Contents

WEIGHTED FORMS OF EULER'S THEOREM

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1 Introduction

The goal of this paper is to state and prove two identities of Ramanujan in terms of weighted forms of Euler's theorem on partitions of integers into distinct and odd parts. In section two we will introduce the terminology necessary for the proof of Euler's theorem equating the number of distinct partitions of an integer with the number of odd partitions of an integer. Section three contains two proofs Euler's theorem, each using bijective techniques. Section four defines and employs the vocabulary of weighted partitions to prove several theorems. Section five resembles section four in form, introducing and using the terms of rooted partitions to prove a few theorems. Finally, we will motivate the corollaries and theorems in section six by stating them in terms of generating functions, yielding the identities of Ramanujan.

To begin here are the identities of Ramanujan that we will prove:

$$\sum_{n=0}^{\infty}[(-q;q)_\infty-(-q;q)_n]=(-q;q)_\infty\left[-\frac{1}{2}+\sum_{d=1}^{\infty}\frac{q^d}{1-q^d}\right]+\frac{1}{2}\left[1+\sum_{n=1}^{\infty}\frac{q^{\binom{n+1}{2}}}{(-q;q)_n}\right] \tag{1}$$

$$\sum_{n=0}^{\infty}\left[\frac{1}{(q;q^2)_\infty}-\frac{1}{(q;q^2)_n}\right]=(-q;q)_\infty\left[-\frac{1}{2}+\sum_{d=1}^{\infty}\frac{q^{2d}}{1-q^{2d}}\right]+\frac{1}{2}\left[1+\sum_{n=1}^{\infty}\frac{q^{\binom{n+1}{2}}}{(-q;q)_n}\right] \tag{2}$$

Note: Define $(x;q)_0 = 1$ and $(x;q)_n$ = (1-x)(1-qx)$\cdots$(1-$q^{n-1}x$) for n $\geq$ 1.

2 Terminology of Partitions

2.1 Simple Terms

Before proving Euler's theorem we need to define some elementary terms [3]. A *partition* λ of a positive integer n is a finite, non-increasing sequence of positive integers λ_1, λ_2,...,λ_r such that $\sum_{i=1}^{r}\lambda_i = n$. We call the λ_i *parts* of the partition. By convention, λ_1 is the largest part. The *length* of λ, $l(\lambda)$, is the number of parts in λ. Define $n_\lambda(d)$ to be the number of parts in λ equal to d. Note that $l(\lambda) = \sum_d n_\lambda(d)$. We define the *weight* of λ to be the sum of the parts of λ, denoted $|\lambda|$.

2.2 Rank and Conjugate

Two less elementary terms will be used heavily: *rank* and *conjugate partition* [3]. The *rank* of a partition λ is λ_1 - $l(\lambda)$, the largest part minus the number of parts. By convention, an empty partition has rank zero. Given a partition $\lambda = (\lambda_1, ..., \lambda_r)$, define the *conjugate partition* λ' of λ to be (λ'_1,λ'_2,...,λ'_t) where λ'_i is the number of parts of λ that are greater than or equal to i. Observe that $l(\lambda) = \lambda'_1$ and $l(\lambda')$= λ_1. Below is an illustration of a partition's Young diagram, which is defined formally two paragraphs down, and the Young diagram of the conjugate partitition.

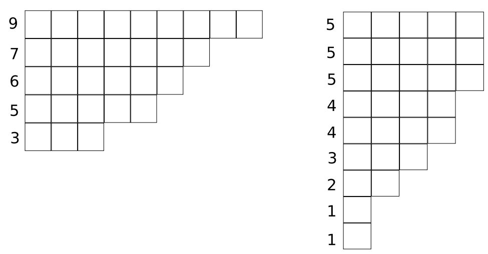

Figure 1: Partition and Conjugate Partition

Observe that conjugating a partition can be illustrated by reflecting a Young diagram about the line running through the upper left corner running down and to the right at a forty-five degree angle below horizontal.

2.3 Young Diagrams

Finally, we need to know what a Young diagram [1], [4] of a partition is. Given a partition λ of n whose parts are $\lambda_1, ..., \lambda_r$ we can construct a diagram as follows. Write a row of λ_1 boxes. Underneath this write a row of λ_2 boxes such that the left-most box in the second row is directly underneath the left-most box of the first row. Continue this process, writing down a left-justified row for each part of λ. The resulting array of boxes is the Young diagram of λ. The second proof we give of Euler's theorem will include illustrations of partitions using their Young diagrams.

2.4 Odd parts and Distinct parts

Before we dive into the proofs we should note that the set of partitions of n into distinct parts will be denoted by D_n, and the set of partitions of n into odd parts will be denoted by O_n. For example (3,3,1) and (4,2,1) are odd and distinct partitions of seven, respectively. Then Euler's theorem may be stated as $|D_n| = |O_n|$ for n $\geq$ 1.

3 Two Proofs of Euler's Theorem

3.1 A Note on the Conventional Proof

The conventional proof of Euler's theorem employing generating functions can be found in Stanton and White's text [4]. The two proofs detailed in this section do not appear in [4]. These proofs, relying on explicit bijections, are of value to us beyond their application to Euler's theorem because the bijections introduce techniques that will be used later in the paper.

3.2 Sylvester's Bijection

The first of the bijective proofs of Euler's theorem relies on Sylvester's bijection, ϕ. It is a map from the set of odd partitions to the set of distinct partitions.

Given odd partition λ of n, represent each part $2m + 1$ with a row of m 2's followed by a 1 on the right-hand end. Arrange the rows in descending order such that the largest part's row is the top row and the smallest part's row is the bottom row. Also arrange the rows so they all start at the same point on the left, forming a column of length $l(\lambda)$ on the left side of our diagram. This diagram is called the 2-modular diagram of the partition. Now decompose the diagram into hooks H_1, H_2,... with the diagonal boxes of our diagram as the corners. We define H_1 to be the hook comprised of the top row and the left most column of our diagram. Define H_i to be comprised of the top row and left most column, after the rows and columns comprising hooks H_k for all k, $i > k \geq 1$ have been removed.

The illustration below shows which boxes comprise each hook. Hook H_i is comprised of the boxes marked with integer i.

3	3	3	3
3	2	2	2
3	2	1	
3	2	1	
3	2	1	
3			

Figure 2: Illustration of hooks

We define a partition μ of n in terms of these hooks. Let μ_1 be the number of squares in H_1, let μ_2 be the number of 2's in H_1 and in general, let μ_{2i-1} be the number of squares in H_i, let μ_{2i} be the number of 2's in H_i. We write our bijection: $\phi(\lambda) = \mu$.

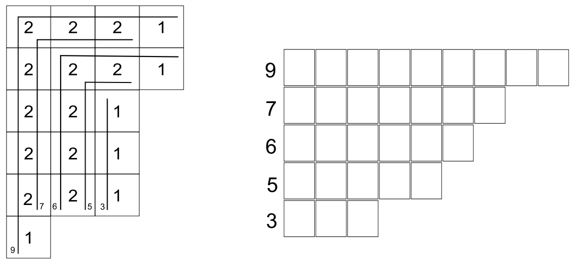

Figure 3: Odd to Distinct Bijection

Now we need to show that μ is a partition into distinct parts. Clearly $\mu_{2i-1} > \mu_{2i}$, because H_i has at least one 1, so by the construction $\mu_{2i-1} > \mu_{2i}$. Next $\mu_{2i} > \mu_{2i+1}$ because the number of squares in H_i is strictly greater than the number of boxes in H_{i+1}. Thus $\mu_k > \mu_{k+1}$ for $1 \leq k \leq l(\mu)$ -1.

To complete the proof of this one-to-one map's bijectivity we need to construct an inverse map ϕ^{-1}.

Let $\mu = (\mu_1,\mu_2,...,\mu_{2k-1},\mu_{2k})$ be a partition of n into distinct parts such that $\mu_i > 0$ for $1 \leq i \leq 2k$-1 and $\mu_{2k} \geq 0$. First look at part μ_{2k}. Write down a row of μ_{2k} 2's and put a 1 at the right-hand end of that row. Next add (μ_{2k-1} - μ_{2k} -1) 1's to the first (left most) column, appending these 1's underneath the bottom entry in the first column. Denote this hook by H_k. The 2's can only appear in the first row of this hook. Now consider parts μ_{2k-2} and μ_{2k-3}. The hook H_k is constructed as follows. There are μ_{2k-2} 2's and μ_{2k-3} - μ_{2k-2} 1's in H_{k-1}. If there is a 1 in the i-th column of H_k, then there must be a 2 immediately to the left of that 1. The rest of the 2's will be put in the first row of H_{k-1}. Next put a 1 at the end of the first row then put the rest of the 1's at the end of the first column. The above procedure is then repeated.

Thus we have a bijection between partitions of n into odd parts and partitions of n into distinct parts.

3.3 Iterated Dyson's Map

Now we offer a second bijective proof of Euler's theorem. In this proof we use a bijection called iterated Dyson's map. This of course should be prefaced with a definition of Dyson's map.

Dyson's map, ψ_r, is a bijection between the sets $H_{n,r+1}$ and $G_{n+r,r-1}$ where the sets $H_{n,r}$ and $G_{n,r}$ are the sets of partitions of n with rank at most r and at least r, respectively.

To demonstrate this, consider the partition $\lambda \in H_{n,r+1}$. Working with the Young diagram of λ, remove the first column. To this new diagram, add a row, on top of the existing top row, consisting of $r + l(\lambda)$ boxes. This new diagram is a Young diagram of a partition $\mu \in G_{n+r,r-1}$. The inverse of this map clear from the illustration. Thus Dyson's map is a bijection.

Now we may define iterated Dyson's map ϕ: $O_n \rightarrow D_n$, another bijection between the odd partitions of n and the distinct partitions of n.

Let $\lambda = (\lambda_1, ..., \lambda_l)$ be a partition of n into odd parts. We construct a partition μ of n from λ as follows. Let $\nu^l = \lambda_l$ and let ν^i denote the partition obtained by applying Dyson's map ψ_{λ_i} to ν^{i+1},

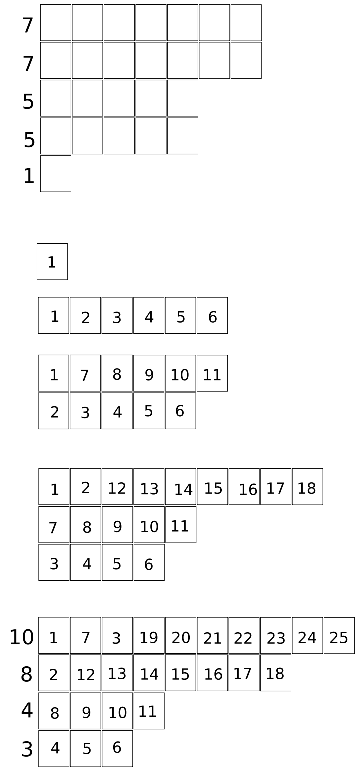

Figure 4: Odd to Distinct map ϕ^{-1}

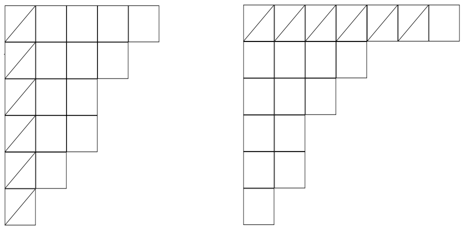

Figure 5: $H_{n,r+1}$ to $G_{n+r,r-1}$ with $n = 18$, $r = 1$

i.e. $\nu^i = \psi_{\lambda_i}(\nu^{i+1})$. Finally, set $\mu = \nu^1$. Because $\nu^i = \lambda_i + \lambda_{i+1} + \cdots + \lambda_l$ we see that $|\mu| = |\lambda|$. It is clear this is a partition into distinct parts as each part is strictly greater than the previous part created by the iterative process.

The inverse of iterated Dyson's map is stated in terms of a recursive procedure. Let $\mu = (\mu_1, \mu_2, ..., \mu_l)$ be a partition of n into distinct parts. Start with $\lambda_1 = r(\mu) = \mu_1 - l(\mu)$ if $\mu_1 - l(\mu)$ is odd. Otherwise set $\lambda_1 = r(\mu) + 1 = \mu_1 - l(\mu) + 1$. Now we apply $\psi^{-1}_{\lambda_i}$ to μ, obtaining partition $\nu^2 = \psi^{-1}_{\lambda_i}(\lambda)$. [Here $\psi^{-1}_{\lambda_i}$ is the inverse of Dyson's ψ_{λ_i} map.] Now we iterate the above procedure, applying it to ν^j for j = 2,3,4,..., we generate the partition $\lambda = (\lambda_1, \lambda_2, ...)$ with odd parts.

4 Weighted Forms of Euler's Theorem

Now that we are fluent these bijections we may make a few observations and introduce some weighted forms of Euler's theorem. First note that Sylvester's bijection shows us that each partition μ of n into distinct parts with maximum part μ_1 corresponds to a partition λ of n into odd parts with maximum part λ_1 such that $\mu_1 = \frac{\lambda_1 - 1}{2} + l(\lambda)$ or $2\mu_1 + 1 = \lambda_1 + 2l(\lambda)$. From this we get the following weighted form of Euler's theorem.

Theorem 4.1 *The sum of μ_1 (or $2\mu_1 + 1$) over all the partitions μ of n into distinct parts equals the sum of $\frac{\lambda_1-1}{2} + l(\lambda)$ (or $\lambda_1 + 2l(\lambda)$) over all partitions λ of n into odd parts, written as*

$$\sum_{\mu \epsilon D_n} \mu_1 = \sum_{\lambda \epsilon O_n} \left(\frac{\lambda_1 - 1}{2} + l(\lambda) \right) \tag{3}$$

or equivalently,

$$\sum_{\mu \epsilon D_n} (2\mu_1 + 1) = \sum_{\lambda \epsilon O_n} (\lambda_1 + 2l(\lambda)) \,. \tag{4}$$

The iterated Dyson's map shows us that a partition λ of n into odd parts with maximum part λ_1 corresponds to a partition μ of n into distinct parts with rank $r(\mu)$ such that

$$r(\mu) + \frac{1 + (-1)^{r(\mu)}}{2} = \lambda_1. \tag{5}$$

This equation is spelled out in the description of the inverse of iterated Dyson's map. In that description is contained "Start with $\lambda_1 = r(\mu) = \mu_1 - l(\mu)$ if $\mu_1 - l(\mu)$ is odd. Otherwise set $\lambda_1 = r(\mu) + 1 = \mu_1 - l(\mu) + 1$." The equation is equivalent to that instruction.

From this we get a second weighted form of Euler's theorem:

Theorem 4.2 *The sum of μ_1 - $l(\mu)$ + $\frac{1+(-1)^{r(\mu)}}{2}$ over all partitions μ into distinct parts equals the sum of λ_1 over all partitions λ of n into odd parts, or,*

$$\sum_{\mu \epsilon D_n} \left(\mu_1 - l(\mu) + \frac{1 + (-1)^{r(\mu)}}{2} \right) = \sum_{\lambda \epsilon O_n} \lambda_1. \tag{6}$$

Now consider the set of partitions of μ of n indo distinct parts with multiplicities $l(\mu) + \mu_1 + \frac{1-(-1)^{r(\mu)}}{2}$. The number of such partitions of n with the multiplicities taken into account equals the number of elements in the set of partitions of n into distinct parts with multiplicities $2\mu_1$ + 1 minus the number of elements in the set of partitions of n into distinct parts with multiplicities μ_1 - $l(\mu)$ + $\frac{1+(-1)^{r(\mu)}}{2}$. Then Theorems 4.1 and 4.2 give us another weighted form of Euler's theorem.

Theorem 4.3 *The sum of $l(\mu) + \mu_1 + \frac{1-(-1)^{r(\mu)}}{2}$ over all partitions μ of n into distinct parts equals the sum of $2l(\lambda)$ over partitions λ of n into odd parts, or,*

$$\sum_{\mu \epsilon D_n} \left(l(\mu) + \mu_1 + \frac{1 - (-1)^{r(\mu)}}{2} \right) = \sum_{\lambda \epsilon O_n} 2l(\lambda). \tag{7}$$

We have one more theorem using weighted partitions. Consider the set of partitions μ of n into distinct parts with multiplicities $l(\mu) + \frac{1-(-1)^{r(\mu)}}{2}$. The number of partitions meeting this demand equals the number of elements in the set of partitions of n into distinct parts with multiplicities μ_1 + 1 minus the number of elements in the set of partitions of n into distinct parts with multiplicities μ_1 - $l(\mu)$ + $\frac{1+(-1)^{r(\mu)}}{2}$. This follows from Euler's theorem and Theorems 1 and 2. From this we obtain the following weighted form of Euler's theorem.

Theorem 4.4 *The sum of $l(\mu) + \frac{1-(-1)^{r(\mu)}}{2}$ over all the partitions μ of n into distinct parts equals the sum of $l(\lambda)$-$\frac{\lambda_1-1}{2}$ over all the partitions λ of n into odd parts, or,*

$$\sum_{\mu \epsilon D_n} \left(l(\mu) + \frac{1-(-1)^{r(\mu)}}{2} \right) = \sum_{\lambda \epsilon O_n} \left(l(\lambda) - \frac{\lambda_1 - 1}{2} \right). \tag{8}$$

Before moving to the next section, consider the following example. We will apply theorems 4.3 and 4.4 to the distinct and odd partitions of seven. First note that the distinct and odd partitions of seven are {(7),(6,1),(5,2),(4,3),(4,2,1)} and {(7),(5,1,1),(3,3,1),(3,1,1,1,1),(1,1,1,1,1,1,1)}, respectively. The left-hand side of equation (7) is in this case 10 + 26 + 2 = 38. The right-hand side of equation (7) is 2(19) = 38! Now let's look at theorem 4.4. The left-hand side of equation (8) is 10 + 2 = 12. The left-hand side of equation (8) is 10 + 2 = 12. The right-hand side is 19 - 7 = 12!

5 Rooted Partitions and Euler's Theorem

5.1 Vocabulary of Rooted Partitions

We are now ready to introduce rooted partitions, their vocabulary and allied theorems. In a sense, a rooted partition is a partition of a partition.

A *rooted partition* of n is a pair of partitions (λ,μ), where $|\lambda| + |\mu| = n$ and μ is a nonempty partition into equal parts. Another way to consider a rooted partition of n is to consider a partition of n in which some nonempty subset of equal parts of the partition is marked. We give some examples using an overline to mark the elements of μ in a rooted partition (λ,μ).

Here are the twelve rooted partitions of 4:

$\overline{4}, \overline{3}+1, 3+\overline{1}, \overline{2}+2, \overline{2}+\overline{2}, \overline{2}+1+1, 2+\overline{1}+1, 2+\overline{1}+\overline{1}, \overline{1}+1+1+1, \overline{1}+\overline{1}+1+1, \overline{1}+\overline{1}+\overline{1}+1, \overline{1}+\overline{1}+\overline{1}+\overline{1}.$

As one might expect, we are interested in rooted partitions with odd parts and rooted partitions with distinct parts. Below are examples of these.

The three rooted partitions of 4 into distinct parts are $\overline{4}, \overline{3}+1, 3+\overline{1}$.

The six rooted partitions of 4 into odd parts are
$\overline{3}+1, 3+\overline{1}, \overline{1}+1+1+1, \overline{1}+\overline{1}+1+1, \overline{1}+\overline{1}+\overline{1}+1, \overline{1}+\overline{1}+\overline{1}+\overline{1}.$

Here is a "constructive" way to look at rooted partitions. Begin with a bipartition of 7 such that the second part is non-empty, say (3,4). Next we may generate a rooted partition of 7 from the bipartition (3,4) by partitioning 3 any way we please and partitioning 4 such that the parts are uniform in size. We call the partition of 3 λ and the partition of 4 μ. Together they form one of the rooted partitions of 7, (λ,μ).

We need two more bits of terminology. A rooted partition (λ,μ) with almost distinct parts is a rooted partition such that the parts of λ are distinct. The root size of a rooted partition (λ,μ) is the number of parts in μ.

5.2 Rooted Partition Theorems

Now we are ready to prove our final batch of theorems. There are four theorems that each require proofs. These four theorems collectively yield four more theorems which are little more than corollaries of the original foursome. Once the corollaries have been recorded we will have all the statements we need for the proofs of Ramanujan's identities. Those proofs involve translating our statements into generating functions, the objects whose form Ramanujan's identities embody. The terms in the following theorems are so lengthy in description that we shall give them single letter names in order that the relationships between the theorems may be easily observed.

Theorem 5.1 *The number of rooted partitions of n into almost distinct parts with odd root size (A) equals the number of rooted partitions of n into almost distinct parts with even root size (B) + the number of rooted partitions of n with distinct parts (C), or A = B + C.*

Proof. We will use an involution τ on the rooted partitions of n with almost distinct parts, excluding those with strictly distinct parts. (These are the partitions (λ, μ) where λ has distinct parts and μ either has multiple parts or λ has a part that is the same size as the parts found in μ). We define τ casewise.

Case 1: Given a rooted partition (λ, μ) with almost distinct parts but not distinct parts, if λ contains the part of μ, move that part to μ.

Case 2: Given a rooted partition (λ, μ) with almost distinct parts but not distinct parts, if λ does not contain the part of μ, move one of the parts of μ to λ.

Clearly this is an involution between the sets of rooted partitions with almost distinct parts but not distinct parts, alternating the parity of the root size. Thus the number of partitions of n with almost distinct parts (but not distinct parts) having even root size equals the number of partitions of n with almost distinct parts (but not distinct parts) having odd size. Add the number of rooted partitions of n with distinct parts to both sides and we get A = B + C, theorem 5.1. QED

Theorem 5.2 *The number of rooted partitions of n into almost distinct parts with odd root size (A) equals the number of rooted partitions of n with odd parts (D), or A = D.*

Proof. Sylvester's bijection returns, this time mapping the set of rooted partitions of n into almost distinct parts with odd root size to the set of partitions of n with odd parts. We will use Sylvester's bijection on the non-root part of the rooted partition, and conjugate the root. We are actually using two bijections to create the bijection we need.

σ: Starting with a rooted partitions (λ, μ) into almost distinct parts with odd root size, apply the inverse map of Sylvester's bijection, ϕ^{-1}, to λ to generate a partition α with odd parts. Let β be the conjugate of μ. Thus β is a partition all of whose parts are odd and equal to each other. The parts of β are equal to $l(\mu)$. Thus our new partition (α, β) is a rooted partition with odd parts.

σ^{-1}: Given a rooted partition $\alpha, \beta)$ with odd parts, apply Sylvester's bijection to α to generate a partition λ with distinct parts. Let μ be the conjugate of β, a partition of equal odd parts. Thus μ has an odd number of parts. We now have a rooted partition (λ, μ) into almost distinct parts with odd root size. QED

Theorem 5.3 *The number of rooted partitions of n into almost distinct parts with even root size (B) plus the number of rooted partitions of n with distinct parts (C) equals the number of rooted partitions of n with odd parts (D), or B + C = D.*

Proof. Theorem 5.1 tells us A = B + C and theorem 5.2 tells us A = D. Thus theorem 5.3, B + C = D, follows easily. QED

Theorem 5.4 *The number of rooted partitions of n with almost distinct parts (A + B) plus the number of rooted partitions of n with distinct parts (C) equals twice the number of rooted partitions of n with odd parts (2D), or A + B + C = 2D.*

Proof. Again, theorem 5.2 says A = D and theorem 5.3 says B + C = D, so clearly A + B + C = 2D. QED

Now we have another pair of theorems whose proofs are a little more sophisticated than the last two.

Theorem 5.5 *The number of rooted partitions of n with distinct parts (C) equals the sum of the lengths over the partitions of n with distinct parts (E), or C = E.*

Proof. Given a partition α with distinct parts, we can construct $l(\alpha)$ distinct rooted partitions (λ, μ) with distinct parts by selecting any part of α to the part of μ, leaving the rest of the parts of α to comprise λ. This map can clearly be reversed, simply by removing the root status from μ. QED

Theorem 5.6 *The number of rooted partitions of n with odd parts (D) equals the sum of the lengths over the partitions of n with odd parts (F), or D = F.*

Proof. Any partition β of n with odd parts can be turned into $l(\beta)$ distinct rooted partitions (λ, μ) of n with odd parts by designating any part of β as the part of μ and leaving the remaining parts of β to form λ. Assume d appears m times ($m \geq 2$) in β. We may choose μ as the partition with d repeated i times, i = 1,2,...,m. QED

Finally, we have two more easy theorems, or theorems whose proofs are easy, given the preceding work.

Theorem 5.7 *The number of rooted partitions of n with almost distinct parts (A + B) equals the sum of twice the lengths over partitions of n with odd parts (2F) minus the sum of the lengths of the partitions of n with distinct parts (E), or A + B = 2F - E.*

Proof. Applying theorems 5.5 and 5.6, D = F and C = E, respectively, to theorem 5.4 which says A + B + C = 2D, we get A + B = 2D - C = 2F - E. QED

Theorem 5.8 *The number of rooted partitions of n into almost distinct parts with even root size (B) equals the sum of lengths over the partitions of n into odd parts (F) minus the sum of lengths over partitions of n into distinct parts (E), or B = F - E.*

Proof. Theorem 5.1 says B + C = D thus B = D - C. Theorems 5.5 and 5.6 permit the substitution of F and E for D and C, respectively, thus B = F - E. QED

6 Generating Functions and Ramanujan's Identities

In this section we rewrite the equations found theorems 4.3 and 4.4 in terms of generating functions with the goal of putting those equations into the algebraic form found in Ramanujan's identities. In this algebraic form we see that theorems 4.3 and 4.4's equations are identical in form and content to Ramanujan's identities.

6.1 Section 5 terms in Brief

Before we begin work with the preceding theorems and quantities in effort to produce Ramanujan's indentities, we shall review those quantities and theorems in brief.

First we will reproduce all the definitions that are introduced in theorems 5.1 through 5.8.

A is the number of rooted partitions of n into almost distinct parts with odd root size.
B is the number of rooted partitions of n into almost distinct parts with even root size.
C is the number of rooted partitions of n into distinct parts.
D is the number of rooted partitions of n with odd parts.
E is the sum of the lengths over the partitions of n with distinct parts.
F is the sum of the lengths over the partitions of n with odd parts.

The theorems have been ordered against their numbering to better illustrate their dependencies. The last four theorems in this list are easily demonstrated using the first four. The first four require proofs independent of the other theorems in the list.

Theorem 5.1 $A = B + C$
Theorem 5.2 $A = D$
Theorem 5.5 $C = E$
Theorem 5.6 $D = F$
Theorem 5.3 $B + C = D$
Theorem 5.4 $A + B + C = 2D$
Theorem 5.7 $A + B = 2F - E$

Theorem 5.8 $B = F - E$

6.2 Generating Functions

Generating functions are algebraic objects whose manipulation can be made useful counting possible solutions to a given problem. In more detail, a generating function is often given as power series expansion of an infinitely differentiable function. The coefficients of the power series of the function are the solutions to the problem for which the function is appropriate [1].

More detail is beyond the scope of this paper, however an example that most people know is worth brief mention. The Fibonacci seqence begins 0,1,1,2,3,5,... Each term is equal to the sum of the previous two terms, a rule that can be formally stated as the recurrence relation $f_n = f_{n-1} + f_{n-2}$. Without going into the details, the generating function for the Fibonacci sequence is

$$g(x) = \sum_{n=0}^{\infty} f_n x^n \tag{9}$$

where

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n, (n \geq 0) \tag{10}$$

The coefficient, f_n, of the nth term of this power series is equal to the nth term of the Fibonacci sequence [2].

The generating functions for the number of odd partitions and the number of disintct partitions are

$$\prod_{i=0}^{\infty}\left(1 - x^{(2i+1)}\right)^{-1} \tag{11}$$

and

$$\prod_{i=0}^{\infty}\left(1 + x^{(i+1)}\right), \tag{12}$$

respectively [4]. Note that $\prod_{i=0}^{\infty}(1 + x^{(i+1)}) = (-x; x)_{\infty}$. This is formally very similar to Ramanujan's formulae. It is this algebraic form that when visited upon theorems 4.3 and 4.4 will show them to be identitical to Ramanujan's identities.

6.3 Ramanujan's First Identity

Ramanujan's first identity, equation (1), may be derived from theorem 4.3 with the aid of the alphabet soup theorems given in section 5. In particular it is theorem 5.7 that will be used.

Theorem 4.3 states

$$\sum_{\mu \epsilon D_n}\left(l(\mu) + \mu_1 + \frac{1-(-1)^{r(\mu)}}{2}\right) = \sum_{\lambda \epsilon O_n} 2l(\lambda). \tag{13}$$

First we will translate this into generating function terminology then manipulate it a little. The generating function form of theorem 4.3 is

$$\sum_{\mu \epsilon D}\left(l(\mu) + \mu_1 + \frac{1-(-1)^{r(\mu)}}{2}\right) q^{|\mu|} = \sum_{\lambda \epsilon O} 2l(\lambda) q^{|\lambda|}. \tag{14}$$

Which may be written

$$\sum_{\mu\epsilon D}\mu_1 q^{|\mu|}+\sum_{\mu\epsilon D}\left(\frac{1-(-1)^{r(\mu)}}{2}\right)q^{|\mu|}=\sum_{\lambda\epsilon O}2l(\lambda)q^{|\lambda|}-\sum_{\mu\epsilon D}l(\mu)q^{|\mu|}. \tag{15}$$

or

$$\sum_{\mu\epsilon D}\mu_1 q^{|\mu|}=\sum_{\lambda\epsilon O}2l(\lambda)q^{|\lambda|}-\sum_{\mu\epsilon D}l(\mu)q^{|\mu|}-\sum_{\mu\epsilon D}\left(\frac{1-(-1)^{r(\mu)}}{2}\right)q^{|\mu|}. \tag{16}$$

or

$$\sum_{\mu\epsilon D}\mu_1 q^{|\mu|}=\sum_{\lambda\epsilon O}2l(\lambda)q^{|\lambda|}-\sum_{\mu\epsilon D}l(\mu)q^{|\mu|}-\sum_{\mu\epsilon D}^{r(\mu)odd}q^{|\mu|}. \tag{17}$$

We may say the following [3] about the left hand side of equation (17)

$$\sum_{\mu\epsilon D}\mu_1 q^{|\mu|}=\sum_{n=0}^{\infty}[(-q;q)_\infty-(-q;q)_n] \tag{18}$$

Thus the left hand side of equation (13) is the left hand side of Ramanujan's first identity. Theorem 5.7 tells us that A + B = 2F - E. We know [3]

$$A+B=(-q;q)_\infty\sum_{d=1}^{\infty}\frac{q^d}{1-q^d} \tag{19}$$

and

$$2F-E=\sum_{\lambda\epsilon O}2l(\lambda)q^{|\lambda|}-\sum_{\mu\epsilon D}l(\mu)q^{|\mu|}. \tag{20}$$

Making substitutions using equations (14) and (16), we transform equation (13) into

$$\sum_{n=0}^{\infty}[(-q;q)_\infty-(-q;q)_n]=(-q;q)_\infty\sum_{d=1}^{\infty}\frac{q^d}{1-q^d}-\sum_{\mu\epsilon D}^{r(\mu)odd}q^{|\mu|}. \tag{21}$$

We have one more expression to substitute. Note that [3]

$$-\frac{1}{2}(-q;q)_\infty+\frac{1}{2}\left[1+\sum_{n=1}^{\infty}\frac{q^{\binom{n+1}{2}}}{(-q;q)_n}\right]=\sum_{\mu\epsilon D}^{r(\mu)odd}q^{|\mu|}. \tag{22}$$

Performing the last substitution using equation (18) gives us Ramanujan's first identity.

$$\sum_{n=0}^{\infty}[(-q;q)_\infty-(-q;q)_n]=(-q;q)_\infty\left[-\frac{1}{2}+\sum_{d=1}^{\infty}\frac{q^d}{1-q^d}\right]+\frac{1}{2}\left[1+\sum_{n=1}^{\infty}\frac{q^{\binom{n+1}{2}}}{(-q;q)_n}\right] \tag{23}$$

6.4 Ramanujan's Second Identity

Ramanujan's second identity, equation (2), may be derived from theorem 4.4. Theorem 4.4 states

$$\sum_{\mu\epsilon D_n}\left(l(\mu)+\frac{1-(-1)^{r(\mu)}}{2}\right)=\sum_{\lambda\epsilon O_n}\left(l(\lambda)-\frac{\lambda_1-1}{2}\right). \tag{24}$$

As we did with theorem 4.3, we will translate this into generating function terminology then manipulate it into a more suitable form.

$$\sum_{\mu\epsilon D}\left(l(\mu)+\frac{1-(-1)^{r(\mu)}}{2}\right)q^{|\mu|}=\sum_{\lambda\epsilon O}\left(l(\lambda)-\frac{\lambda_1-1}{2}\right)q^{|\lambda|}. \tag{25}$$

We can rewrite this equation as

$$\sum_{\lambda\epsilon O}\left(\frac{\lambda_1-1}{2}\right)q^{|\lambda|}=\sum_{\lambda\epsilon O}l(\lambda)q^{|\lambda|}-\sum_{\mu\epsilon D}l(\mu)q^{|\mu|}-\sum_{\mu\epsilon D}\left(\frac{1-(-1)^{r(\mu)}}{2}\right)q^{|\mu|}. \tag{26}$$

Which can be stated more simply as

$$\sum_{\lambda\epsilon O}\binom{\frac{\lambda_1-1}{2}}{}q^{|\lambda|}=\sum_{\lambda\epsilon O}l(\lambda)q^{|\lambda|}-\sum_{\mu\epsilon D}l(\mu)q^{|\mu|}-\sum_{\mu\epsilon D}^{r(\mu)odd}q^{|\mu|}. \tag{27}$$

Translating each piece of this into q-series form [3] yeilds the identity

$$\sum_{n=0}^{\infty}\left[\frac{1}{(q;q^2)_\infty}-\frac{1}{(q;q^2)_n}\right]=(-q;q)_\infty\left[-\frac{1}{2}+\sum_{d=1}^{\infty}\frac{q^{2d}}{1-q^{2d}}\right]+\frac{1}{2}\left[1+\sum_{n=1}^{\infty}\frac{q^{\binom{n+1}{2}}}{(-q;q)_n}\right]. \tag{28}$$

Thus the two identities of Ramanujan say something about partitions as well as "merely" being formal statements about generating functions.

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