A.9) and (A.10) into (A.1), we have

$$\int_V \left\{ \frac{\partial \phi}{\partial t} + \nabla \cdot (\mathbf{F} + \phi \mathbf{V}) - H \right\} dV = 0 \quad (\text{A.11})$$

Because  $V$  is an arbitrary volume, equation (A.11) can only be true if the term in brackets is zero for our little subdivided volumes (note that this “zero everywhere” does not necessarily apply at scales smaller than the limiting resolution discussed earlier). At scales we care about, we can write:

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\mathbf{F} + \phi \mathbf{V}) - H = 0 \quad (\text{A.12})$$

This is the general form for all conservation laws in continuum mechanics.