A Questionable Distance-Regular Graph

Rebecca Ross

Abstract In this paper, we introduce distance-regular graphs and develop the intersection algebra for these graphs which is based upon its intersection numbers. We discuss results following from the definition of the intersection algebra. We investigate two examples of distance-regular graphs and show how these results apply. Finally, we introduce parameters that determine intersection numbers. We investigate if these intersection numbers are nonnegative and feasible.

1. Introduction

In this paper, we introduce distance-regular graphs and develop the intersection algebra for these graphs which is based upon its intersection numbers. We discuss results following from the definition of the intersection algebra. We investigate two examples of distance-regular graphs and show how these results apply. Finally, we introduce parameters that determine intersection numbers. We investigate if these intersection numbers are nonnegative and feasible.

2. Preliminaries

In this section we fix notation and review the basic definitions regarding graphs, distance-regularity graphs, and intersection numbers. For more information concerning graphs, see West [1] or Biggs [3].

2.1 Graphs

A *graph* Γ consists of a finite set V, whose elements are called *vertices*, another finite set E, whose elements are called *edges*, and a subset I of V x E, called the *incidence* relation such that each edge is incident with exactly two vertices, and no pair of edges is incident with the same pair of vertices. Two distinct vertices are said to be *adjacent* if there exists an edge incident to them both.

A *walk* is a sequence of vertices such that consecutive vertices in the sequence are adjacent. A *path* is a walk with distinct vertices. A graph is *connected* if for each pair of distinct vertices there exists a path containing them. The *length* of a walk with n vertices is $n-1$. The *distance* between vertices u and v, denoted by $\partial(u, v)$, is the length of the shortest path containing the vertices. The *diameter*, d, of a graph is the maximum distance between any two vertices in $V(\Gamma)$.

A graph *automorphism* is a permutation of the vertex set that preserves the adjacency relation. The group of automorphisms of a graph, Γ, is denoted by $\mathrm{Aut}(\Gamma)$. A graph Γ is *vertex-transitive* if for every $u, v \in V(\Gamma)$, there exists an automorphism mapping u to

v. A connected graph Γ is *distance-transitive* if for any vertices $u, v, x, y \in V(\Gamma)$ such that $\partial(u, v) = \partial(x, y)$ there exists an automorphism $g \in \mathrm{Aut}(\Gamma)$ that maps u to x and v to y.

2.2 Distance-regular graphs

Let Γ be a connected graph with diameter d. For each $u \in V(\Gamma)$ and each integer i, define

$$\Gamma_i(u) := \{v \in V(\Gamma) \mid \partial(u, v) = i\}.$$

If k is a nonnegative integer, we say Γ is *regular*, with *valency* k, if for all $u \in V(\Gamma)$, $|\Gamma_1(u)| = k$. We say Γ is *distance-regular*, with *intersection numbers* p^h_{ij} $(0 \leq h, i, j \leq d)$, whenever for all integers h, i, j $(0 \leq h, i, j \leq d)$ and all $u, v \in V(\Gamma)$ with $\partial(u, v) = h$,

$$|\Gamma_i(u) \cap \Gamma_j(v)| = p^h_{ij}. \tag{1}$$

These intersection numbers must be nonnegative and satisfy a triangle inequality, in that

$$p^h_{ij} = 0$$

if one of h, i, j is greater than the sum of the other two. Necessary conditions such as these on the intersection numbers are often called *feasibility conditions.* Unless a set of parameters meets these criteria, it is impossible (infeasible) to have a distance-regular graph with these intersection numbers. For more information on feasibility conditions, see [2]

For convenience, we define

$$\begin{aligned} c_i &:= p^i_{i-1,1} && (1 \leq i \leq d), \\ b_i &:= p^i_{i+1,1} && (0 \leq i \leq d-1), \\ a_i &:= p^i_{i1} && (1 \leq i \leq d) \end{aligned}$$

and define $c_0 = 0$ and $b_d = 0$. Note that $a_0 = 0$ and $c_1 = 1$. Also, for Γ regular with valency k, we have $k = b_0$. We can also observe that

$$c_i + a_i + b_i = k \qquad (0 \leq i \leq d). \tag{2}$$

For any vertex v of a distance-regular graph Γ with diameter d, define $k_i = |\Gamma_i(v)|$ for $1 \leq i \leq d$. It follows then that there are k_{i-1} vertices in $\Gamma_{i-1}(v)$ and each of those vertices is adjacent to b_{i-1} vertices in $\Gamma_i(v)$. Also, there k_i vertices in $\Gamma_i(v)$ that are adjacent to c_i vertices in $\Gamma_{i-1}(v)$. So counting the number of edges incident with both $\Gamma_{i-1}(v)$ and $\Gamma_i(v)$ is $k_{i-1}b_{i-1} = k_i c_i$. Thus,

$$k_i = \frac{k b_1 b_2 \ldots b_{i-1}}{c_2 c_3 \ldots c_i} \tag{3}$$

For the remainder of this paper, we will assume Γ is distance-regular with diameter $d \geq 3$.

3. The Adjacency Algebra, $\mathcal{A}$

Given a graph, Γ with vertex set $\{v_1, v_2, ..., v_n\}$, the *adjacency matrix* of Γ is the $n \times n$ matrix, A, whose entry $(A)_{ij}$ is the number of edges in Γ with endpoints $\{v_i, v_j\}$. We notice that by our definition of graph, A is a symmetric $0-1$ matrix. It also follows then that if λ is an eigenvalue of A, then λ is real and that the multiplicity of λ as a root of $\det(\lambda I - A) = 0$ is equal to the dimension of the space generated by the eigenvector corresponding to λ. We will usually refer to the eigenvalues of A as the *eigenvalues of* Γ.

Definition 3.0.1 Let Γ be a distance-regular graph with diameter d and adjacency matrix A. Define the *adjacency algebra* of Γ to be the algebra given by $\text{span}\{A^0, A, A^2, \ldots\}$ and denoted by $\mathcal{A}$.

Proposition 3.0.2 *For any nonnegative integer k, the (i,j)th entry of A^k counts the number of v_i, v_j-walks of length k.*

Proof. This holds for $k = 0$ since A^0 is the identity matrix, I. For $k = 1$, $A^1 = A$ is the adjacency matrix, which by definition counts the number of walks of length 1 between any two vertices. Suppose true for A^l for $l < k$. Observe that every walk of length k from v_i to v_j consists of a walk of length $k-1$ from v_i to a vertex v_l that is adjacent to v_j. By the induction hypothesis, $(A^{k-1})_{il}$ is the number of walks of length $k-1$ with endpoints $\{v_i, v_l\}$ for $0 \leq l < k$. Thus

$$\begin{aligned}
(A^k)_{ij} &= (A^{k-1}A)_{ij} \\
&= \sum_{l=1}^{n} (A^{k-1})_{il}(A)_{lj} \\
&= (A^{k-1})_{i1}(A)_{1k} + (A^{k-1})_{i2}(A)_{2k} + \cdots + (A^{k-1})_{in}(A)_{nj}
\end{aligned}$$

which equals the number of walks of length k with endpoints $\{v_i, v_j\}$. $\square$

Lemma 3.0.3 *Let Γ be a distance-regular graph with diameter d and adjacency algebra $\mathcal{A}$. Then dim($\mathcal{A}$) $\geq d+1$.*

Proof. Pick any $u, v \in V(\Gamma)$ and let $\partial(u,v) = k$. Then for all $i < k$ the uv-entry of A^i will be zero since the shortest path connecting u to v is of length k and the uv-entry of A^k is nonzero. Since the diameter of Γ is d, there exist vertices $x, y \in V(\Gamma)$ such that $\partial(x,y) = i$ for $0 \leq i \leq d$. So for all $0 \leq i \leq d$, A^i is not contained in the span of $A^0, \cdots, A^{i-1}$. Thus, $\{A^0, A, \ldots, A^d\}$ are linearly independent and $\dim \mathcal{A} \geq d+1$. $\square$

Corollary 3.0.4 *A connected graph with diameter d has at least $d+1$ distinct eigenvalues.*

Proof. Let Γ be a connected graph with adjacency matrix, A. By Lemma 3.0.3, the dimension of $\mathcal{A}$ is at least $d+1$, thus the minimum polynomial of A has degree at least $d+1$. It follows then that A has at least $d+1$ distinct eigenvalues. $\square$

3.1 Bose-Mesner Algebra, $\mathcal{M}$

We next recall the Bose-Mesner algebra of Γ. For each integer i $(0 \leq i \leq d)$, let A_i be the i^{th} *distance matrix*, which is the $n \times n$ matrix where each rs-entry is given by

$$(A_i)_{rs} = \begin{cases} 1 & \text{if } \partial(v_r, v_s) = i, \\ 0 & \text{if } \partial(v_r, v_s) \neq i \end{cases} \qquad (v_r, v_s \in V(\Gamma)).$$

We notice that $A_0 = I$ and that A_1 is the adjacency matrix A of Γ. Also,

$$A_0 + A_1 + ... + A_d = J,$$

where J is the all-1 matrix of size n. It is also clear that $A_i^t = A_i$ for $(0 \leq i \leq d)$.

Lemma 3.1.1 *Let Γ be a distance-regular graph with diameter d. For $1 \leq i, j \leq d$,*

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h.$$

Proof. Let Γ be a distance-regular graph with diameter d and A_k the k^{th} distance matrix of Γ for $(0 \leq k \leq d)$. Pick any $x, y \in V(\Gamma)$ and let $t = \partial(x, y)$. Then

$$\begin{aligned} (A_i A_j)_{xy} &= \sum_{z \in V(\Gamma)} (A_i)_{xz} (A_j)_{zy} \\ &= |\{z \in V(\Gamma) \,|\, \partial(x, z) = i \text{ and } \partial(z, y) = j\}| \\ &= p_{ij}^t. \end{aligned}$$

On the other hand, the xy-entry of $\sum_{h=0}^{d} p_{ij}^h A_h$ is equal to

$$\begin{aligned} \sum_{h=0}^{d} p_{ij}^h (A_h)_{xy} &= p_{ij}^t (A_t)_{xy} \\ &= p_{ij}^t (1) \\ &= p_{ij}^t, \end{aligned}$$

since $(A_h)_{xy} = 0$ if $h \neq t$. $\square$

Corollary 3.1.2 *Let Γ be a distance-regular graph. For $1 \leq i \leq d$,*

$$AA_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}.$$

Proof. By applying Lemma 3.1.1 and the definition of b_i, a_i, and c_i, the equation follows easily. $\square$

It follows then that

$$A_{i+1} = \frac{1}{c_{i+1}} [AA_j - b_{j-1} A_{j-1} - a_j A_j],$$

which is a polyomial in A with degree $j+1$.

Now, the distance matrices are linearly independent (exactly one matrix has a nonzero entry in every position) and, by Lemma 3.1.1, closed under multiplication. Thus, we have that $\{A_0, A_1, ..., A_d\}$ is a basis for what is called the *Bose-Mesner algebra of* Γ, denoted by $\mathcal{M}$. It follows that $\mathcal{M}$ has dimension $d+1$. Furthermore, A, the adjacency matrix of distance-regular graph Γ, is an element of $\mathcal{M}$. Thus, $\mathcal{A} \subseteq \mathcal{M}$. Therefore,by Lemma 3.0.3, it follows that dim $\mathcal{A} = d+1$ and that $\{A_0, A_1, ..., A_d\}$ is a basis for $\mathcal{A}$.

Corollary 3.1.3 *For a distance-regular graph* Γ *with diameter d,* $\mathcal{M} = \mathcal{A}$.

Proof. Clear since $\mathcal{A} \subseteq \mathcal{M}$ and dim $\mathcal{A}$ =dim $\mathcal{M}$. $\square$

Corollary 3.1.4 *If* Γ *is a distance-regular graph with diameter d, then* Γ *has* $d+1$ *distinct eigenvalues.*

Proof. Follows from Lemma 3.0.3 and Lemma 3.1.3. $\square$

We define the *eigenmatrix* of distance-regular graph, Γ, to be the $(d+1)\times(d+1)$ matrix given by the the l-th column of P consists of the eigenvalues of A_l. The il-th entry of P is denoted $P_l(i)$.

4. The Intersection Algebra, $\mathcal{B}$

In this section we will introduce the intersection algebra. We begin by defining the intersection matrices. For more information on the intersection algebra, see [3] and [5].

Definition 4.0.5 Let Γ be a distance-regular graph with diameter d. For any j $(0 \le j \le d)$, we define *the intersection matrix*, B_j, to be the $(d+1)\times(d+1)$ matrix with entries given by:

$$(B_j)_{ih} := p^h_{ij}. \qquad (0 \le h, i \le d) \tag{4}$$

Define $\mathcal{B} := \text{span}\{B_0, B_1, \ldots, B_d\}$. Also, let $B_1 := B$. By the triangle inequality, $p^h_{0j} = \delta_{hj}$, so that the $0h$-entry of B_j is nonzero if and only if $j = h$. It follows then that $B_0, B, \ldots, B_d$ are linearly independent. We will show that $\mathcal{B}$ is closed under matrix multiplication but first we need the following lemma.

Lemma 4.0.6 *For a distance-regular graph* Γ *with diameter d, then*

$$\sum_{h=0}^{d} p^h_{mi} p^n_{hj} = \sum_{l=0}^{d} p^l_{ij} p^n_{ml} \tag{5}$$

Proof. Given vertices w, x, y, z such that $\partial(w, x) = n$, we will show that both sides of equation (5) count the number of ordered pairs (y, z) such that $\partial(x, y) = m, \partial(w, z) = j$, and $\partial(y, z) = i$.

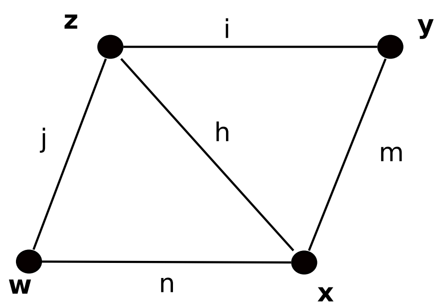

Figure 1: Counting Quadrilaterals by $\partial(x,z)$

Let h be an integer such that $0 \leq h \leq d$. Suppose $\partial(x,z) = h$. Then since $\partial(w,x) = n$ and $\partial(w,z) = j$, the number of choices we have for such a vertex z is p^n_{jh}. See Figure 1. Given such a z, since $\partial(x,z) = h$ and we also want $\partial(y,z) = i$ and $\partial(x,y) = m$, there are p^h_{mi} choices for y. So, if we let $h = 0$, the number of pairs (y,z) is $p^n_{j0}p^0_{mi}$ by the multiplication rule for counting. Or if we let $h = 1$, the number of pairs (y,z) is $p^n_{j1}p^1_{mi}$ and so on up to $h = d$. Summing from 0 to d, yields the total number of such pairs. Thus the total number of pairs (y,z) that satisfy the given conditions is equal to $\sum_{h=0}^{d} p^h_{mi}p^n_{hj}$.

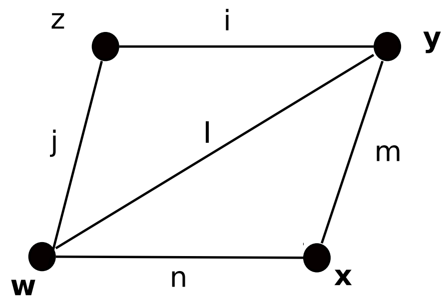

Figure 2: Counting Quadrilaterals by $\partial(w,y)$.

Now let l be an integer such that $0 \leq l \leq d$. Suppose $\partial(w,y) = l$. Then since $\partial(w,x) = n$ and $\partial(x,y) = m$, the number of choices we have for such a vertex y, is p^n_{ml}. See Figure 2. Given such a y, since $\partial(w,y) = l$ and we also want $\partial(y,z) = i$ and $\partial(x,y) = j$, there are p^l_{ij} choices for z. So, if we let $l = 0$, the number of pairs (y,z) is $p^0_{ij}p^n_{m0}$ by the multiplication rule for counting. Or if we let $l = 1$, the number of pairs (y,z) is $p^1_{ij}p^n_{m1}$ and so on up to $l = d$. Summing from 0 to d, yields the total number of such pairs. Thus the total number of pairs (y,z) that satisfy the given conditions is equal to $\sum_{l=0}^{d} p^l_{ij}p^n_{ml}$. $\square$

Proposition 4.0.7 *For a distance-regular graph* Γ *with diameter* d*,*

$$B_i B_j = \sum_{h=0}^{d} p^h_{ij} B_h \tag{6}$$

Proof. We will show that the mn-th entry of both matrices are the same.

$$\begin{aligned}(B_iB_j)_{mn} &= \sum_{h=0}^{d}(B_i)_{mh}(B_j)_{hn} = \sum_{h=0}^{d} p_{mi}^{h}p_{hj}^{n} \\ &= \sum_{l=0}^{d} p_{ij}^{l}p_{ml}^{n} = \sum_{h=0}^{d} p_{ij}^{h}(B_h)_{mn}\end{aligned}$$

by equation (4) and Lemma 4.0.6 above. □

Corollary 4.0.8 *For a distance-regular graph* Γ, $\mathcal{M} \cong \mathcal{B}$.

Proof. Let $\phi : \mathcal{M} \to \mathcal{B}$ be the linear map given by $\phi(A_i) = B_i$ for all i. Then ϕ is an isomorphism of vector spaces. Furthermore, by (6),

$$\begin{aligned}\phi(A_iA_j) &= \phi(\sum_{h=0}^{d} p_{ij}^{h}A_h) = \sum_{h=0}^{d} p_{ij}^{h}\phi(A_h) \\ &= \sum_{h=0}^{d} p_{ij}^{h}B_h = B_iB_j = \phi(A_i)\phi(A_j).\end{aligned}$$

Therefore, ϕ is an isomorphism of algebras. □

Corollary 4.0.9 *Let* Γ *be a distance-regular graph. For* $1 \le i \le d-1$,

$$BB_i = b_{i-1}B_{i-1} + a_iB_i + c_{i+1}B_{i+1}. \tag{7}$$

Proof. Due to isomorphism and Corollary 3.1.2. □

Corollary 4.0.10 *Let* Γ *be a distance-regular graph with diameter* d. *Then*

$$p_{ij}^{h} = \frac{1}{c_j}[b_{i-1}p_{i-1j-1}^{h} + (a_i - a_{j-1})p_{ij-1}^{h} + c_{i+1}p_{i+1j-1}^{h} - b_{j-2}p_{ij-2}^{h}] \tag{8}$$

Proof. Obtaining the intersection number p_{ij}^{h} is equivalent to finding the ih-entry of the matrix B_j. Solving for B_{i+1} in Corollary 4.0.9, yields

$$B_{i+1} = \frac{1}{c_{i+1}}[BB_i - b_{i-1}B_{i-1} - a_iB_i]$$

from which it follows that

$$B_i = \frac{1}{c_i}[BB_{i-1} - b_{i-2}B_{i-2} - a_{i-1}B_{i-1}]. \tag{9}$$

Thus,

$$\begin{aligned}p_{ij}^{h} = (B_j)_{ih} &= \frac{1}{c_j}(BB_{j-1} - b_{j-2}B_{j-2} - a_{j-1}B_{j-1})_{ih} \\ &= \frac{1}{c_j}[(BB_{j-1})_{ih} - b_{j-2}p_{ij-2}^{h} - a_{j-1}p_{ij-1}^{h}].\end{aligned}$$

And

$$
\begin{aligned}
(BB_{j-1})_{ih} &= \sum_{k=0}^{d} (B)_{jk}(B_{j-1})_{kh} \\
&= \sum_{k=0}^{d} p_{i1}^{k} p_{kj-1}^{h} \\
&= b_i p_{i-1j-1}^{h} + a_i p_{ij-1}^{h} + c_{i+1} p_{i+1j-1}^{h},
\end{aligned}
$$

by the triangle inequality. Therefore, the result follows. □

Corollary 4.0.11 *Let Γ be a distance-regular graph with diameter d. Then Γ has $d+1$ distinct eigenvalues which are the eigenvalues of the intersection matrix B.*

Proof. Since $\mathcal{M} \cong \mathcal{B}$, the minimum polynomals of A and B are the same. Thus they have the same eigenvalues. □

5. Examples

5.1 The Hamming Graph

The Hamming graph $H(n,q)$ is the graph whose vertices are sequences of length n from an alphabet of size q. We observe that $|V(H(n,q))| = q^n$. Two vertices are considered adjacent if the n-tuples differ in exactly one term. We will show that the Hamming graph is distance-regular. First, we need the following results.

Lemma 5.1.1 *For all vertices x, y of $H(n,q)$, $\partial(x,y) = i$ if and only if $n(x,y) = i$, where $n(x,y)$ is defined to be the number of coordinates vertices x and y differ when considered as n-tuples.*

Proof. Let x and y be vertices of Hamming graph, $H(n,q)$. Then by the adjacency relation, if $\partial(x,y) = 0$ then x and y are the same vertices and therefore differ in 0 coordinates. Similarly, if $\partial(x,y) = 1$ then x and y are adjacent and by the adjacency relation differ in exactly one term. Suppose this holds for $\partial(x,y) < i$. Consider $\partial(x,y) = i$. Then by definition of distance, there exists a path between x and y of length i. So there exists a vertex,z, that is distance $i-1$ from x and distance 1 from y. By the induction hypothesis, z differs from x in exactly $i-1$ terms. It also differs from y in exactly 1 term by the adjacency relation. Thus, y differs from x in exactly $i-1+1 = i$ terms. □

Lemma 5.1.2 *The Hamming graph is vertex-transitive.*

Proof. By definition of vertex-transitive, H(n,q) is vertex-transitive if for all pairs of vertices x, y there exists an automorphism of the graph that maps x to y. Let v be a fixed vertex and $x \in V(H(n,q))$. Then the mapping $\rho_v : x \to x+v$, where addition is done modular q, will be an automorphism of the graph since if the n-tuples x, y differ in exactly 1 term, then the n-tuples $x+v$ and $y+v$ will differ in exactly 1 term thus preserving the adjacency relation. And for any two vertices, $x, y \in V(H(n,q))$, the automorphism ρ_{y-x} maps x to y. Thus, the Hamming graph is vertex-transitive. □

Lemma 5.1.3 *The Hamming graph is distance-regular.*

Proof. For a graph to be distance-regular, it is enough to show that for any vertex, the intersection numbers a_i, b_i, and c_i are independent of choice of vertex. Pick vertices x, y such that $\partial(x,y) = i$. Since $H(n,q)$ is vertex transitive, suppose, without loss of generality, that vertex x is the sequence $\{00000\ldots 0\}$. By Lemma 5.1.1, y will have i nonzero entries. Now, a_i is the number of neighbors of y that are also distance i from x. So we have i choices of coordinate in which to differ from y and $q-2$ letters of the alphabet to choose from. Thus, $a_i = \binom{i}{1}(q-2) = i(q-2)$. And b_i will be the number of neighbors of y that are also distance $i+1$ from x. So there are $n-i$ places in which to differ from x and y and $q-1$ letters to choose from. So $b_i = (n-i)(q-1)$. In counting c_i, we are counting the number of vertices that are distance $i-1$ from x and adjacent to y. So we can change any of the i nonzero terms to choose to turn back to zero. So $c_i = i$. Thus the Hamming graph is distance-regular. □

It can be proven (see [4]) that P, the eigenmatrix of $H(3,2)$ is given by

$$P_l(i) = \sum_{\alpha=0}^{l} (-q)^{\alpha}(q-1)^{l-\alpha}\binom{n-\alpha}{l-\alpha}\binom{i}{\alpha}. \tag{10}$$

Restricting our attention to the first column of this matix, we have

$$\begin{aligned} P_1(i) &= \sum_{\alpha=0}^{1} (-q)^{\alpha}(q-1)^{1-\alpha}\binom{n-\alpha}{1-\alpha}\binom{i}{\alpha} \\ &= qn - n - qi, \end{aligned} \tag{11}$$

which gives the eigenvalues for A. By Lemma 4.0.11, these eigenvalues will be the same as the eigenvalues of the intersection matrix B. If we let θ_i denote the i-th eigenvalue of B, we have

$$\theta_i = q(n-i) - n.$$

5.2 H(3,2)

Let's look at a concrete example of the Hamming graph, $H(3,2)$. The vertices of this graph will be 3-term sequences from a binary alphabet. So $|V(H(3,2))| = 8$ and each vertex will have degree 3 since there are 3 terms in which any two vertices can differ.

The adjacency matrix for $H(3,2)$ is given below.

$$A = \left(\begin{array}{c|cccccccc}
 & 000 & 100 & 010 & 001 & 110 & 101 & 011 & 111 \\
\hline
000 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
100 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
010 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
001 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
110 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
101 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
011 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
111 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{array}\right)$$

We can also determine the intersection matrix, B. The diameter of $H(3,2)$ is 3 since vertices can differ at most by 3 terms. So B will be a 4×4 matrix. Using the intersection numbers derived in Lemma 5.1.1, we can generate B by calculating up to $i = 3$. Thus the intersection matrix B for $H(3,2)$ is:

$$B = \begin{pmatrix} p_{01}^0 & p_{01}^1 & p_{01}^2 & p_{01}^3 \\ p_{11}^0 & p_{11}^1 & p_{11}^2 & p_{11}^3 \\ p_{21}^0 & p_{21}^1 & p_{21}^2 & p_{21}^3 \\ p_{31}^0 & p_{31}^1 & p_{31}^2 & p_{31}^3 \end{pmatrix} = \begin{pmatrix} a_0 & c_1 & 0 & 0 \\ b_0 & a_1 & c_2 & 0 \\ 0 & b_1 & a_2 & c_3 \\ 0 & 0 & b_2 & a_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We can also use equation (9) to generate the rest of the intersection matrices. These will be B_2 and B_3 since the diameter of $H(3,2)$ is equal to 3. So

$$B_2 = B_{1+1} = \frac{1}{c_2}[BB - b_0 B_0 - a_1 B] = \frac{1}{2}[B^2 - 3B_0 - 0B]$$

and

$$B_3 = B_{2+1} = \frac{1}{c_3}[BB_2 - b_{2-1}B_{2-1} - a_2 B_2] = \frac{1}{3}[BB_2 - 2B - 0B_2].$$

So we have that

$$B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 \\ 3 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Now that we have the intersection matrices, we can calculate the eigenvalues and eigenmatrix for $H(3,2)$. As predicted by equation (10), the eigenmatrix P for the distance-regular graph $H(3,2)$ is given by

$$P(\Gamma) = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -3 & -1 & -1 \end{pmatrix}$$

5.3 The Johnson Graph

The Johnson graph $J(n, r)$, is defined to be the graph whose vertices are the r-element subsets of a n-element set S. We observe that $|V(J(n, r))| = \binom{n}{r}$. Two vertices are adjacent if the size of their intersection is exactly $r - 1$. Said another way, vertices are adjacent in they differ in only one term. We will show the Johnson graphs are distance-regular but the need the following Lemma first.

Lemma 5.3.1 *If x, y are vertices of a Johnson graph, then $\partial(x, y) = i$ if and only if $|x \cap y| = r - i$, when considering x, y to be r-sets.*

Proof. Let x, y be vertices of $J(n, r)$. Then $\partial(x, y) = 0$ if and only if x and y are the same vertex, which holds if and only if $|x \cap y| = r = r - 0$. Suppose the result holds for any x, y with $\partial(x, y) < i$.

If $\partial(x, y) = i$, then $\partial(x, y) > i - 1$, so $|x \cap y| < r - i + 1$ by the induction hypothesis. So $|x \cap y| \leq r - i$. By definition of distance, there exists a path of length i from x to y. Thus there exists a vertex z that is distance $i - 1$ from x and adjacent to y. So by the induction hypothesis

$$|z \setminus x| = i - 1 \quad \text{and} \quad |y \setminus z| = 1.$$

Now we notice that

$$|y \setminus x| = |(y \setminus x) \cap z| + |(y \setminus x) \setminus z|.$$

Since $(y \setminus x) \cap z \subseteq z \setminus x$ and $(y \setminus x) \setminus z \subseteq y \setminus z$,

$$\begin{aligned} |y \setminus x| &\leq |z \setminus x| + |y \setminus z| \\ &= (i - 1) + 1, \end{aligned}$$

so $|y \setminus x| \leq i$ which implies $|x \cap y| \geq r - i$. We conclude $|x \cap y| = r - i$ as desired.

Now suppose $|x \cap y| = r - i$. We need to show that $\partial(x, y) = i$. If $\partial(x, y) \leq i$ then, by the induction hypothesis, $|x \cap y| > r - i$, a contradiction. So $\partial(x, y) \geq i$. On the other hand, if we let

$$x \setminus y = \{x_1, \ldots, x_i\} \quad \text{and} \quad y \setminus x = \{y_1, \ldots, y_i\},$$

then we can define, for each j $(0 \leq j \leq i)$,

$$z_j = (x \setminus \{x_1, \ldots, x_j\}) \cup \{y_1, \ldots, y_j\},$$

and the sequence $x = z_0, z_1, \ldots, z_i = y$ is an xy-path of length i. So $\partial(x, y) \leq i$, forcing $\partial(x, y) = i$ as desired. $\square$

Lemma 5.3.2 *Johnson graphs are distance-regular.*

Proof. Again, it is enough to show that the intersection numbers for Johnson graphs are independent of choice of vertex for the graph to be distance-regular. Let x, y be vertices of $J(n, r)$ such that $\partial(x, y) = i$. Considered as r-element subsets of $\{1, 2, \ldots, n\}$, to get a neighbor of y, we need to pick an element of y, say a, and change it an element that is not in y, say to b. There are four ways this can be done.

Case 1: If a is an element of $x \cap y$ and b is an element of $x \setminus y$, then z will still differ from x by $r - i$ terms since a was common to both x and y but b is not common to x. This gives a neighbor of y such that $\partial(x, z) = i$.

Case 2: If a is an element of $x \cap y$ and b is not an element of $x \cup y$, then z will be a neighbor of y that differs from y in 1 term and from x in $r - (i + 1)$ terms. So $\partial(x, z) = i + 1$.

Case 3: If a is an element of $y \setminus x$ and b is an element of $x \setminus y$, then z will differ from y in 1 term and from x in only $r - (i - 1)$ since we are changing a to a term that is already in x. Thus $\partial(x, z) = i - 1$.

Case 4: If a is an element of $y \setminus x$ and b is not an element of $x \cup y$, then z will differ from y by 1 term and from x in $r - i$ terms since a was not in x and neither is b. Thus $\partial(x, z) = i$.

Now, by definition the intersection number a_i is given by $|\Gamma_i(x) \cap \Gamma_1(y)|$. So we want all vertices, z, such that $\partial(x, z) = i$ and $\partial(z, y) = 1$. These are given by Case 1 and Case 4. From Case 1, we have that there are $r - 1$ choices for a and i choices for b. From Case 4 we have i choices for a and $n - r - i$ choices from b. Thus $a_i = (r - i)i + i(n - r - i)$. The intersection number b_i is given by $|\Gamma_{i+1}(x) \cap \Gamma_1(y)|$. So we want all vertices, z, such that $\partial(x, z) = i + 1$ and $\partial(z, y) = 1$. These are given by Case 2. We have $r - i$ choices for a and since we must pick z not in the union of x and y, we have $n - 2r + (r - i) = n - k - i$ choices for b. Thus $b_i = (r - i)(n - r - i)$. The intersection number c_i is given by $|\Gamma_{i-1}(x) \cap \Gamma_1(y)|$. So we want all vertices, z, such that $\partial(x, z) = i - 1$ and $\partial(z, y) = 1$. These are given by Case 3. We have i choices for a and i choices for b, thus $c_i = i^2$. Since the intersection numbers for $J(n, r)$ are independent of choice of vertex, the Johnson graph is distance-regular. $\square$

It can be shown (see [4]) that the eigenvalue matrix for the Johnson graph is given by

$$P_l(i) = \sum_{\alpha=0}^{l} (-1)^{l-\alpha} \binom{r-\alpha}{l-\alpha} \binom{r-i}{\alpha} \binom{n-r+\alpha-i}{\alpha} \tag{12}$$

It follows then at the eigenvalues for the Johnson graph are given by the first column of this matrix which is

$$P_1(i) = \sum_{\alpha=0}^{1} (-1)^{1-\alpha} \binom{r-\alpha}{1-\alpha} \binom{r-i}{\alpha} \binom{n-r+\alpha-i}{\alpha}.$$

Thus the eigenvalues for the adjacency matrix A for Johnson graph $J(n, r)$ is given by

$$\theta_i = -r + (r - i)(n - r + 1 - i). \tag{13}$$

5.3.3 Johnson Graph J(4,2)

Let's look at a concrete example of the Johnson graph, $J(4, 2)$. The vertices of this graph will be 2 element subsets of the numbers $\{1, 2, 3, 4\}$. So $|V(J(4, 2))| = 6$ and each vertex will have degree 4 since each vertex with differ in term with exactly 4 other vertices. The adjacency matrix for $J(4, 2)$ is given below.

$$A = \begin{array}{c|cccccc} & 12 & 13 & 14 & 23 & 24 & 34 \\ \hline 12 & 0 & 1 & 1 & 1 & 1 & 0 \\ 13 & 1 & 0 & 1 & 1 & 0 & 1 \\ 14 & 1 & 1 & 0 & 0 & 1 & 1 \\ 23 & 1 & 1 & 0 & 0 & 1 & 1 \\ 24 & 1 & 0 & 1 & 1 & 0 & 1 \\ 34 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}$$

As we did with the Hamming graph H(3,2), we can calculate the intersection matrices. The diameter of $J(4,2)$ is 2. So we will only have intersection matrices B and B_2 which will be 3×3 matrices. Using the intersection numbers derived in Lemma 5.3.1, we can generate B by calculating up to $i = 2$. Thus,

$$B = \begin{pmatrix} p_{01}^0 & p_{01}^1 & p_{01}^2 \\ p_{11}^0 & p_{11}^1 & p_{11}^2 \\ p_{21}^0 & p_{21}^1 & p_{21}^2 \end{pmatrix} = \begin{pmatrix} a_0 & c_1 & 0 \\ b_0 & a_1 & c_2 \\ 0 & b_1 & a_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 4 & 2 & 4 \\ 0 & 1 & 0 \end{pmatrix}$$

Continuing as we did for the $H(3,2)$, we will use the recurrence relation on the intersection matrices to generate B_2. So,

$$\begin{aligned} B_2 &= B_{1+1} \\ &= \frac{1}{c_2}[BB - b_0 B_0 - a_1 B] \\ &= \frac{1}{4}[B^2 - 4B_0 - 2B] \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{aligned}$$

Now that we have the intersection matrices, we can calculate the eigenvalues and eigenmatrix for $J(4,2)$. As predicted by equation (12), the eigenmatrix P for the distance-regular graph $J(4,2)$ is given by

$$P(\Gamma) = \begin{pmatrix} 1 & 4 & 1 \\ 1 & 0 & 1 \\ 1 & -2 & -1 \end{pmatrix}$$

6. Feasibility of Given Parameters

We were given specific intersection numbers and worked to determine if a distance-regular graph with diameter d was feasible. In general, there are some basic conditions that must hold for a graph to be feasible. We have already mentioned that the intersection numbers, p_{ij}^h, must be nonnegative integers. Also, we have that for any vertex v of a distance-regular graph Γ with diameter d, k_i, the number of vertices in $\Gamma_i(v)$, must be an integer for $(2 \le i \le d)$.

The parameters we were given are:

$$c_i = \frac{3i(d-i+1)(d+i+1)}{d(d+2)(2i+1)}, \quad b_i = \frac{3(d-i)(i+1)(d+i+2)}{d(d+2)(2i+1)} \tag{14}$$

We were able to determine the following results.

Proposition 6.0.4 *Given the parameters defined in (14), then* $k = 3$.

Proof. By definition, $k = b_0$. Thus for the given parameters,

$$k = b_0 = \frac{3(d-0)(0+1)(d+0+2)}{d(d+2)(0+1)} = 3.$$

□

Proposition 6.0.5 *Given the parameters defined in (14), then* $k_i = 2i+1$.

Proof. This result follows by induction on i. For $k_0 = 1$ and $k_1 = k = 3$ the recurrence holds. Suppose true for i and consider k_{i+1}. From equation (3), $k_{i+1} = k_i \frac{b_i}{c_{i+1}}$, which equals $2i+3$ by (14). □

So one of our feasibility constraints is meet. Now to get the last parameter, a_i and begin checking the other constraint.

Proposition 6.0.6 *Given the parameters defined in (14), then*

$$a_i = \frac{3i(i+1)}{d(d+2)}$$

.

Proof. From equation (2), $a_i + b_i + c_i = k$, so it follows that given our parameters,

$$\begin{aligned} a_i &= k - b_i - c_i \\ &= k - \frac{3i(d-i+1)(d+i+1)}{d(d+2)(2i+1)} - \frac{3(d-i)(i+1)(d+i+2)}{d(d+2)(2i+1)} \\ &= \frac{d^2(k-3) + 2d(k-3) + 3i(i+1)}{d(d+2)} \\ &= \frac{3i(i+1)}{d(d+2)}, \end{aligned}$$

when $k = 3$. □

6.1 Intersection Numbers Investigated

We can set up the intersection matrix, B for our given parameters although we won't have a set diameter and so the dimension for the intersection matrix is undetermined.

$$B = \begin{pmatrix} a_0 & c_1 & 0 & \cdots & 0 \\ b_0 & a_1 & c_2 & \cdots & 0 \\ 0 & b_1 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & c_d \\ 0 & 0 & \cdots & b_{d-1} & a_d \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 3 & \frac{6}{d(d+2)} & \frac{6(d-1)(d+3)}{5d(d+2)} & \cdots & 0 \\ 0 & \frac{6(d-1)(d+3)}{3d(d+2)} & \frac{18}{d(d+2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \frac{3}{d+2} \\ 0 & 0 & \cdots & \frac{3d(2d+1)}{d(d+2)(2d-1)} & \frac{3(d+1)}{d+2} \end{pmatrix} \qquad (15)$$

For our parameters to be considered feasible, each entry of the intersection matrices must be nonnegative. Recall the entries of intersection matrices are the intersection numbers for the graph. These can be determined using the recursive formula given in equation (8).

Conjecture 6.1.1 *The entries in $B_0, B, B_2, \ldots, B_d$ are all nonnegative.*

Partial results that support the conjecture.

Proposition 6.1.2 *Intersection matrices B_0 and B are nonnegative.*

Proof. Clearly $B_0 = I$ is nonnegative. B is as equation (15) and since a_i, b_i, and c_i are nonnegative so will the matrix. □

Proposition 6.1.3 *Intersection matrix B_2 is nonnegative.*

Proof. This result follows by applying the triangle inequality to the recurrence relation on the intersection numbers given in 8. Specifically, the intersection numbers found in B_2 are given by:

$$p_{i2}^h = \frac{1}{c_2}[b_{i-1}p_{i-11}^h + (a_i - a_1)p_{i1}^h + c_{i+1}p_{i+11}^h - b_0 p_{i0}^h] \qquad (16)$$

By the triangle inequality, the only nonzero intersection numbers will be when the sum of any two of i, h, and 2 do no exceed the third term. Thus we have the following cases.

Case 1: $h = i-2$. This reduces equation (16) to

$$\begin{aligned} p_{i2}^{i-2} &= \frac{1}{c_2}[b_{i-1}b_{i-2}] \\ &= 15/2\,\frac{i\,(i-1)\,(i-1-d)\,(i-2-d)\,(d+i+1)\,(d+i)}{d\,(d+2)\,(2\,i-1)\,(2\,i-3)\,(d-1)\,(d+3)} \end{aligned}$$

which is nonnegative since $i \le d$ for all i.

Case 2: $h = i-1$. This reduces equation (16)

$$\begin{aligned} p_{i2}^{i-1} &= \frac{1}{c_2}[b_{i-1}(a_{i-1} + a_i - a_1)] \\ &= -15\,\frac{i\,(i-1)\,(i+1)\,(d+i+1)\,(i-1-d)}{(2\,i-1)\,(d-1)\,(d+3)\,d\,(d+2)} \end{aligned}$$

which by observation is nonnegative.
Case 3: $h = i$. This reduces equation (16) to

$$\begin{aligned} p^i_{i2} &= \frac{1}{c_2}[b_{i-1}c_i + (a_i - a_1)a_i + c_{i+1}b_i - b_0)] \\ &= 5\,\frac{i\,(i+1)\,(3\,i^2+3\,i-3-2\,d-d^2)^2}{(2\,i-1)\,(2\,i+3)\,(d+2)\,d\,(d-1)\,(d+3)} \end{aligned}$$

which we can observe to be nonnegative.
Case 4: $h = i + 1$. This reduces equation (16) to

$$\begin{aligned} p^{i+1}_{i2} &= \frac{c_{i+1}}{c_2}[a_{i+1} + a_i - a_1] \\ &= -15\,\frac{i\,(i+2)\,(i+1)\,(d+i+2)\,(i-d)}{(2\,i+3)\,(d+2)\,d\,(d-1)\,(d+3)} \end{aligned}$$

Case 5: $h = i + 2$. This reduces equation (16) to

$$\begin{aligned} p^{i+2}_{i2} &= \frac{1}{c_2}[c_{i+1}c_{i+2}] \\ &= 15/2\,\frac{(i+2)\,(i+1)\,(i-d)\,(d+i+3)\,(d+i+2)\,(i+1-d)}{d\,(d+2)\,(2\,i+3)\,(2\,i+5)\,(d-1)\,(d+3)} \end{aligned}$$

If $d > i$, then p^{i+2}_{i2} is nonnegative and if $d = i$ then $p^{i+2}_{i2} = 0$. Thus p^{i+2}_{i2} is nonnegative. Therefore, the entries of B_2 are nonnegative. □

Conjecture 6.1.4 *The entries of the intersection matrix B_3 are nonnegative.*

Proof. The intersection numbers found in B_3 are given by the recurrence relation of equation (8). Specifically, the intersection numbers found in B_3 are given by

$$p^h_{i3} = \frac{1}{c_3}[b_{i-1}p^h_{i-12} + (a_i - a_2)p^h_{i2} + c_{i+1}p^h_{i+12} - b_1 p^h_{i1}]. \tag{17}$$

By applying the triangle inequality, we have the following cases.
Case 1: $h = 1 - 3$. This reduces equation (17) to

$$\begin{aligned} p^{i-3}_{i3} &= \frac{1}{c_3}[b_{i-1}p^{i-3}_{i3}] \\ &= \frac{1}{c_3 c_2}[b_{i-3}b_{i-2}b_{i-1}] \\ &= -\frac{35}{2}\,\frac{i\,(i-1)\,(i-2)\,(i-1-d)\,(i-2-d)\,(i-3-d)\,(d+i+1)\,(d+i)\,(d+i-1)}{d\,(d+2)\,(2\,i-1)\,(2\,i-3)\,(2\,i-5)\,(d-1)\,(d+3)\,(d-2)\,(d+4)} \end{aligned}$$

Case 2: $h = i - 2$. This reduces equation (17) to

$$p^{i-2}_{i3} = \frac{1}{c_3}[b_{i-1}p^{i-2}_{i-12} + (a_i - a_2)p^{i-2}_{i2}]$$

$$
\begin{aligned}
&= \frac{1}{c_3}[b_{i-1}b_{i-2}(a_{i-2}+a_{i-1}-a_1)+(a_i-a_2)b_{i-1}b_{i-2}] \\
&= \frac{b_{i-1}b_{i-2}}{c_3c_2}[a_{i-2}+a_{i-1}+a_i-a_1-a_2] \\
&= \frac{105}{2}\,\frac{i\,(i-1)\,(i-2)\,(i+1)\,(i-1-d)\,(i-2-d)\,(d+i+1)\,(d+i)}{(2\,i-1)\,(2\,i-3)\,(d-1)\,(d+3)\,(d-2)\,(d+4)\,d\,(d+2)}
\end{aligned}
$$

Case 3: $h = i-1$. This reduces equation (17) to

$$
\begin{aligned}
p_{i3}^{i-1} &= \frac{1}{c_3}[b_{i-1}p_{i-12}^{i-2}+(a_i-a_2)p_{i2}^{i-2}+c_{i+1}p_{i+12}^{i-1}] \\
&= \frac{1}{c_3c_2}[b_{i-1}b_{i-2}(a_{i-2}+a_{i-1}-a_1)+(a_i-a_2)b_{i-1}b_{i-2}+c_{i+1}b_ib_{i-1}] \\
&= \frac{1}{c_3c_2}[b_{i-1}b_{i-2}(a_{i-2}+a_{i-1}+a_i-a_1-a_2)+c_{i+1}b_ib_{i-1}] \\
&= -21/2\,\frac{(i-1)\,(i+1)\,(d+i+1)\,(i-1-d)\,(5\,i^2-12-2\,d-d^2)^2\,i}{d\,(2\,i-3)\,(2\,i+3)\,(d+2)\,(2\,i-1)\,(d-1)\,(d+3)\,(d-2)\,(d+4)}
\end{aligned}
$$

Case 4: $h = i$. This reduces equation (17) to

$$
\begin{aligned}
p_{i3}^{i} &= \frac{1}{c_3}[b_{i-1}p_{i-12}^{i}+(a_i-a_2)p_{i2}^{i}+c_{i+1}p_{i+12}^{i}-b_1p_{i1}^{i}] \\
&= \frac{1}{c_3}[\frac{b_{i-1}c_ic_{i+1}}{c_2}+a_i-a_2c_2[b_{i-1}c_i+(a_i-a_1)a_i+c_{i+1}b_i-b_0] \\
&\qquad +\frac{c_{i+1}}{c_2}[b_ib_{i-1}]-b_1a_i] \\
&= \frac{1}{c_3c_2}[b_{i-1}c_ic_{i+1}+(a_i-a_2)(b_{i-1}c_i+(a_i-a_1)a_i+c_{i+1}b_i-b_0)+c_{i+1}b_ib_{i-1}] \\
&\qquad -\frac{b_1a_i}{c_3} \\
&= 7\,\frac{i\,(i-1)\,(i+2)\,(i+1)\,(5\,i^2+5\,i-6-6\,d-3\,d^2)^2}{(2\,i+3)\,(d+2)\,(d+4)\,(d-2)\,(d+3)\,(d-1)\,(2\,i-1)\,d}
\end{aligned}
$$

Case 5: $h = i+1$. This reduces equation (17) to

$$
\begin{aligned}
p_{i3}^{i+1} &= \frac{1}{c_3}[b_{i-1}p_{i-12}^{i+1}+(a_i-a_2)p_{i2}^{i+1}+c_{i+1}p_{i+12}^{i+1}-b_1p_{i1}^{i+1}] \\
&= \frac{c_{i+1}}{c_3c_2}[b_{i-1}c_i+(a_{i+1}+a_i-a_1)(a_i-a_2)] \\
&\quad +\frac{c_{i+1}}{c_3c_2}[[b_ic_{i+1}+(a_{i+1}-a_1)a_{i+1}+c_{i+2}b_{i+1}-b_0]-b_1(a_{i+1}+a_i-a_1)] \\
&= -21/2\,\frac{i\,(i+2)\,(i+1)\,(d+i+2)\,(i-d)\,(5\,i^2+10\,i-7-2\,d-d^2)^2}{(2\,i-1)\,d\,(d+3)\,(d-1)\,(2\,i+5)\,(d-2)\,(d+4)\,(d+2)\,(2\,i+3)}
\end{aligned}
$$

Case 6: $h = i + 2$. This reduces equation (17) to

$$\begin{aligned} p_{i3}^{i+2} &= \frac{1}{c_3}[(a_i - a_2)p_{i2}^{i+2} + c_{i+1}p_{i+12}^{i+1}] \\ &= \frac{1}{c_3 c_2}[(a_i - a_2)c_{i+1}c_{i+1} + c_{i+1}[b_i c_{i+1} + (a_{i+1} - a_1)a_{i+1} + c_{i+2}b_{i+1} - b_0]] \\ &= \frac{105}{2}\,\frac{(i+3)\,(i+2)\,(i+1)\,(d+i+3)\,(d+i+2)\,(i+1-d)\,(i-d)\,i}{d\,(d+3)\,(d-1)\,(2\,i+5)\,(d-2)\,(d+4)\,(d+2)\,(2\,i+3)} \end{aligned}$$

Case 7: $h = i + 3$. This reduces equation (17) to

$$\begin{aligned} p_{i3}^{i+3} &= \frac{1}{c_3}[c_{i+1}p_{i+12}^{i+3}] \\ &= \frac{c_{i+2}c_{i+3}}{c_3 c_2} \end{aligned}$$

which Maple simplified to

$$-\frac{35}{2}\,\frac{(i+3)\,(i+2)\,(i+1)\,(d+i+4)\,(d+i+3)\,(d+i+2)\,(i+2-d)\,(i+1-d)\,(i-d)}{d\,(d+2)\,(2\,i+5)\,(2\,i+7)\,(d-1)\,(d+3)\,(2\,i+3)\,(d-2)\,(d+4)}$$

□

Thus, B_3 is nonnegative.

Proposition 6.1.5

For distance-regular graph Γ with intersection matrix B as given, and diameter d, the i^{th} eigenvalue of B is given by

$$\theta_i = \theta_0 - \frac{6i(i+1)}{d(d+2)}$$

where $\theta_0 = 3$.

Proof. plug in a's, b's, and c's and

7. Conclusions

References

[1] Douglas B. West. *Introduction to Graph Theory*, Prentice Hall, New Jersey, 2001, 1-28.

[2] A.E. Brouwer, A.M. Cohen, A. Neumaier. *Distance-Regular Graphs*, Springer-Verlag, New York, 1980, 126-135.

[3] Norman Biggs. *Algebraic Graph Theory*, Cambridge University Press, Cambridge, 164-169.

[4] J. H. van Lint, R. M. Wilson. *A Course in Combinatorics*, Cambridge University Press, 1992, 364-374.

[5] Chris Godsil, Gordon Royle, *Algebraic Graph Theory*, Springer-Verlag, New York, 2001, 163-170.

[6] Lowell W. Beineke, Robin J. Wilson. *Topics in Algebraic Graph Theory*, Cambridge University Press, Cambridge, 2004, 30-50.