ALTERNATING SIGN MATRICES AND SYMMETRY

By

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Introduction

An Alternating Sign Matrix (or ASM) is an $n \times n$ matrix whose entries are each 0, 1, or -1 with the property that the sum of each row or column is 1, and the non-zero entries in any row or column alternate in sign. An example of a 6×6 ASM is

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The study of ASM's arose out of analyzing Dodgson's method for evaluating determinants [1]. The simplest of questions was posed: How many $n \times n$ ASM's are there? This question proved difficult to answer. David Robbins, Howard Rumsey, and William Mills conjectured in [2] that the number of $n \times n$ ASM's is given by the product

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

and it was more than 10 years before a satisfactory proof was given by Doron Zeilberger in [3].

The sequence (A_n) grows very quickly, its first ten terms being 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700. Surprisingly, this sequence was recognized by other mathematicians at the time, in particular George Andrews, as that one counting another class of objects: Descending Plane Partitions. In fact, counting ASMs is equivalent to many other counting problems. See [4] for a partial list of these.

Square ice

Near the time Zeilberger was presenting his proof, Greg Kuperberg made an important discovery. Alternating sign matrices had actually been studied before in detail, but under the radar of the mathematical community. What's more, this alternate theory on ASMs was unbeknownst even to its purveyors. What Kuperberg discovered was that physicists had long studied ASMs as objects they referred to as "square ice" states. Though physicists had not produced a formula for the number of square ice states, their work suffered mathematicians to "use a previous exercise" when they identified square ice with ASMs. This discovery allowed for a considerably shorter proof of the ASM conjecture using the pre-existing theory from physics. His proof can be found in [5].

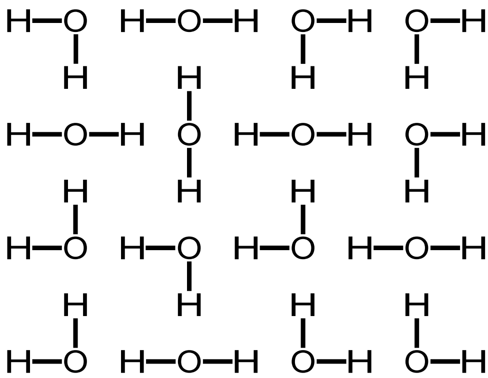


FIGURE 1. A Square Ice State

What is square ice? It is a 2-dimensional square lattice of water molecules with Oxygen atoms on the vertices of the grid and a Hydrogen atom between neighboring Oxygen atoms. In Figure 1, bonds between Oxygen and Hydrogen atoms are represented by a line.

This gives rise to a connected directed graph whose vertices represent the Oxygen atoms and directed edges represent the Hydrogen atoms. Edges are oriented so that their terminal vertex is the Oxygen atom to which the Hydrogen atom is bonded. This graph is the view of square ice we will take in this paper. Figure 2 below shows the directed graph of the square ice from Figure 1.

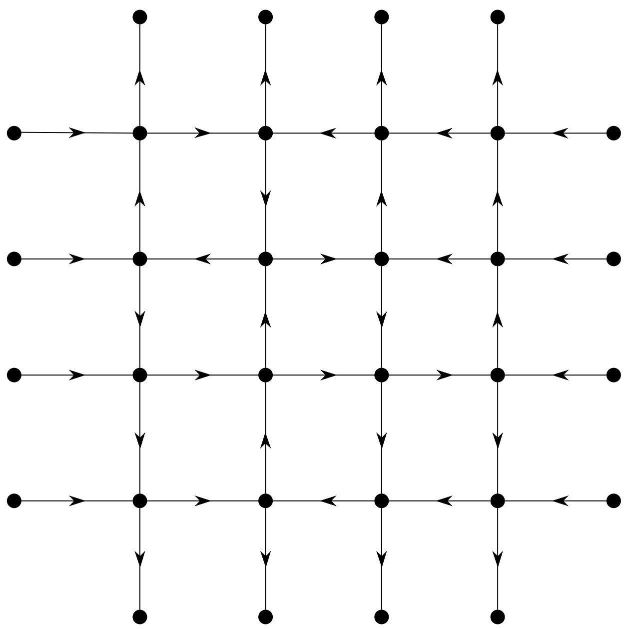


FIGURE 2. Square Ice viewed as a directed graph

To produce a bijection between square ice and ASMs we must impose what are known as *domain wall boundary conditions* to the square ice, requiring that there are Hydrogen atoms bonded all along the left and right sides and none bonded along the top and bottom.

A few things are apparent by the construction of our graph. First, each *interior vertex* (those of degree > 1) has two inward and two outward pointing edges. So in terms of which edges point in or out, these vertices may come in only $\binom{4}{2} = 6$ possible configurations. Second, our boundary condition assures that the edges on the boundary point inward in the horizontal direction and outward in the vertical direction.

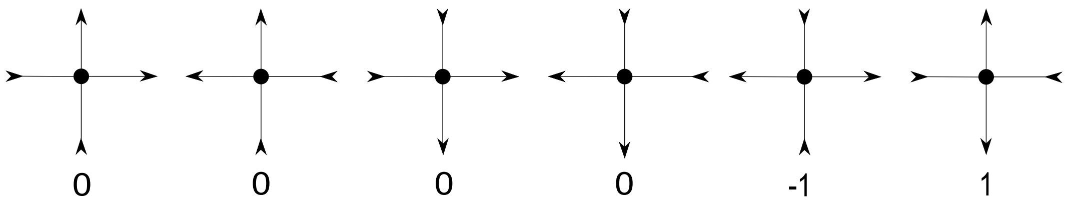


FIGURE 3. The six possible vertex types

Now the way to turn square ice into an ASM is by assigning to each of the interior vertices the value 1, 0, or -1, depending on the configuration of its inward and outward pointing edges. Figure 3 shows each vertex configuration along with its corresponding value. The alternating sign matrix for our square ice example becomes

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Turning an ASM into square ice with boundary conditions is basically the reverse of this process. Deciding which of the 4 vertex configurations to assign to a given zero entry in the ASM would be the only noteworthy thing here. It turns out that once the 1s and -1s have been given their unique vertex configurations, the graph is completely determined by the following: Consider a 0 in the ASM and look at the sums of the entries to the left of it in the same row and also above it in the same column. The sum must be 0 or 1 in each case by the alternating property, and herein lie our four choices: If the row sum is 1 or 0, choose from the vertex configurations which horizontally point all-left or all-right, respectively. Then if the column sum is 1 or 0, choose the vertex configurations which vertically point all-down or all-up, respectively.

This can be thought of another way. Imagine traversing the graph from left to right along a row, beginning with an exterior edge, and keep track of direction changes of edges along the way. Any 1s in this row of the matrix will change the direction from right-to-left, and -1s will change from left-to-right. Entries other than this will need to preserve direction and are the zero configurations with no horizontal direction change. Similarly, taking a vertical path from top to bottom will begin with an edge pointing up, with 1s and -1s changing direction from up-to-down and down-to-up, respectively. Again, the 0s must not change direction.

This will determine exactly which zero configurations are needed to complete the graph once the 1s and -1s have been positioned.

Colored ice

Let us impose coordinates on our graph and define the PARITY OF A VERTEX (x, y) to be the parity of $x + y$. Now we color an edge blue if it points from an odd vertex to an even vertex, and green if it points from an even to an odd. Figure 4 shows how this works on our square ice example where the blue edges are shown solid and the green edges dashed.

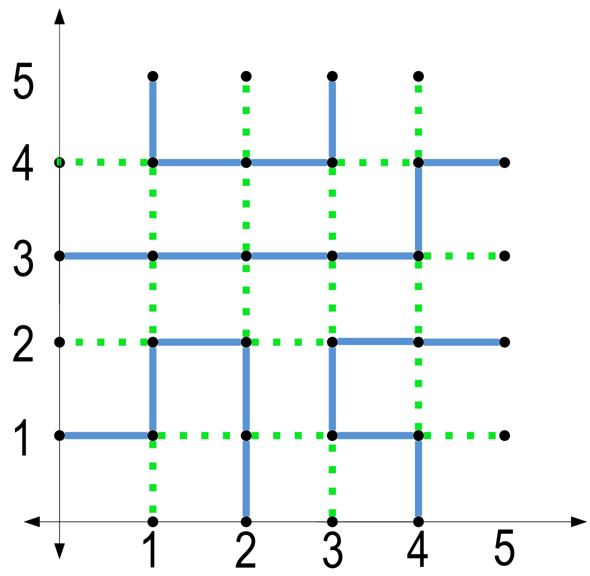


FIGURE 4. A 2-coloring of square ice with coordinates

It is obvious by our boundary conditions that the $4n$ exterior edges will alternate in color around the graph. Also, since each vertex has in-degree and out-degree 2, we now have that each interior vertex is on 2 blue and 2 green edges. In addition, let's agree to give each exterior vertex the color of its incident edge.

Proposition 1. *A monochromatic component in a 2-colored square ice graph is either a path connecting same-colored exterior vertices or it is a cycle.*

Proof. By symmetry, we may consider only the blue components. The vertices of the blue subgraph are either degree 1 or 2. By construction, the degree 1 vertices are exactly the exterior vertices so any component containing an exterior edge must be a path and hence connects distinct blue exterior vertices. If a component does not contain an exterior edge, then each of its vertices has exactly 2 neighbors and is a cycle by definition. □

Definition Call exterior vertices which are joined by a monochromatic path PAIRED and the induced partition of the set of exterior vertices a PAIRING.

A WEALTH OF SYMMETRIES

Shown in Figure 5 are the seven 3×3 alternating sign matrices along with their square ice blue subgraphs. The exterior blue vertices have been numbered clockwise beginning at the point $(0, 1)$. See that each vertex is paired with its neighbor on the left in three cases and with its neighbor on the right in three cases. The remaining case shows the vertex paired with the one on the opposite side.

Now we know that the set of ASMs is closed under the symmetry group D_8 since neither a rotation of $\frac{n}{2}\pi$ or taking the transpose will change the fact that rows and columns sum to 1 and nonzero entries alternate in sign. What is not as obvious is that ASMs, as a set, possess more symmetries than this! We will use our 2-colored square ice graphs to examine these symmetries. Let's look at the case $n = 3$ to shed some light here.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

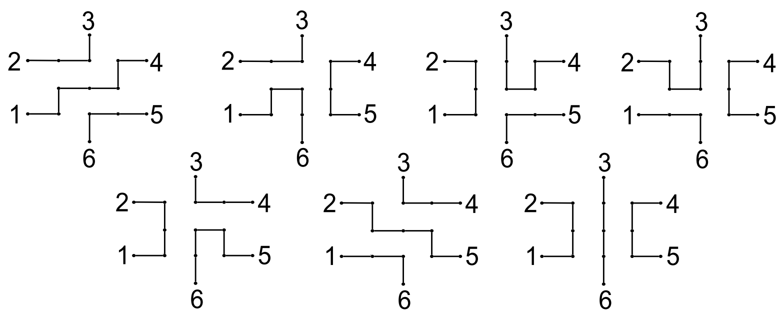


FIGURE 5. The seven 3×3 ASMs and their associated blue subgraphs

The blue subgraphs in Figure 5 have an interesting property which will lead us to our main result. Consider these numbered blue vertices not as sitting on the edges of a square, but rather as the vertices of a hexagon. Notice that the second and last graphs on the top row have the same pairings: $\{(1, 6), (2, 3), (4, 5)\}$. Now imagine leaving the graph alone, but rotating the numbers on the hexagon counterclockwise by one position. The graphs with this new numbering would now have the pairing $\{(1, 2), (3, 4), (5, 6)\}$ which amounts to rotating the pairing *clockwise.* But among our seven possibilities we already have two graphs with this pairing. The fact that there were 2 graphs with that pairing *before* and *after* rotating vertices illustrates a general property that these graphs have, namely that the number of $n \times n$ ASMs with a given pairing remains the same after its exterior vertices are rearranged by any rotation in D_{2n}. We restate this as Theorem 2 [6].

Theorem 2. *Let $\mathfrak{A}_n(\pi_B, \pi_G, l)$ be the set of $n\times n$ ASMs in which the blue subgraph induces the pairing π_B, the green subgraph induces the pairing π_G, and the sum of the number of cycles in the two subgraphs is l. If π'_B is π_B rotated clockwise, and π'_G is π_G rotated counterclockwise, then the sets $\mathfrak{A}_n(\pi_B, \pi_G, l)$ and $\mathfrak{A}_n(\pi'_B, \pi'_G, l)$ are in bijection.*

To prove Theorem 2, we shall produce a bijection between the sets of $n \times n$ ASMs whose induced pairings differ by a rotation in D_{2n}. This will be accomplished by focusing our attention first on the interior vertices of the blue subgraph and then on the exterior paired vertices. In what follows, consider an $n \times n$ ASM and let Γ be the coordinatized 2-coloring of its square ice graph as in Figure 4.

Definition For each unit square subgraph S of Γ, define the PARITY OF S to be the parity of its lower-left vertex. We refer to a square of parity k as a k-SQUARE.

Partitioning the squares of Γ by parity will allow us to produce our bijection in two stages - treating first the 1-squares and then the 0-squares.

An interior square S (containing only interior vertices) in Γ may have any one of $2^4 = 16$ color configurations for its edges. If the 4 edges of S alternate in color around the square, call this an ALTERNATING SQUARE. Note that only interior squares may be alternating. Now for $k \in \{0, 1\}$, define functions $G_k : \Gamma \to \Gamma$ which switch the edge-colors of all alternating k-squares. These functions are well-defined since squares of the same parity can only meet at vertices and not edges. Therefore what G_k does to one k-square does not affect what it does to any other k-square.

Next, define the composite functions $H_k = G_k \circ R$ where R is the function which switches the color of every edge in Γ. The result of applying H_k to Γ is that all edges in Γ will switch colors *except* for those in alternating k-squares. Finally, we will want to affect both 0-squares and 1-squares in this way, so we define the function $G = H_0 \circ H_1$. This function has been called *gyration* and the meaning for this will be seen in its action on the pairing of Γ. Figure 6 shows the function G applied to our running example of square ice.

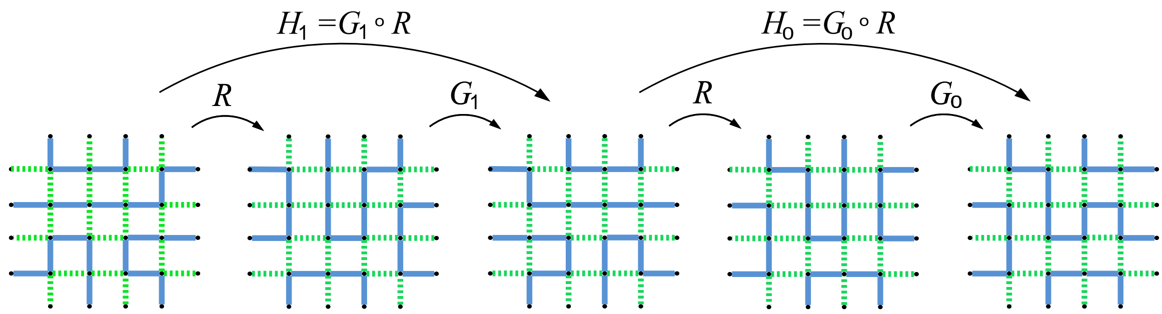
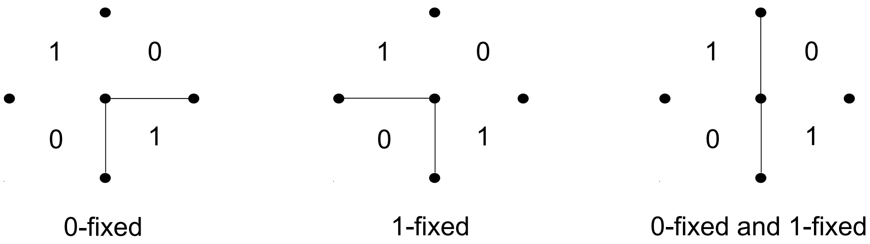


FIGURE 6. Example of G acting on 2-coloring of square ice

Notice that after H_1, the blue pairing has been rotated clockwise, though it is now a pairing on formerly green vertices. The green pairing has been rotated counterclockwise, now on formerly blue vertices. The map H_0 repeats this effect, where the blue and green pairings have returned to their original colors, but rotated. The alternating k-squares are like "fixed points" for the H_k. But since our result regards pairings, and pairings paths, we will characterize blue components (and hence paths) by certain subsets of the vertex set of Γ.

Definition Define a k-FIXED VERTEX as an interior vertex whose 2 blue edges are contained in different k-squares.

By construction of Γ, an interior vertex must be 0-fixed, 1-fixed, or both:



Since k-fixed vertices are where a blue (respectively green) component moves from one k-square to another, we will examine their behavior with respect to the functions H_k.

Proposition 3. *A vertex is k-fixed in Γ iff it is k-fixed in $H_k(\Gamma)$.*

Proof. As each internal vertex in Γ is contained in some internal k-square, we need only consider the internal squares. The possibilities of having a k-fixed vertex in an interior k-square are shown in Figure 7 where the darkened vertices are k-fixed. Therefore a vertex in a k-square is k-fixed before and after the application of H_k.

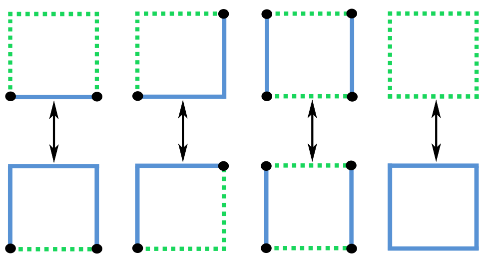


FIGURE 7. k-fixed vertices in a k-square before and after the application of H_k

□

Remark. An internal k-square cannot possess only 1 or only 3 k-fixed vertices. If v was the unique k-fixed vertex in a k-square S, it would have an incident blue edge on S and one off S. Then all of the other (non k-fixed) vertices would have their incident blue edges all on S, resulting in three blue edges incident to v. Having only 3 k-fixed vertices gives a contradiction with a similar argument.

It is clear that the application of H_k will reverse the edge colors in all but the third case above, the alternating square: First R changes every edge color, and then G_k changes back only those in alternating squares. Thus H_k leaves alternating k-squares alternating and k-fixed vertices k-fixed.

Proposition 4. *Two k-fixed vertices are in the same blue component of Γ after the action of H_k if and only if they were in the same blue component before.*

Proof. We have seen that if 2 k-fixed vertices are adjacent in the blue subgraph (Figure 7) then they remain fixed after the application of H_k since there is a k-square containing them both. If u and v are k-fixed vertices connected by a blue path of non-k-fixed vertices, then that path must be contained in a single k-square since the vertices between u and v, by definition, have their incident blue edges both on a single k-square. Now Figure 7 shows that u and v would still be connected by a blue path after the application of H_k since the colors would switch in all cases except the third.

Now we may take a blue path of any length connecting 2 k-fixed vertices and divide it at each intermediate k-fixed vertex into subpaths. Each subpath is contained in a single k-square and may or may not change after applying H_k, depending on whether it is part of an alternating square or not. In this way, H_k may alter the path overall, but *not* the k-fixed vertices it contains.

Finally, since H_k^2 is the identity on Γ, the argument above is bi-directional.

□

So a monochromatic path passes through the same sequence of k-fixed vertices before and after H_k. We know what will happen at each interior subpath, but not what happens before the first and after the last k-fixed vertices. We know both that the endpoints paired by that path will rotate around the graph and, because of the order in which G is constructed - first H_1 and then H_0, the direction this will happen.

Recall that we originally colored blue the edges which point from a 1-vertex to a 0-vertex, so by the domain wall boundary conditions, the bottom left horizontal edge is always blue. Then H_1 will "move" this edge up one position and H_0 will "move" it up again, thereby rotating it clockwise around the graph. Similarly, the bottom left vertical edge is always green in Γ, so H_1 will move it one position to the right and H_0 will move it right again, thereby rotating it anticlockwise around the graph. As the exterior edges alternate in color around the graph, this will be true in general: Blue edges rotate clockwise and green edges rotate anticlockwise.

The one question remaining is, what happens before the first and after the last k-fixed vertex in a monochromatic path? The answer is: not much. We will see this in the following proposition.

Proposition 5. *An exterior k-square contains exactly one k-fixed vertex which is connected to each endpoint in Γ by monochromatic paths.*

Proof. The squares on the corners of Γ have only one interior vertex which clearly must be fixed and connected to each of the exterior vertices in that square by a blue and a green path, each of length 1.

So suppose S is a non-corner k-square on the boundary of Γ, and that none of its vertices is k-fixed. Exactly one external edge in S must be blue by construction. As neither of the internal vertices on S is fixed, the path beginning with this blue edge must never leave S. Thus S is all blue, contradicting the fact that the exterior edges must alternate in color around Γ. So S must have a k-fixed vertex.

If S had 2 k-fixed vertices, then the blue path beginning in S must leave S at its first internal vertex v, but the other k-fixed vertex w must also have a blue component passing between different k-squares there. This component may not include the other external edge in S by the alternating condition, so it must also be connected to v, giving v 3 incident blue edges. Thus S must have exactly 1 k-fixed vertex.

The fact that this vertex is connected to each of the exterior vertices by monochromatic paths requires checking 2 cases. *Case 1:* The k-fixed vertex is on the exterior blue edge. Then the blue path leaves S and the other two edges in S are necessarily green. *Case 2:* Reverse *blue* and *green* in Case 1. □

Now before we prove Theorem 2, let us summarize what we have shown. A component in the blue subgraph is either a cycle or a path connecting exterior blue vertices. If it is a path, then it passes through a sequence of k-fixed vertices (one sequence for each parity). These 1-fixed vertices are still 1-fixed and on the path after the application of H_1. Likewise, the 0-fixed vertices remain 0-fixed and on the path after the application of H_0. These sequences of vertices both begin and end in exterior squares. Thus paths become paths and cycles become cycles after H_k, and paths are identified by their k-fixed vertices before and after H_k.

Proof of Theorem 2. Let P be a blue path in Γ connecting exterior blue vertices u and v. By Proposition 5, P contains k-fixed vertices and Propositions 3 and 4 give that H_k leaves these vertices k-fixed and connected in the image of P. Moreover, Proposition 5 gives that $H_k(u)$ and $H_k(v)$ are connected. Since each of these results was bijective and applied to both parities, we have a bijection between the set of blue paths in Γ and $G(\Gamma)$. This, in turn, gives a bijection of pairings before and after G. Finally, we saw that by the construction of Γ and the order we composed the function G that the blue and green pairings rotate clockwise and anticlockwise, respectively, after the application of G.

Next, let l be the total number of cycles in Γ. By Propositions 3 and 4, a cycle containing k-fixed vertices will be a cycle of the same color after the application of H_k. If a cycle contains no k-fixed vertices, it must be contained in a single k-square by definition of k-fixed. Cycles of this type will merely change color under H_k. Thus we have a bijection between the set of cycles in Γ and $G(\Gamma)$. The colors may change, but the overall number does not. □

Remark. Consider the function D which reflects Γ over the line $y = x$. This will switch the colors of all exterior paths and vertices. Now for either parity, the function $H_k \circ D$ will reflect the pairings over the line $y = x$ (with blue and green pairings on formerly green and blue vertices, respectively) and rotate them (clockwise if blue, anticlockwise if green) back to their original colors. The choice of which H_k to compose with D will merely affect which

2 external vertices remain fixed.

Corollary 6. *If the blue square ice subgraph of an $n \times n$ ASM has its external vertices as the vertices of a regular n-gon, and if its blue pairing is rearranged by an element of D_{2n}, then the number of ASMs with a given pairing remains the same.*

Proof. This is a direct result of Theorem 2 and its subsequent remark as the rotation and reflection described there generate D_{2n}. □

An example

To give an example showing the function G at work on a large example of square ice, we give a 15×15 ASM and its associated colored graph. To properly see what is happening at each step of the process, first the blue subgraph will be shown and then the green subgraph.

Some things to notice: There are 7 total cycles in the graph - 3 blue and 4 green. Notice how the single-square blue cycle (contained in a 1-square) changes to green after H_1 and remains green after H_0, but it has grown to a two-square cycle. It is also interesting to see how the larger cycles change shape and move, though you can check that they still (after each H_k) contain the same k-fixed vertices. Last but not least, the movement of the paths is delightful to watch. The construction shows the blue and green paths rotating clockwise and anticlockwise, respectively, all the while "dodging" the moving cycles.

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

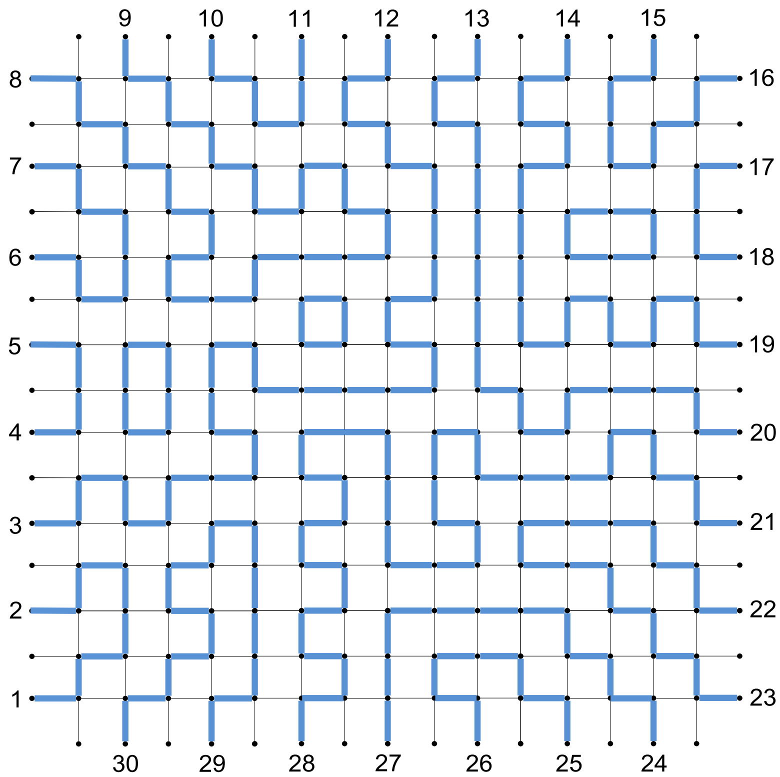


FIGURE 8. blue graph

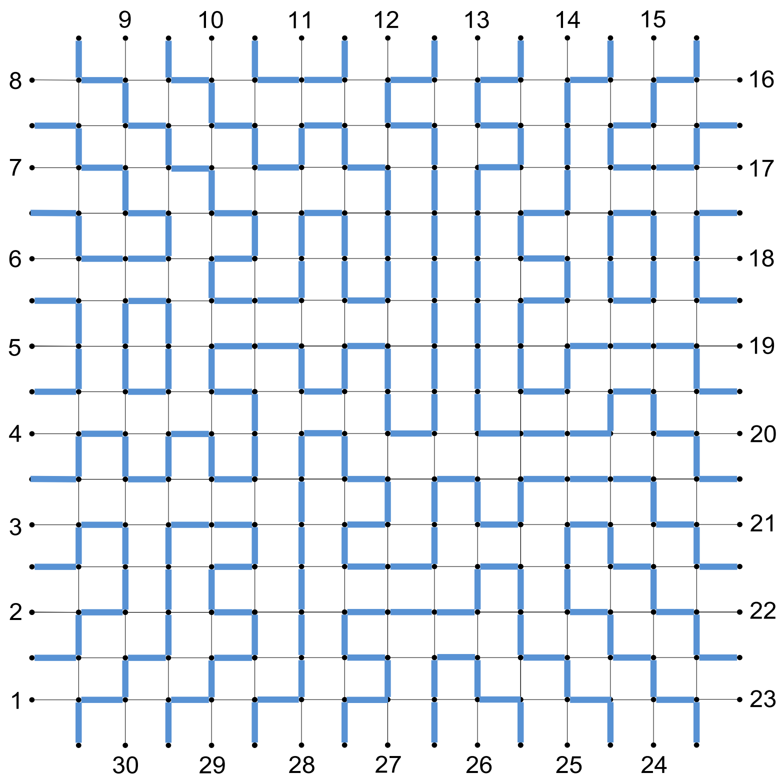


FIGURE 9. blue graph after H_1

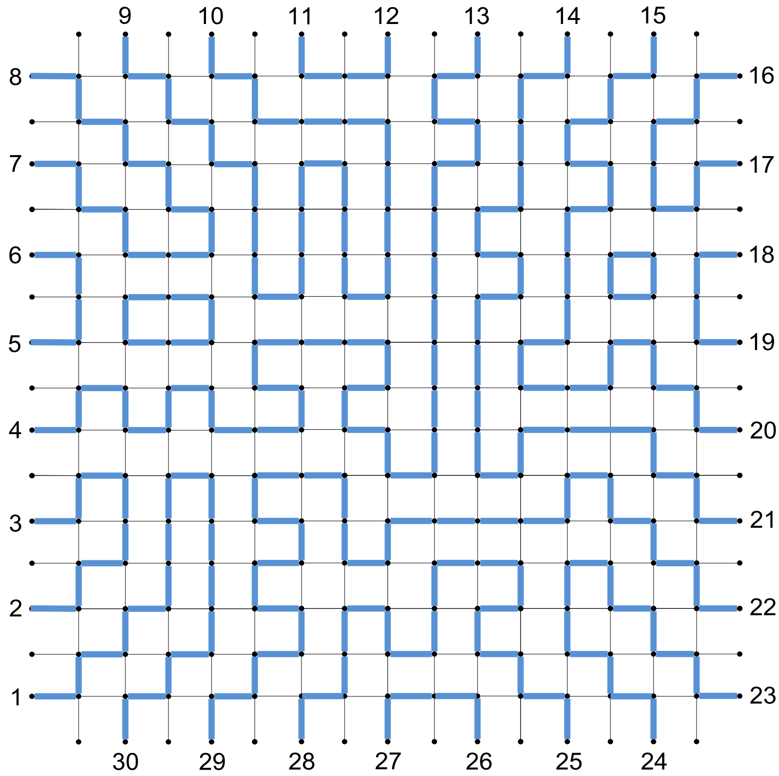


FIGURE 10. blue graph after H_0

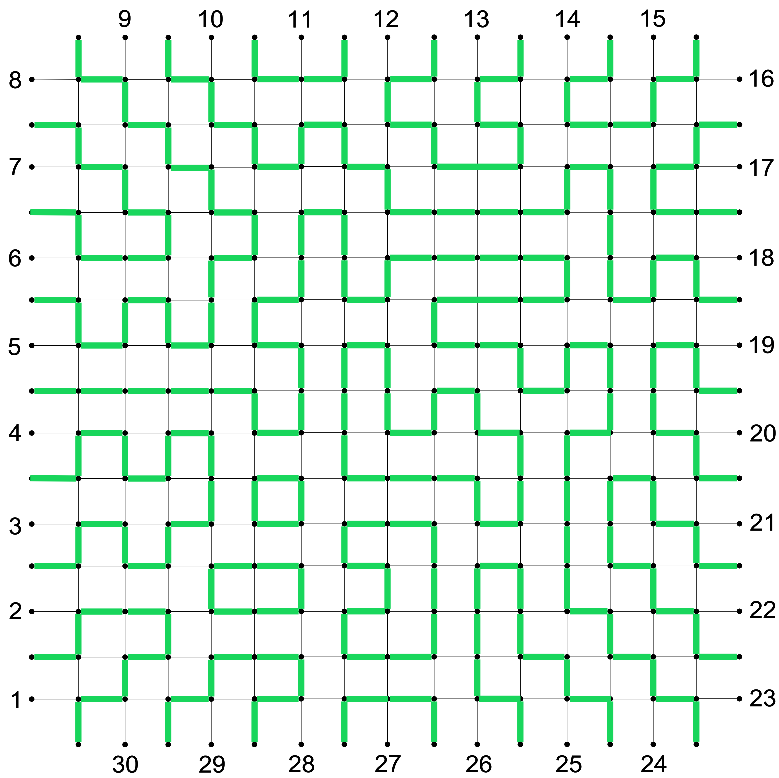


FIGURE 11. green graph

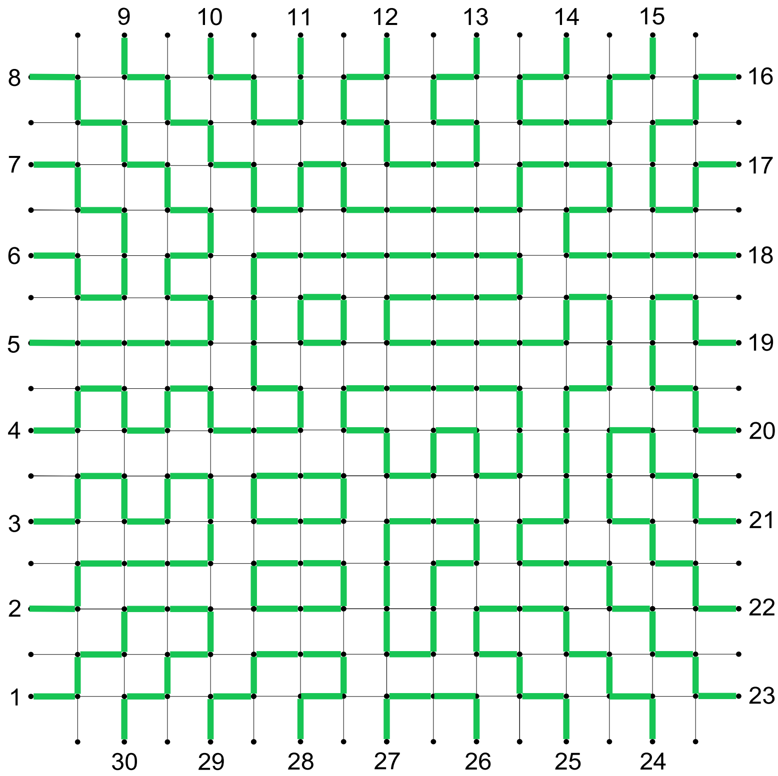


FIGURE 12. green graph after H_1

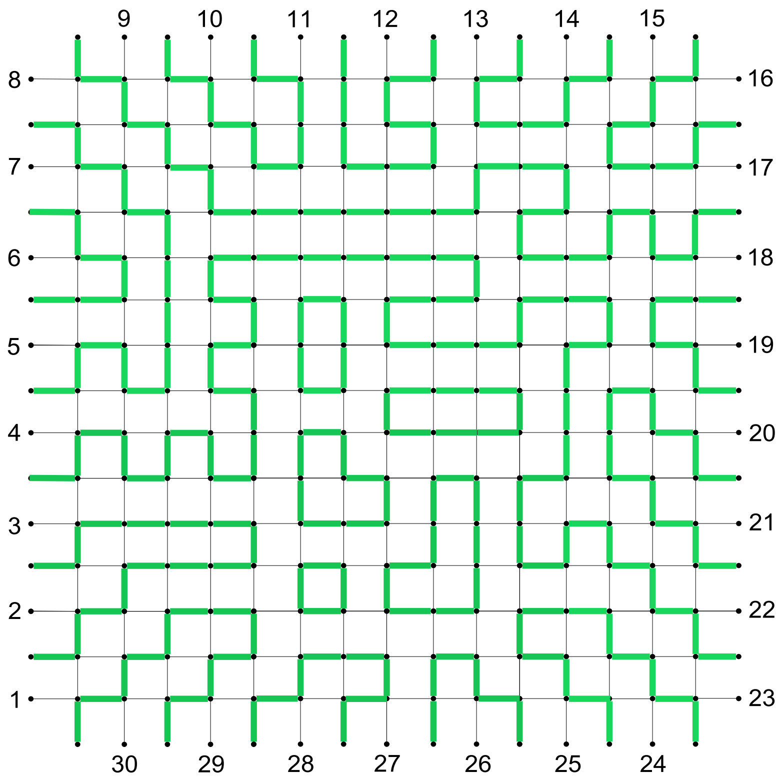


FIGURE 13. green graph after H_0

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