WHERE'S THE FIRE? AN EXACT BOUND ON THE EXTREMAL CASES OF THE IDENTIFYING CODE PROBLEM

WIL LANGFORD

Under the direction of

Dr. John Caughman

with second reader

Dr. M. Paul Latiolais

In partial fulfillment of the requirements for the degree of

Masters of Science in Mathematics

Portland State University

Fariborz Maseeh Department of Mathematics and Statistics

Contents

Part 1. Introduction

Suppose you had a simple fire sensor that could detect a fire burning in the room in which it was placed or in any adjacent room. Suppose also that the sensor had a binary output and could only say, "there is a fire," or "there is no fire." In some buildings, putting such a detector in every room would allow you to pinpoint a fire based on which sensors activated. The design of other buildings would still allow such a fire safety system to notify you of a fire, but the room containing the fire may not be determined exactly. What building designs would allow such a system to precisely locate a fire? Can you cut costs by omitting some sensors and still locate a fire?

Alternatively, suppose that you had a microprocessor in an array of microprocessors that could test itself and nearby processors for faults, but could only report whether or not a fault was detected. What circuit designs allow faults to be precisely located? Can fewer sensors be used and still pinpoint the faults?

Both of these problems are addressed by identifying codes of graphs. In the fire detection system example, the rooms correspond to vertices, the doors to edges, and the *code* is the subset of rooms containing sensors.

In this paper, I'll present Gravier and Moncel's result from their 2007 article "On graphs having a $V - \{x\}$ set as an identifying code" [1]. My intent is to expand their exposition by providing illustrations and examples from the literature. In terms of the examples above, their result states that if the structure of the building makes it possible to pinpoint the fire, it is always safe to leave at least one room without a sensor.

Part 2. Definitions

The definitions of more generic graph theory terms such as VERTEX or DEGREE are outside the scope of this paper. Hence we shall assume a familiarity with the definitions used by West[2]. We note, however, that our graphs will be undirected and simple, so they will have neither loops nor multiedges. We will not, in general, assume that the vertex set is necessarily finite.

1. Terminology for codes in graphs

Definition 1. Fix any vertex v in a graph G, and any integer $r \geq 1$. Denote by $B_r(v)$ the BALL OF RADIUS r centered at vertex v, that is, the set of all vertices in $V(G)$ that have graph-theoretic distance at most r from v.

Consider $B_1(v) = \{v, a, b, c\}$ and $B_2(g) = \{g, e, f, x, b, c, h\}$ in the graph shown in Figure 1.1 below. Note that in Figure 1.1, successively larger r-balls centered at v are obtained by adding the boxed sets of vertices as you move right across the diagram. For example, $B_2(v)$ includes v and the first two distance boxes.

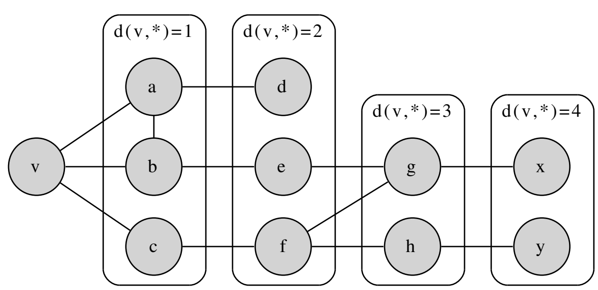

FIGURE 1.1.
A distance partition of a graph with root vertex v.

Definition 2. Fix an integer $r \geq 1$. An r-CODE of G is a subset C of the vertex set of G such that there is at least one code vertex within the r-ball of every vertex in the graph.

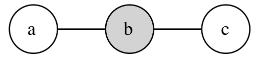

FIGURE 1.2.
A labeled P_3 with 1-code $C = \{a, c\}$.

As an example, consider a labeled path on three vertices as in Figure 1.2. Note that $V(G) = \{a, b, c\}$ and $E(G) = \{ab, bc\}$. Then $C = \{a, c\} \subset V(G)$ is an r-code of G, with radius $r = 1$.

In contrast, the set $\{a\} \subset V(G)$ of the graph in Figure 1.3 is *not* a 1-code because $B_1(c)$ contains no code vertices.

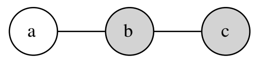

FIGURE 1.3.
A labeled P_3 with (invalid) 1-code $C = \{a\}$.

Charon et al[3] and Bertrand et al[4] use the term *r-dominating code* to denote the same concept that I'm simply calling an *r-code*. These articles deal with more granular concepts that aren't relevant in this context.

In terms of the examples of the introduction, a set that is not an r-code wouldn't alert you at all in case of a fire or faulty microprocessor in a vertex that was too far from a code vertex.

Definition 3. Given an r-code C, the SIGNATURE $s_r(v)$ of a vertex v is the intersection of the r-ball centered at v with C; that is,

$$s_r(v) = B_r(v) \cap C.$$

As an example, observe that $B_1(a) = \{a, b\}$ and $C = \{a, c\}$ in Figure 1.2. Then $s_1(a) = \{a, b\} \cap \{a, c\} = \{a\}$. Similarly, because $B_1(b) = \{a, b, c\}$, we have $s_1(b) = \{a, b, c\} \cap \{a, c\} = \{a, c\}$. Note that a subset of the vertex set of a graph is an r-code if and only if the signature $s_r(v)$ of every vertex v is nonempty.

Definition 4. Given an r-code C, a pair of vertices $u, v \in V(G)$ are called TWIN VERTICES whenever $s_r(u) = s_r(v)$.

In K_3 (or more generally *any* complete graph on more than one vertex), $B_1(x) = B_1(y)$ for all vertices x and y. In Figure 1.4, every vertex is a twin of every other vertex.

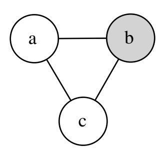

FIGURE 1.4.
A labeled K_3 with vertices a and c in the 1-code and b not in the 1-code. Any two vertices in this graph are twins.

Definition 5. The SYMMETRIC DIFFERENCE OF SETS A AND B is the set $(A \backslash B) \cup (B \backslash A)$ denoted by $A \Delta B$. Two vertices are twins if and only if their signatures are equal. Equivalently, two vertices $u, v \in V(G)$ are twins if and only if the symmetric difference of their signatures is empty. That is, u is a twin of v if and only if

$$s_r(u) \Delta s_r(v) = \emptyset.$$

In Figure 1.5, $s_1(a) = \{a, b, c\}$ and $s_1(b) = \{a, b, d, e\}$, so $s_1(a) \Delta s_1(b) = \{c, d, e\}$.

Definition 6. An r-IDENTIFYING CODE of G is an r-code of G such that no twin vertices exist. A graph that admits an r-identifying code is called r-IDENTIFIABLE.

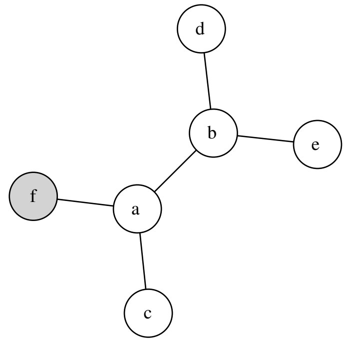

FIGURE 1.5.
A labeled graph with vertices with 1-code $C = \{a, b, c, d, e\}$.

Referring to Figures 1.2 and 1.4, notice that P_3 is 1-identifiable and K_3 is not. Neither are r-identifiable for any $r > 1$, because when $r > 1$ the r-balls of every vertex would contain the entire code (and thus their signatures would equal each other.) When the r-balls of two vertices are equal, their signatures are necessarily equal.

Figure 1.6 illustrates a 1-code that does not 1-identify the graph. Note that while c is in the symmetric difference $s_1(a) \Delta s_1(b)$, the signatures $s_1(b)$ and $s_1(c)$ are equal, hence b and c are twins.

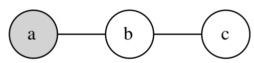

FIGURE 1.6.
A labeled P_3 with vertices with 1-code $C = \{b, c\}$.

The examples from the introduction are applications of 1-identifying codes (often referred to simply as *identifying codes*) because they consider only the 1-balls centered at each sensor.

Definition 7. Let u and v be vertices in a graph G with r-code C. Any vertex in $s_r(u) \Delta s_r(v)$ is said to r-SEPARATE u from v. When $s_r(u) \Delta s_r(v)$ is nonempty, u and v are said to be r-SEPARATED.

Note that this means that if a vertex v separates two other vertices, v must be in the code. In Figure 1.5, vertex d 1-separates vertex b from vertices e and a. Though it may not be intuitive at a glance, vertex e 1-separates itself from vertex f.

If the distance between two vertices is strictly greater than $2r$, then those two vertices are necessarily r-separated because their r-balls do not intersect. Often 1-separation is referred to simply as *separation.*

Definition 8. The r-TRANSITIVE CLOSURE of a graph G is denoted G^r, has vertex set $V(G^r) = V(G)$, and edge $uv \in E(G^r)$ if and only if there exists a path on at most r edges from u to v in G.

Definition 9. The COMPLETE CODE of a graph is the entire vertex set of that graph.

2. A GLOBAL ASSUMPTION

We will consider only simple graphs, because multiple edges and loops have no effect on identifying codes.

If a graph is disconnected, then the r-ball of each vertex is a subset of the vertex set of exactly one connected component. Therefore, the components must be considered individually when constructing an r-identifying code. To that end, we'll be considering only simple, *connected* graphs.

As mentioned in the discussion following Definition 4, a complete graph K_n never admits an $r-$identifying code when $n > 1$, because the r-ball of every vertex is $V(K_n)$, so all r-balls (and hence all vertex signatures) are equal. A single vertex graph must have its only vertex in any r-identifying code. A connected graph on two vertices is isomorphic to K_2, so it does not admit an r-identifying code. Thus, we will be considering only simple, connected graphs *on at least three vertices.*

A graph containing vertices of infinite degree can require every vertex to be included in any r-identifying code. An example of such a graph is given in Part 4. Therefore, we will consider only simple, connected graphs on at least three vertices that have *bounded maximum degree.*

The following definition summarizes and formalizes the discussion above into a global assumption:

Definition 10. Unless specified otherwise, the graph G is defined to be

- simple and connected,
- on at least three vertices,
- with r-identifying, complete code $C = V(G)$, and
- bounded maximum degree.

Proposition 11 clarifies the assumption that the complete code is r-identifying.

Part 3. Results

We will start by considering 1-identifying codes and then generalize to r-identifying codes.

3. Adding vertices to a code is always safe

The following pair of results justify the choice to let $C = V(G)$ in the global Definition 10.

Proposition 11. *Assume a graph G is simple and connected. If C is a 1-identifying code of G and S is a set of vertices not in the code, then $C \cup S$ is a 1-identifying code of G.*

Proof. Let $u, v \in V(G)$ be arbitrarily chosen. Let $S \subseteq V(G) \backslash C$. Because C is a 1-identifying code of G, there exists a vertex $a \in s_1(u) \Delta s_1(v)$ that 1-separates u from v. If the code is altered to include an arbitrary vertex $x \in S$, then the 1-separation of u from v by a still holds. Because u, v, and x were chosen arbitrarily, this is true for all pairs of vertices in $V(G)$ and all vertices in S, hence $C \cup S$ is a 1-identifying code of G. □

Corollary 12. *If G is a 1-identifiable graph, then $C = V(G)$ is a 1-identifying code of G.*

Proof. If we let $S = V(G) \backslash C$, then the corollary follows immediately from Proposition 11. □

4. When a vertex v is required in a 1-identifying code

We will now consider a specific vertex $v \in V(G)$ and some consequences of it being required in any 1-identifying code. That is, if $C \backslash \{v\}$ is not a 1-identifying code (and by Definition 10, C is), then there are some important stepping stone lemmas to be found concerning v and vertices nearby. Figure 4.1 illustrates both Lemma 13 and Corollary 14.

Lemma 13. *Assume Definition 10 with $r = 1$. If vertices x and y are twins in $C - v$, then $x \sim y$.*

Proof. By the assumption that $C = V(G)$, it must be the case that $x, y \in C$. Since x and y are twins in $C - v$, it must also be the case that except for v, $s_1(x) = s_1(y)$. Observe that $x \in s_1(x) = s_1(y)$, so $x \in B_1(y)$ and x is adjacent to y. □

Corollary 14. *Assume Definition 10 with $r = 1$. If vertices x and y are twins in $C - v$, then*

(1) *exactly one of x and y is adjacent*[1] *to v and*
(2) *the distance from v to the other is 2, and there exists a length 2 path through the adjacent vertex in part (1) above.*

[1] Bold edge in Figure 4.1.

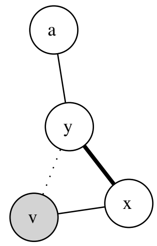

FIGURE 4.1.
Illustration for Lemma 13 and Corollary 14.

Proof. Note that v must 1-separate x and y, so it must be in exactly one of their 1-balls by definition. Without loss of generality, let $v \in B_1(x)$. Then $v \sim x$ and part (i) is proven.
Because v 1-separates x and y and $v \sim x$, it must be the case that v is not adjacent[2] to y. By Lemma 13, $x \sim y$, so $v \sim x \sim y$ is a vy-path of length 2. □

5. Conditions on vertex removal

Lemma 15. *Assume Definition 10 with $r = 1$. The set $C \backslash \{v\}$ is a 1-identifying code of G if and only if all vertices of $B_1(v)$ are 1-separated from all of the vertices of $V(G) \backslash B_1(v)$.*

Proof. Suppose that $C \backslash \{v\}$ is a 1-identifying code of G. Then all vertices of G must be 1-separated from each other pairwise, and in particular all vertices of $B_1(v)$ are 1-separated from all vertices of $V(G) \backslash B_1(v)$.

Contrapositively suppose that $C \backslash \{v\}$ is not a 1-identifying code of G. Then by Corollary 14, a vertex in $B_1(v)$ must be the twin of a vertex in $V(G) \backslash B_1(v)$. By definition, twin vertices are not 1-separated. □

6. Removal of at least one vertex is possible when $r = 1$

We now turn to the main result of the paper. Figure 6.1 is illustrative of the vertices referred to throughout the theorem.

Theorem 16. *(Gravier and Moncel,* [1]*) Assume Definition 10 with $r = 1$. There exists a vertex v such that $V(G) \backslash \{v\}$ is a 1-identifying code of G.*

Proof. Because the maximum degree of G is bounded by assumption in Definition 10, there exists at least one vertex in G of maximum degree. Let a be a vertex having maximum degree in G. If $C \backslash \{a\}$ is a 1-identifying code of G, then the proof is complete. If not, then there exist twin vertices in G given 1-code $C \backslash \{a\}$.

By Corollary 14 and Lemma 15, one of these twins is in $B_1(a)$; call it x. The other, which we'll denote by y, is in $V(G) \backslash B_1(a)$. Because x and y were not twins in C but are in $C \backslash \{a\}$,

$$s_1(x) = s_1(y) \cup \{a\}$$

and because $C = V(G)$,

$$B_1(y) \subsetneq B_1(x). \tag{6.1}$$

We claim that since $C \backslash \{a\}$ is not a 1-identifying code of G, then $C \backslash \{y\}$ must be a 1-identifying code of G. By Lemma 15, it suffices to show that each vertex of $B_1(y)$ is 1-separated from each vertex of $V(G) \backslash B_1(y)$.

We start by 1-separating all vertices in $B_1(y)$ from everything outside of that ball *except* a. Let $z \in B_1(y)$. By (6.1), it must be the case that $z \in B_1(x)$. Let

$$b \in (V(G) \backslash B_1(y)) \backslash \{a\} = V(G) \backslash (B_1(x) \cup B_1(y)) = V(G) \backslash B_1(x),$$

that is, let b fall outside of the union of the 1-balls around x and y. Because a is the only difference between $B_1(y)$ and $B_1(x)$, we see that $b \notin B_1(x)$ and symmetrically $x \notin B_1(b)$. Thus, x 1-separates z from b.

[2] Dotted non-edge in Figure 4.1.

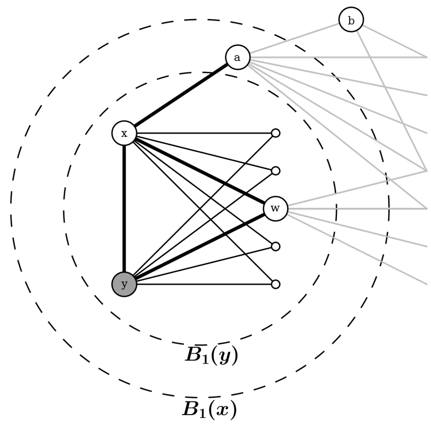

FIGURE 6.1.
The 1-code $C = V(G) \setminus \{y\}$. The gray lines indicate potential sets of edges to regions of the graph that are not relevant. The small vertices are elements of $B_1(y) \setminus \{w, x, y\}$.

We continue by 1-separating all vertices in $B_1(y)$ from a. First, note that $a \in V(G) \setminus B_1(y)$, so a 1-separates itself from y because $a \not\sim y$. If the only neighbors of a were in $B_1(x)$, then $x \sim y$ and $a \not\sim y$ would make the degree of x strictly greater than that of a, a contradiction. Therefore, a has a neighbor in $V(G) \setminus B_1(x)$ and that neighbor separates a from x.

Let w be any non-y, non-x vertex of $B_1(y)$, that is, $w \in B_1(y) \setminus \{x, y\}$. By way of contradiction, suppose that a and w are not 1-separated by 1-code $C \setminus \{y\}$ but that they were by 1-code C. Then $B_1(w) = B_1(a) \cup \{y\}$, and the degree of w is greater than the degree of a, a contradiction.

Hence, every vertex of $B_1(y)$ is 1-separated from every vertex of $V(G) \setminus B_1(y)$ and so by Lemma 15, the 1-code $V(G) \setminus \{y\}$ is a 1-identifying code of G. □

7. Extension of the $r = 1$ results to $r > 1$

We now extend the previous results to show that it is possible to remove at least one vertex from a graph that meets the conditions of Definition 10 when $r > 1$.

Lemma 17. *Recall Definition 8. The maximum degree of a graph G is bounded if and only if the maximum degree of G^r is bounded.*

Proof. By definition, $V(G) = V(G^r)$ and $E(G) \subseteq E(G^r)$, so G has bounded maximum degree whenever G^r does.
Conversely, suppose that G has bounded maximum degree d, and let $v \in V(G)$ be any vertex. Then the degree of v in G^r can be no greater than d^r, and G^r has bounded maximum degree. □

Lemma 18. *A simple, connected graph G is r-identifiable by a r-code if and only if G^r is 1-identifiable by the same set C.*

Proof. Given vertices $u, v \in V(G)$, Definition 8 implies that $u \in B_r(v)$ in G if and only if $u \in B_1(v)$ in G^r. Indeed for any vertex x, it follows that $B_r(x)$ in G is equal to $B_1(x)$ in G^r. The equality of $s_r(x)$ in G and $s_1(x)$ in G^r follows.

Suppose that G is r-identifiable. Then for all distinct $x, y \in V(G)$, $s_r(x) \neq s_r(y)$. Then in G^r, $s_1(x) \neq s_1(y)$ in G^r.

Conversely, suppose that G^r is 1-identifiable. Then similar to the above argument, for all distinct $a, b \in V(G^r)$, we see that $s_1(a) \neq s_1(b)$ in G^r, so $s_r(x) \neq s_r(y)$ in G.

Therefore, any r-identifying code of a graph G is a 1-identifying code of G^r and vice versa. □

Corollary 19. *Assume Definition 10. There exists a vertex v such that $V(G) \setminus \{v\}$ is an r-identifying code of G.*

Proof. Consider G^r. By Lemma 17, G^r has bounded maximum degree and by Lemma 18, G^r is 1-identifiable. Thus, G^r meets the criteria of Theorem 16 and there exists a vertex $v \in G^r$ such that $C \setminus \{v\}$ is a 1-identifying code of G^r. By Lemma 18, the set $C \setminus \{v\}$ is an r-identifying code of G. □

Part 4. Exhibits of extremal cases

8. Requiring a code of cardinality $n - 1$

As an example of a family of graphs that require all but one vertex in any 1-identifying code, consider the stars. That is, consider a single central vertex adjacent to every other vertex in the graph, with every other vertex adjacent to only the central vertex.

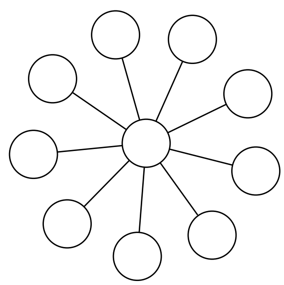

FIGURE 8.1.
A star graph on 10 vertices.

If the central vertex of a star is removed from an otherwise complete code, then each leaf 1-separates itself from all other leaves and the central vertex is 1-separated from each leaf v by all non-v leaves.

On the other hand, if a leaf u is removed from an otherwise complete code, then every non-u leaf 1-separates itself from all other leaves (including u,) and the central vertex is 1-separated from u by all non-u leaves.

Thus any single vertex removed from a star graph's complete code leaves a 1-identifying code.

However, if two leaves are removed, then their signatures are both exactly the central vertex. If a leaf and the central vertex are removed, then the leaf's signature is empty.

Hence, all stars on $n \geq 3$ vertices require exactly $n - 1$ vertices in their minimum-cardinality 1-identifying codes.

All finite graphs whose 1-identifying codes require all but one vertex are classified by Foucaud et al [5].

9. Unbounded maximum degree

We now explore an example justifying our global assumption of bounded degree. Section 5 of [3] gives the following example of a graph of unbounded maximum degree.

Let G be a graph with vertex set $V(G) = \mathbb{Z}$ and the following edges:

- Each even integer has an edge to every other even integer.
- Each odd integer has an edge to every other odd integer.
- Every even integer has an edge to every *larger* odd integer.

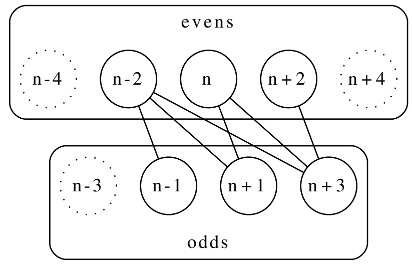

FIGURE 9.1.
A finite subgraph of a graph with unbounded maximum degree.

That is, the evens form a complete graph, the odds form a complete graph, and the edges from evens to odds are all to the "right" on the number line. See Figure 9.1 for an illustration of a small, finite portion of G.

Let n be any even integer. Consider what 1-separates n from $n-2$. No even can 1-separate them, as all even numbers are in $B_1(n) \cap B_1(n-2)$. No odd numbers less than n are in $B_1(n)$. All odd numbers greater than $n-2$ are in $B_1(n) \cap B_1(n-2)$. In fact, the only thing 1-separating n from $n-2$ is $n-1$. Because n was chosen arbitrarily, this means that every odd integer must be contained in any 1-identifying code.

Now consider what 1-separates $n+1$ from $n-1$. Similar to the evens, all evens less than $n-1$ and all odds are in $B_1(n-1)$. The only number in $B_1(n+1)$ that is not in $B_1(n-1)$ is n itself, so all even integers must be in any 1-identifying code.

Hence, the only 1-identifying code of G is $V(G)$.

Part 5. Further questions and conjectures

As the primary result of this paper shows, a 1-identifiable graph contains at least one vertex that is removable from the complete code. Future research could explore the classification of graphs for which the removable vertex is unique.

At the other extreme, there exist graphs for which any single vertex could be removed from the complete code. Examples include stars on at least four vertices and vertex transitive graphs.

Part 6. References

References

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