The Golden Root of Chromatic Polynomials

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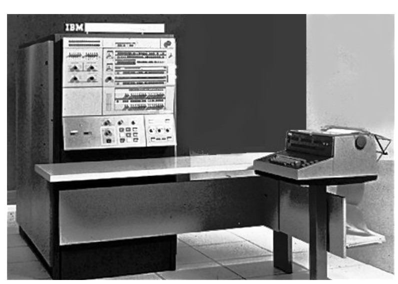
Contents

1 Abstract

In this 501 paper, I will present the results found in the article "On Chromatic Polynomials and the Golden Ratio," by W.T. Tutte. Specifically, I outline how the Golden Ratio, Φ, nearly appears as a root, the Golden Root, to the chromatic polynomials of a particular class of graphs. Specifically, when the n-vertex planar triangulation is evaluated at $1+\Phi$, the value must be less than Φ^{5-n} in absolute value. In this way, Tutte provides a theoretical explanation for what was originally a purely empirical observation. In a later chapter, I also present a few results about the Four Color Theorem which show that the Golden Root can actually never be a root of such a chromatic polynomial.

2 History of the Problem

In 1966, a woman by the name of Ruth Bari earned her doctorate in Mathematics at Johns Hopkins University with the completion of her thesis, "Absolute Reducibility of Maps of at Most 19 Regions." Bari's paper covers regular major maps.
The chromatic polynomials, introduced by George David Berkhoff in 1912 to further the study of the Four Color problem, of all the aforementioned maps were calculated and it was this list of results that caught the eye of renowned mathematician William Tutte.



During the Summer of 1968 in Waterloo, Ontario, Canada, Tutte used the now antiquated IBM360 computer to determine all zeroes of Bari's chromatic polynomials.

As a general rule, Tutte found that there is just one negative non-integral value for which the polynomial vanishes and this zero is close to

$$u = (-3 + \sqrt{5})/2 = -0.38196601....$$

Immediate observation reveals the fact that u differs only the number 2 from the number of the Golden Ratio ϕ. Tutte refers to this particular zero as the golden root of the polynomial.

Bari was using a slight variation of the polynomial which we now commonly refer to as the chromatic polynomial. In modern notation, her chromatic polynomials would be shifted and we would have a root near the value $1 + \Phi$.

For most of Bari's polynomials, the golden root agrees with u to five or more decimal places. Tutte also found the zeroes of those cubic maps of up to eleven faces for which there is no ring of three faces and these zeroes have agreement of up to two decimal places of the golden root. The main body of this paper is a theoretical explanation of these empirical observations that these chromatic polynomials tend to have a root near $\phi + 1$. After the main theorem, I have included an Epilogue that discusses a related result to Tutte's found in the study of the Four Color Problem proving that $\phi + 1$ can never actually be a root to a chromatic polynomial.

3 Academic Motivation

My personal motivation for studying Tutte's result and its correlations is threefold: one part world-famous constant, one part discrete mathematics, and one part love of color. And it was with a broad search consisting of subject words Golden Ratio and Graph Theory that landed me pinpoint in the world of the chromatic polynomials and some striking properties with respect to their roots... or impossible roots for that matter.

As a child, I was infatuated with a number of things, but I had no idea that many of them would re-emerge in my life in the concept of a Master's Degree research paper. There was my obsession with the integers from as far back as I can remember, mainly as a result from being born on a Friday the 13th and the obvious attention it brought me as a child. There was my love of the concept of color and its prevalence in everything. Colors had personalities for me as a child, much the same as the integers did. Add to that an affinity for puzzles, mazes and other thinking games, one might not have been surprised I would develop an interest in the discipline of Discrete Mathematics.

In addition, I adored the realm of fantasy and its dragons and such. Naturally, I ventured into the realm of Dungeons and Dragons and was immediately fascinated with the dice that accompany the game.

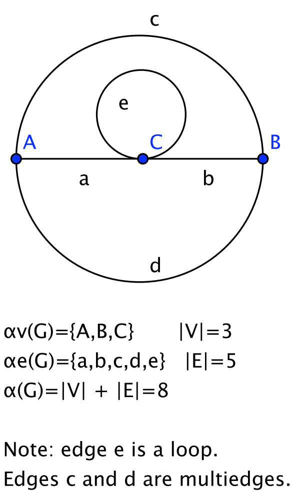

Not until later in life did I realize the behind-the-scenes beauty of the Platonic Solids, but the 5-headed Chromatic Dragon (all 5 heads being of different color) and the icosahedron would play a role for me 25 years later.

Chromatic polynomials or dragons, truncated icosahedrons or 20-sided dice, shapes or mysterious numbers, or connect the dots or planar triangulations... it makes no difference in the grand scheme of things because in the end everything is connected.

4 Definitions

Graph Constraints A *graph* G is a triple consisting of a *vertex set* $V(G)$, an *edge* set $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its *endpoints*. A *loop* is an edge whose endpoints are equal. *Multiple Edges* are edges having the same pair of endpoints. A *simple graph* is a graph having no loops or multiple edges.

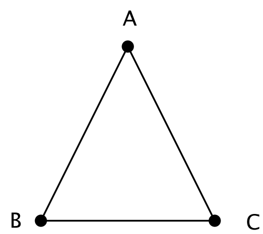
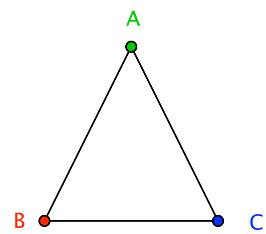
When u and v are the endpoints of an edge, they are *adjacent* and are *neighbors*. We write $x \leftrightarrow y$ for "x is adjacent to y". The graphs considered throughout this paper are finite and loopless. They may, however, have multiple edges.



Set Cardinalities Each vertex set and edge set has a finite number of elements in it. Edges from a vertex to itself are forbidden, but multigraphs are allowed. i.e. There may exist more than one edge from vertex to another. We denote the numbers of vertices and edges of a graph G by $\alpha_v(G)$ and $\alpha_e(G)$, respectively. We also write $\alpha(G)$ for the sum of these two numbers.

$$\alpha_v(G) = |V| \;\; where \; V = \{v_i \;\; | \;\; v \in V \; i \in \mathbb{Z}\}$$
$$\alpha_e(G) = |E| \;\; where \; E = \{e_i \mid v_i \sim v_j \longleftrightarrow \;\; e_i \; connects \; v_i \; to \; v_j\}$$
$$\alpha(G) = |V| + |E|$$

Vertex Coloring If G is a graph and n is a positive integer, we write $P(G,n)$ for the number of ways of properly coloring the vertices of G with n given colors so that no edge has both of its ends the same color. In such a coloring not all the n colors need be used. A permutation of the n colors that affects the colors actually used is considered to give a new coloring of the graph.

Deletion and Contraction Let A be an edge, with ends x and y, in a graph G. We write G'_A for the graph obtained from G by deleting A, but retaining its ends x and y. If A is the only edge of G with ends x and y, we write G''_A for the graph obtained from G'_A by identifying x and y. We say that G''_A is obtained from G by *contracting* A.
The following is a formal algorithm for the Deletion and Contraction method of breaking down a graph into its respective null graphs. We will soon see that determining *P(G,n)* with respect to null graphs is quite simple and thus the algorithm provides us with a basis of determining the number of ways of coloring the vertices, *P(G,n)* of any graph G.

Null Graph Algorithm

1- Give G a positive value.

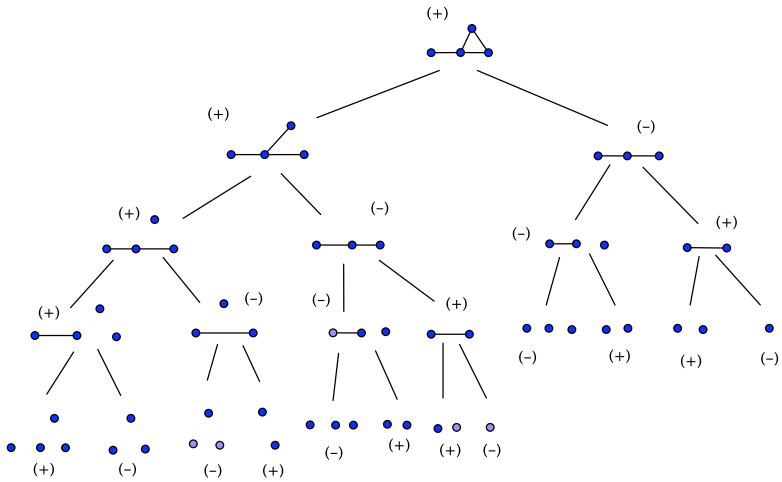
2- While there is a signed graph, and an edge, α, in the signed graph, do:

i- Choose a non-null signed graph and an edge α.

ii- Remove α from the graph, while keeping its sign if α was deleted, and negating the sign if α was contracted.

3- Sum up all the chromatic polynomials of the null graphs with the appropriate signs.

In the figure below, a step to the left is a deletion and a step to the right is a contraction. After each step, if the graph is not reduced down to a null graph, the algorithm is repeated. At the end of the algorithm, only null graphs remain.



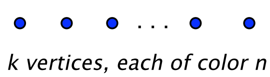
5 Set-up Theorems

From the definition of *P(G,n)* and *Deletion/Contraction*, we may derive the following theorems:

Theorem 1 *If G has $k > 0$ vertices and no edges, then $P(G,n) = n^k$.*

Proof. Suppose G were a graph with $k > 0$ vertices and no edges. Since no edges exist, surely no edge will have the same edge color regardless of any coloring. Thus, we have k vertices and n choices of colors for coloring each vertex. Therefore,

$P(G,n) = n \times n \times ... \times n$, k times $= n^k$



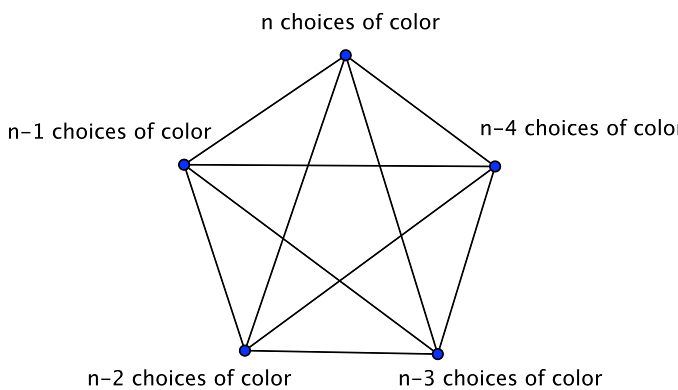
and we are done.

□

Theorem 2 *If G is a complete k-graph where $k > 0$ then*
$P(G,n) = n(n-1)(n-2)...(n-k+1)$.

Proof. Suppose G were a complete *k-graph* where $k > 0$. By definition, a complete *k-graph* is a graph with k vertices, each with valency $k-1$ and thereby adjacent to every other vertex in G. Choose any arbitrary vertex and color it some color of the n choices. Since this vertex is adjacent to each other vertex, then the next vertex must be of some color different than our first choice. Thus we have $n-1$ remaining colors to choose from. Continuing in this manner and counting each vertex along the way until all k vertices are colored, we obtain the following for $P(G,n)$:

$$P(G,n) = P(n,k) = \frac{n!}{(n-k)!} = \frac{n(n-1)(n-2)...(n-k+1)(n-k)...3\times2\times1}{(n-k)(n-k-1)(n-k-2)...3\times2\times1}.$$



Indeed,

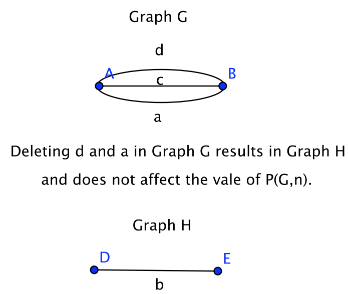
$$\frac{n(n-1)(n-2)...(n-k+1)(n-k)...3\times2\times1}{(n-k)(n-k-1)(n-k-2)...3\times2\times1} = n(n-1)(n-2)...(n-k+1)$$

and we are done.

□

Theorem 3 *If G has two edges with the same pair of ends, then the deletion of one of them does not affect the value of $P(G,n)$.*

Proof. Suppose G is a multigraph and contains multiple edges between the same vertices. Coloring has nothing to do with how more or less connected one vertex is to another, just whether or not it is connected at all. Therefore, without loss of generality, deleting all extra edges between the same of pair vertices so as to leave just one still keeps those two vertices adjacent and thereby a different color as required and not altering the value of $P(G,n)$.



$\square$

By a natural extension of the results of Theorems 1 and 2, we define $P(G, n)$ to be 1 when G is a null graph. It is convenient to refer to a null graph as the complete 0-graph. We use these conventions in the next theorem.

Theorem 4 *Let G be the union of two subgraphs H and K whose intersection is a complete k-graph ($k \geq 0$). Then*

$$P(G, n) \cdot P(H \bigcap K, n) = P(H, n) \cdot P(K, n).$$

Proof. Suppose G is a graph described as above. Then there would exist two disjoint sets of vertices whose colorings were not determined by one another, call them G_H and G_K. Since the vertex set of $H \bigcap K$ is that of a complete *k-graph*, it is disjoint from either G_H or G_K. Thus

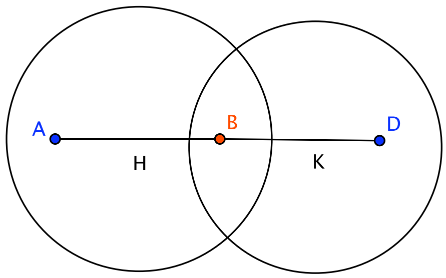
$P(H, n) = P(K_k, n) \cdot P(G_H) = n(n-1)(n-2)(n-r)(n-r-1) \cdot P(G_H)$
and

$P(K, n) = P(K_k, n) \cdot P(G_K) = n(n-1)(n-2)(n-r)(n-r-1) \cdot P(G_K)$

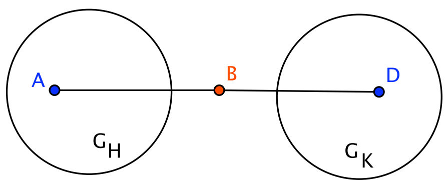
$P(H \bigcap K, n) = n(n-1)(n-2)(n-r)(n-r-1) \cdot$
$P(G, n) \cdot n(n-1)(n-2)(n-r)(n-r-1) =$

$= [n(n-1)(n-2)(n-r)(n-r-1)]^2 \cdot P(G_H) \cdot P(G_K) = P(H, n) \cdot P(K, n)$

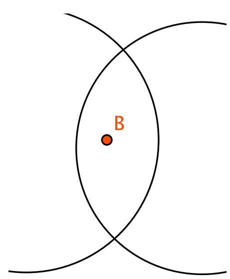
Graph, G



Note that H and K are both subgraphs of G,
where H=Path on 2 vertices and K=Path on 2 vertices



Two disjoint sets of vertices that are not
dependent on one another for their colorings



The vertex set of H
intersected with that
of K is that of a
complete graph

$$P(G,n) \cdot P(H \bigcap K, n) = P(H,n) \cdot P(K,n)$$

$$= P(G,n) \cdot P(K_1, n) = P(H,n) \cdot P(K,n)$$

$$= P(G,n) \cdot n = n(n-1) \cdot n(n-1)$$

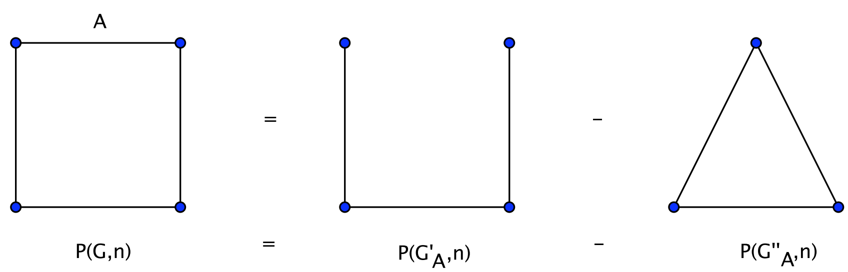
$$P(G,n) = \frac{n(n-1) \cdot n(n-1)}{n} = n(n-1)^2$$

$\square$

Theorem 5 *If G is a simple graph and A is an edge of G, then*

$$P(G,n) = P(G'_A,n) - P(G''_A,n).$$

Proof. Each proper $n - coloring$ of G is a proper $n - coloring$ of G'_A. A proper $n - coloring$ of G'_A is a proper $n - coloring$ of G if and only if it gives distinct colors to the endpoints of b,c of A. Hence we can count the proper n-colorings of G by subtracting from $P(G'_A,n)$ the number of proper n-colorings of G'_A that give b and c the same color.
Colorings of G'_A in which b and c have the same color correspond directly to proper $n - colorings$ of G_A", in which the color of the contracted vertex is the common color of b and c. Such a coloring properly colors all of the edges of G_A" if and only if it properly colors all the edges of G other than A.



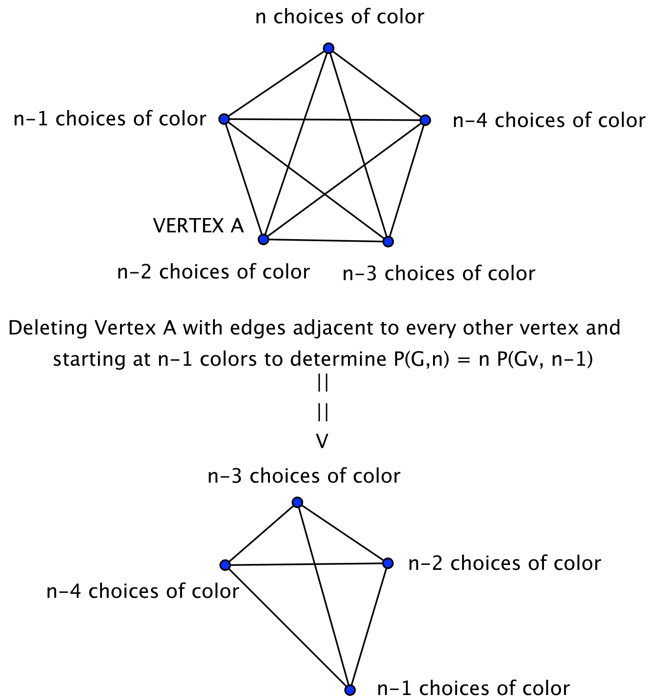
□

Theorem 6 *Let A be a vertex of G, and let it be joined by an edge to every other vertex of G. Then, if* G_A *is the graph obtained from G by deleting A and its incident edges, we have*

$$P(G,n) = n \cdot P(G_A, n-1).$$

Proof. Suppose G is a graph with a vertex A as described above. Then $A \in V | A_i \sim A_j, \ \ \forall i \neq j$. If G_A is defined as above, then we know that no other vertex in G has the same color as A, which is n, by definition of A's degree. Thus, since $P(G,n)$ is an arrangement with respect to n colors, we simply have selected a color from one of those n choices for the first slot in our arrangement. The rest of the arrangement is solely determined by the coloring of G_A without use of the one out of n colors just chosen. Thus, $n \cdot P(G_A, n-1)$ is the representation of our permutation of colorings on n colors and we are done.

□



We now make use of Theorems 1 through 6 to determine $P(G, n)$ in some simple cases.

Theorem 7 *Let T_k be a tree with exactly k vertices. Then*

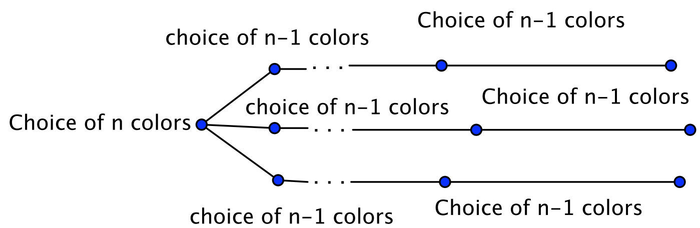
$$P(T_k, n) = n(n-1)^{k-1}.$$

Proof. We note the fact that any tree with $k \geq 2$ vertices has a vertex of degree 1. This proof is by induction on the number of vertices. The result is true for $n = 1$ since $P(T_k, 1) = (1) \cdot (1-1)^0 = 1$. Suppose it were true for a tree with T with $n = N - 1$ vertices, e an edge of T incident with a vertex of degree 1. Then T'_e has two components, one of which is an isolated vertex and the other a tree of $N - 1$ vertices. In this case, the latter case is the tree T''_e. Hence

$$P(T'_e, n) = nP(T''_e, n).$$

Using Theorem 5 and our induction hypothesis we obtain

$$P(T, n) = (n-1) \cdot P(T_e'', n) = (n-1)n(n-1)^{N-2} = n(n-1)^{N-1}.$$



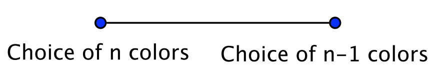
Thus the result is true when $n = N$, and therefore for all n by the principle of induction.

□

Let C_k denote a circuit of k edges where $k \geq 2$. Then by Theorems 2 and 3, we have the following lemma.

Lemma 1 $P(C_2, n) = n(n-1)$

Proof. Suppose our graph was C_2. Then we have a graph of two vertices and two edges, each edge connecting the vertices twice. We have already shown in Theorem 3 that the deletion of one of these edges does not affect the value $P(C_2, n)$. We have reduced this to a K_2 graph, a complete 2-graph, where $P(K_2, n) = n(n-1)$ by Theorem 2. The result follows immediately.

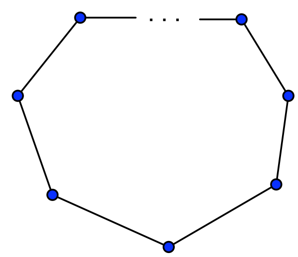


□

Moreover, the deletion-contraction method yields a more general result for any cycle graph on k edges. For $k \gtrsim 2$ and by Theorems 5 and 7, we obtain a nice theorem.

Theorem 8 *Let C_k be a cycle graph with exactly k vertices or edges. Then*

$$P(C_k, n) = (n-1)^{k-1} + (-1)^k(n-1).$$



Proof. If $k \geq 3$ the deletion of any edge from C_k results in a Path graph P_k where P_k is a tree on k vertices. The contraction of any edge results in a cycle graph C_{k-1} and thus by Theorem 5

$$P(C_k, n) = n(n-1)^{k-1} - P(C_{k-1}, n).$$

One may observe that $C_3 = K_3$ and thus, we have

$$P(C_3, n) = n(n-1)(n-2) = (n-1)^3 - (n-1).$$

Using $P(C_3, n)$ as an initial condition, we may solve the recursion given above and obtain our result:

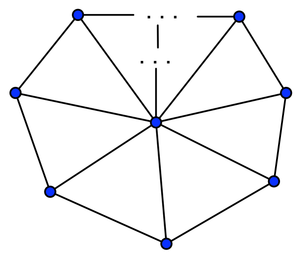
$$P(C_k, n) = (n-1)^k + (-1)^k(n-1).$$

□

Let W_k, be a *wheel* of k *spokes* . This means that W_k is obtained from a circuit C_k of k edges adjoining a new vertex v called the *hub*, and then joining v to the k vertices of C_k by k new edges called *spokes*. By Theorems 6 and 8 we have another theorem.

Theorem 9 *Let W_k be a wheel graph on k spokes. Then*

$$P(W_k, n) = n(n-2)^k + (-1)^k(n-2).$$



Proof. Suppose we have a graph that is a wheel defined above. Then by definition, there exists some circuit graph C_k with k edges and k vertices that creates W_k. Thus, using Theorem 6, we have $P(W_k, n) = nP(G_v, n-1)$ where G_v is the graph made when deleting vertex v and all edges incident to v. If we assume that v is the center of our wheel, then deleting v and all of the spokes results in the circuit graph C_k. Hence $P(W_k, n) = nP(G_v, n-1) = nP(C_k, n-1)$ and by Theorem 8 we already know that $P(C_k, n) = (n-1)^k + (-1)^k(n-1)$ Surely, substituting $n-1$ for n in the preceding equation results in our desired formula. $P(W_k, n) = nP(G_v, n-1) = nP(C_k, n-1) = n(n-2)^k + (-1)^k(n-2)$ and we are done.

□

6 Chromatic Polynomial

Chromatic Polynomial It is known that $P(G, n)$ is a polynomial in n, for each G, and that its degree is equal to the number of vertices of G. This result is proved by induction, using Theorems 1, 3, and 5. *Proof.* By Theorem 1, if a graph G has k vertices and no edges, then $P(G, n) = n^k$. This is obviously a polynomial in n and of degree k. By Theorem 2, if G is a complete K-graph, where $K > 0$, then $P(G, n) = n \cdot (n-1) \cdots (n-k+1)$. Again, we have a resulting polynomial in n with a product of k terms, and is thus of degree k. By Theorem 7, if G is a tree with k vertices, then $P(G, n) = n(n-1)^{(k-1)} \cdot n^{(k+1)} \cdot \cdots$.
$P(G, n)$ is known as the *chromatic polynomial* of G. As a polynomial it has a value not only when n is a positive integer but when n is any real or complex number, while providing us with an empirical number of proper colorings of any graph. .

7 Planar Graphs

Planar Graphs Tutte lightly states that it is known that a planar triangulation must have at least one vertex whose valency is less than 6. To prove this aside, let us first introduce the Handshaking lemma and a few corollaries:

Lemma 2 *In any graph, the sum of all the vertex degrees is equal to twice the number of edges.*

Proof. Since each edge has two ends (adjacent vertices), it must contribute exactly 2 to the sum of the degrees and the result follows immediately.

$$\sum \delta(G_v) = 2e$$

□

Lemma 3 *Euler's Formula: If a connected plane graph G has exactly n vertices, e edges, and f faces, then*

$$n - e + f = 2.$$

Proof. We induct on the number of vertices, n. The basis step for $n = 1$ gives us a bouquet of loops. But loops are not allowed so we simply have a null graph, resulting in $e = 0$ edges and giving us $f = 1$ face. Indeed, $1 + 0 + 1 = 2$ and the trivial base holds.

For our Induction step, suppose the statement is true for some $n>1$. Since G is connected and has more than one vertex, there exists some edge and that edge is not a loop. When contracting such an edge, we obtain a plane graph *G'* with *n'* vertices, *e'* edges, and *f'* faces. The contraction does not change the number of faces since we merely shortened boundaries, but it reduces the number of edges and vertices by 1, so $n' = n-1$,$e' = e-1$, and $f' = f$. Applying our induction hypothesis yields the following:

$n - e + f = n' + 1 - (e' + 1) + f' = n' - e' + f' = 2$
and our result is proven.

□

Corollary 1 *Let G be a connected planar simple graph with n vertices, where $n \geq 3$, and has m edges. Then $m \geq 3n - 6$.*

Proof. Let G have f faces. Then the Handshaking lemma allows for the planar graph that $2m \geq 3f$ (since the degree of each face of a simple graph is at least 3). So $f \leq 2/3\,m$. Apply Euler's formula:

$$n - m + f = 2 \Rightarrow m - n + 2 \leq 2/3\,m \Rightarrow m \leq 3n - 6$$

and the result is proven.

□

Corollary 2 *Let G be a connected planar simple graph with n vertices, m edges and NO triangles. Then* $m \leq 2n - 4$.

Proof. For a graph G with f faces, the Handshaking lemma shows for planar graphs that $2m \geq 4f$ (since the degree of each face of a simple graph without triangles is at least 4) so that $f \leq 1/2\,m$. Using Euler's formula shows

$$n - m + f = 2 \Rightarrow m - n + 2 = f \Rightarrow m - n + 2 \leq 1/2\,m \Rightarrow m \leq 2n - 4$$

However, suppose there are triangles as faces, more specifically all faces were triangles, then the Handshaking lemma implies

$2m = 3f \Rightarrow m = 3/2\,f \Rightarrow n - m + f = 2 \Rightarrow n - 3/2\,f + f = 2 \Rightarrow n - 1/2\,f = 2 \Rightarrow n - 2 = 1/2\,f \Rightarrow f = 2n - 4$
And now we finally come back to Tutte's claim:

Corollary 3 *Let G be a connected planar simple graph. Then G contains at least one vertex of degree 5 or less*

Proof. Suppose G is planar. Then by Corollary 1 above, we have $m \leq 3n - 6$. Now suppose that every vertex in G has degree 6 or more. By the Handshaking lemma,

$$\sum \delta(G_v) = 2e \leq 2(3n - 6) \leq 6n - 12.$$

. Therefore, the degree sum is at most $6n - 12$. Invoking the Pigeonhole Principle, we can divide $6n - 12$ by the number of vertices n to obtain the average vertex degree in G.

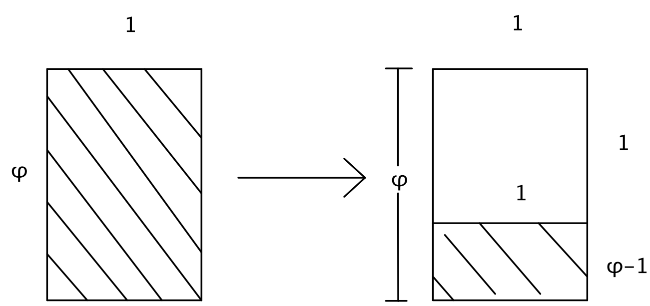
$$\frac{6n-12}{n} = 6 - \frac{12}{n} < 6$$

However, this is a contradiction since we assumed that every vertex had degree 6 or greater thus proving that there exists at least one vertex of degree 5 in the graph G.

$\square$

8 Phi, Φ

The Golden Ratio Nothing extraordinary about the the now popular ϕ is stated here, but for a the chance newcomer, let's take a quick look at where this remarkable concept comes from.



Given a rectangle having sides in the ratio $1 : x$, ϕ is defined as the unique number x such that partitioning the original rectangle into a square and new rectangle as illustrated above results in a new rectangle which also has sides in the ratio $1 : x$ (i.e., such that the shaded rectangles shown above are similar). Based on the above definition, we immediately see the ratio:

$$\frac{\phi}{1} = \frac{1}{\phi - 1}$$

resulting in the quadratic polynomial

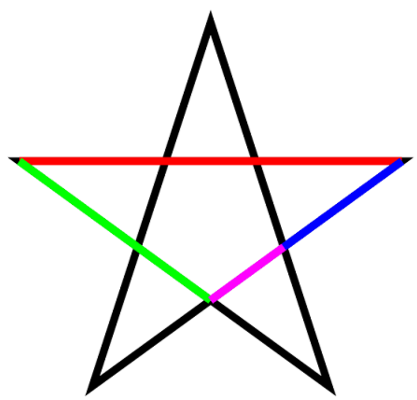
$$\phi^2 - \phi - 1 = 0.$$

Using the quadratic equation and solving the polynomial for its roots, we obtain the exact value of ϕ, namely

$\phi = 1/2 + \sqrt{5}/2$

$= 1.6180339887498948482045868343...$

There are number of ways in which to arrive at the Golden Ratio, ranging from Euclid's extreme and mean ratios of a line segment to nested radicals and recurrence relations to infinite series related to the Fibonacci numbers; however, my personal favorite was that of the Pythagoreans and their Pentagram, determining that each intersection of edges sections the edges in the golden ratio: the ratio of the length of the edge to the longer segment is ϕ, as is the length of the longer segment to the shorter. Also, the ratio of the length of the shorter segment to the segment bounded by the 2 intersecting edges (a side of the pentagon in the pentagram's center) is ϕ. The pentagram includes ten isosceles triangles, five acute and five obtuse isosceles triangles. In all of them, the ratio of the longer side to the shorter side is ϕ.



9 A Non-Integral Value of n

Let ϕ denote the golden ratio. Thus

$$\phi^2 = \phi + 1, \tag{1}$$

$$\phi = (1 + \sqrt{5})/2 = 1.61803398874989484820... \tag{2}$$

In this section we put $n = 1 + \phi$. Thus

$$n = \phi^2 = \phi + 1, \qquad n - 1 = \phi, \qquad n - 2 = \phi^{-1}, \tag{3}$$

by (1). Applying this result to Theorem 4 we obtain the following rule:

Theorem 10 *Let G be the union of two subgraphs H and K whose intersection is a complete k-graph, where $k = 0, 1, 2, or\ 3$. Then*

$$P(G, 1 + \phi) = \phi^{-\theta} P(H, 1 + \phi) P(K, 1 + \phi),$$

where $\theta = 0, 2, 3$ or 2, respectively.

Proof. There are four cases here:

Case 1: $k = 0,\ \theta = 0.\ H \bigcap K = \varnothing$
In the first case the intersection of H an K is the null graph. We have already defined the null graph to be the complete 0-graph and have $P(\varnothing, n) = 1$. Therefore, using Theorem 4, we may write $P(G, n)P(H \bigcap K, n) = P(H, n)P(K, n) = P(G, n)1 = P(H, n)P(K, n)$ and obtain $P(G, n) = P(H, n)P(K, n)$ Evaluating for $\theta = 0$, and substituting $n = \phi +\ 1$ from (3) we see that

$P(G, \phi + 1) = \phi^0 P(H, \phi + 1)P(K, \phi + 1) = P(H, \phi + 1)P(K, \phi + 1)$
and the result is proven.

Case 2: $k = 1,\ \theta = 2.\ H \bigcap K =$ complete 1-graph, K_1
A complete 1-graph has $P(K_1, n) = n$. Therefore $P(G, n)P(H \bigcap K, n) = P(H, n)P(K, n) \longleftrightarrow P(G, n)n = P(H, n)P(K, n)$ by direct result of Theorem 4. Substituting $n = \phi + 1 = \phi^2$ from (3), we obtain $P(G, \phi + 1)\phi^2 = P(H, \phi + 1)P(K, \phi + 1)$. Multiplying both sides by ϕ^{-2} yields our result for $\theta = 2$,

$P(G, \phi + 1) = \phi^{-2} P(H, \phi + 1)P(K, \phi + 1)$
and the result is proven.

Case 3: $k = 2,\ \theta = 3.\ H \bigcap K =$ complete 2-graph, K_2
A complete 2-graph has $P(K_2, n) = n(n-1)$. Therefore $P(G, n)P(H \bigcap K, n) = P(H, n)P(K, n) \longleftrightarrow P(G, n)n(n - 1) = P(H, n)P(K, n)$ by direct result of Theorem 4. Substituting $n = \phi + 1 = \phi^2$ from (3), we obtain $P(G, \phi + 1)\phi^2\phi = P(G, \phi + 1)\phi^3 = P(H, \phi + 1)P(K, \phi + 1)$. Multiplying both sides by ϕ^{-3} yields our result for $\theta = 3$,

$P(G, \phi+1) = \phi^{-3} P(H, \phi+1) P(K, \phi+1)$
and the result is proven.

Case 4: $k = 3,\ \theta = 2$. $H \bigcap K$ = complete 3-graph, K_3
A complete 3-graph has $P(K_3, n) = n(n-1)(n-2)$. Therefore $P(G, n)P(H \bigcap K, n) = P(H, n)P(K, n) \longleftrightarrow P(G, n)n(n-1)(n-2) = P(H, n)P(K, n)$ by direct result of Theorem 4. Substituting $n = \phi + 1 = \phi^2$ from (3), we obtain $P(G, \phi+1)\phi^2\phi\phi^{-1} = P(G, \phi+1)\phi^2 = P(H, \phi+1)P(K, \phi+1)$. Multiplying both sides by ϕ^{-2} yields our result for $\theta = 2$,

$P(G, \phi+1) = \phi^{-2} P(H, \phi+1) P(K, \phi+1)$
and the result is proven.

□

If v is a vertex of a graph G we write G_v for the graph obtained from G by deleting the vertex v and its incident edges.

We say that G is *wheel-like* at a vertex v if there is a circuit C of G satisfying the following conditions:

- v is not a vertex of C.
- Each vertex of C is joined to v by at least one edge of G.
- No edge of G joins v to a vertex not belonging in C.

Under these conditions we say that the circuit C *encloses* v and we obtain the following:

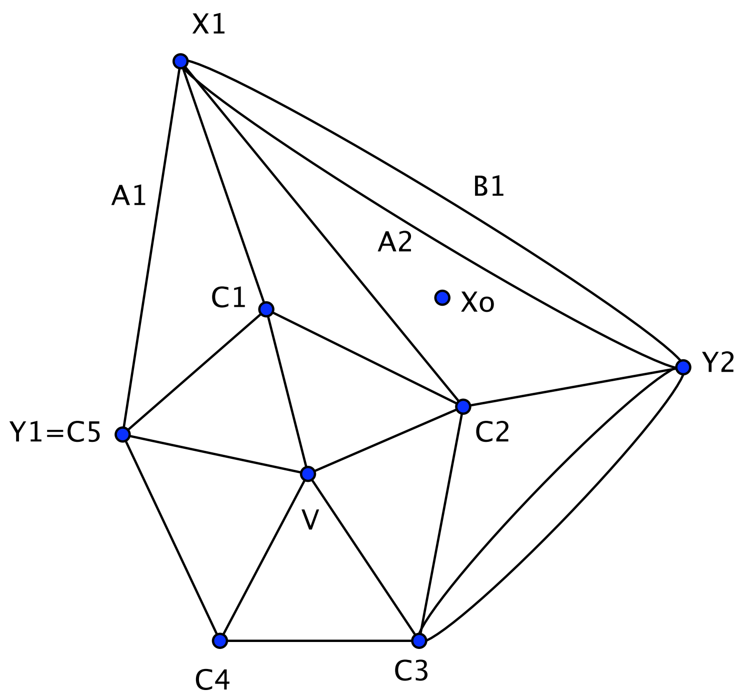
Theorem 11 *Let a vertex v of a planar graph G be enclosed by a circuit C of m vertices. Then*

$$P(G, 1+\phi) = (-1)^m \phi^{1-m} P(G_v, 1+\phi).$$

Proof. Proceed with induction. The theorem is trivially true when $\alpha(G) = 0$ since then no vertex v or circuit C can exist. Thus, m = 0 and

$P(G, 1+\phi) = (-1)^m \phi^{1-m} P(G_v, 1 + \phi) = (-1)^0 \phi^{1-0} P(G_v, 1 + \phi) = \phi P(G_v, 1 + \phi) = P(G, 1+\phi)$

Now we may assume as an inductive hypothesis that it is true whenever $\alpha(G)$ is less than some positive integer q and consider the case $\alpha(G) = q$. This now breaks into a few different cases that represent the possibilities that can happen.



CASE A. Suppose first that there is a vertex x of G that is distinct from v and does not belong to C.

CASE A1. It may happen that X_0 is incident with no edges of G. In this case we write H for the edgeless graph defined by the vertex X_0 and K for the graph G_{x_0}. Clearly, K is planar by our original hypothesis and v is still enclosed by the circuit C in K. Moreover, G_v is the union of H and K_v. We then have

$P(G, 1+\phi) = P(H, 1+\phi) \cdot P(K, 1+\phi)$ by Theorem 10,

$P(G, 1+\phi) = (-1)^m \phi^{1-m} P(H, 1+\phi) P(K_v, 1+\phi)$ by the inductive hypothesis,

$P(G, 1+\phi) = (-1)^m \phi^{1-m} P(G_v, 1+\phi)$ also by Theorem 10.

and the theorem is true for G.

CASE A2. Another possibility is that X_1 is joined to another vertex Y_2, necessarily not v, by two distinct edges A_2 and B_1. Let H be the graph obtained from G by deleting B_1. Then H_v is obtained from G_v by deleting B_1 and in neither case does the deletion affect the chromatic polynomial by Theorem 3. But v is still enclosed by the circuit C in H and the theorem is true for H by the inductive hypothesis. Hence, the theorem is also true for G.

CASE A3. In the remaining case X_1 is joined to another vertex $Y_1 \neq v$ by exactly one edge A_1. We consider the graphs G'_{A_1} and $G"_{A_1}$. These graphs are planar since the property of planarity is invariant with respect to deletions and contractions of edges. We note that

$$\alpha(G"_{A_1}) < \alpha(G'_{A_1}) < \alpha(G). \tag{4}$$

We see that

$$(G_v)'_A = (G'_A)_v, \qquad (G_v)''_A = (G''_A)_v. \tag{5}$$

Moreover, v is enclosed by the circuit C in each of the graphs G'_{A_1} and $G"_{A_1}$. Accordingly,

$P(G, 1+\phi) = P(G'_A, 1+\phi) - P(G''_A, 1+\phi)$ by Theorem 5,

$P(G, 1+\phi) = (-1)^m \cdot \phi^{1-m} \cdot P((G_v)'_A, 1+\phi) - P((G_v)''_A, 1+\phi)$

by (4), (5), and the inductive hypothesis,

$P(G, 1+\phi) = (-1)^m \cdot \phi^{1-m} \cdot P(G_v, 1+\phi)$ again by Theorem 5 and the theorem is true for G.

CASE B. We may now assume that G has no vertices other than v and the vertices of C. Let the vertices of C be enumerated as v_1, v_2,..., v_m in their cyclic order in C. By definition of a circuit, we can clearly deduce that $m \geq 2$.

CASE B1. Suppose first that there is an edge A of G whose ends are non-consecutive vertices of C. We may write them as v_i and v_j, where $2 < j < m$. There are edges A_1 and A_j joining v to v_1 and v_j, respectively. We consider the complete 3-graph T defined by the edges A, A_1, and A_j.

Since G is planar it is the union of two proper subgraphs H and K whose intersection is the circuit T and since G is wheel-like at v, we may assume H to include v_2 but not v_{j+1}, and K to include v_{j+1} but not v_2. We have

$$\alpha(H), \alpha(K) < \alpha(G). \tag{6}$$

We observe that v is enclosed in H by a circuit of j edges, and in K by a circuit of $m-j+2$ edges. Moreover G_v is the union of H_v and K_v, two subgraphs whose intersection is a complete 2-graph with edge A. Hence,

$P(G, 1+\phi) = \phi^{-2} \cdot P(H, 1+\phi) \cdot P(K, 1+\phi)$ by Theorem 10,

$$P(G, 1+\phi) = \phi^{-2} \cdot (-1)^j \cdot \phi^{1-j} \cdot P(H_v, 1+\phi) \cdot (-1)^{m-j+2} \cdot \phi^{-1-m+j} \cdot P(K_v, 1+\phi)$$

by (6) and the inductive hypothesis. Thus

$$P(G, 1+\phi) = (-1)^m \cdot \phi^{-2-m} \cdot P(H_v, 1+\phi) \cdot P(K_v, 1+\phi)$$

$$P(G, 1+\phi) = (-1)^m \cdot \phi^{1-m} \cdot P(G_v, 1+\phi)$$

again by Theorem 10 and the theorem is true for G.

CASE B2. In the remaining case G is equivalent to a wheel of m spokes and C to a circuit of m edges by Theorem 3. Then,

$P(G, 1+\phi) = \phi^2 \cdot \phi^{-m} + (-1)^m \cdot \phi^{-1}$, by Theorem 9 and (3),

$$P(G, 1+\phi) = (-1)^m \cdot \phi^{1-m} \cdot \phi^m + (-1)^m \cdot \phi$$

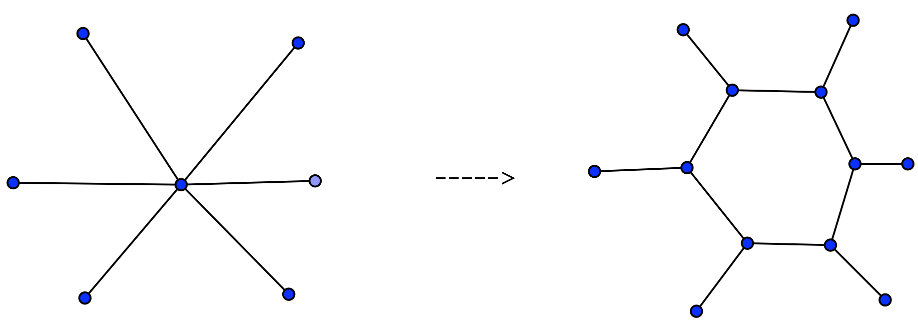
$P(G, 1+\phi) = (-1)^m \cdot \phi^{1-m} \cdot P(G_v, 1+\phi)$ by Theorem 8 and the theorem is true for G.

We have now shown that if the theorem holds when $\alpha(G) < q$, it holds also for $\alpha(G) = q$. Since it is trivially true when $\alpha(G) = 1$ it follows in general by induction and we are done.

□

10 Maps

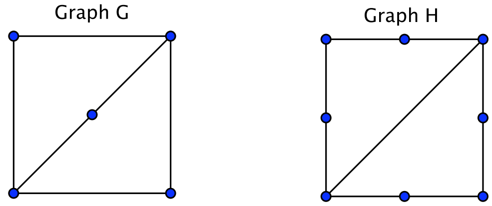
A regular map is a cubic map of simply connected regions without proper 2-rings or proper 3-rings. A cubic map is a collection of vertices, joined together by edges, to create enclosed faces with the restriction that exactly three edges meet at each vertex. Note that if a map has a vertex with more than three edges meeting at it, an entire new face can be added over the vertex. This result is due to the work the esteemed Arthur Cayley in 1879. This result is important to this paper as it allows us to determine a coloring of the newer cubic map and making the task of determining the original coloring trivial since we can simply shrink each face back down to the vertex it replaced.



Major maps are those regular maps which contain no regions with fewer than five sides. By deriving some of the properties of regular major maps, all such maps with fewer than twenty regions were determined within homeomorphisms in her work. Note that two graphs are homeomorphic if there is an isomorphism from some subdivision of one graph to some subdivision of the other graph.

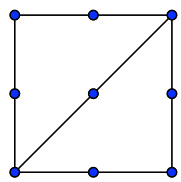
In the following example, graph G and graph H are homoeomorphic:

Graph G Graph H



If G' is the graph created by subdivision of the outer edges of G and H' is the graph created by subdivision of the inner edge of H, then G' and H' have a similar graph drawing:

G', H'



Therefore, there exists an isomorphism between G' and H', meaning G and H are homeomorphic.

Let a graph G be realized in the 2-sphere or closed plane so as to dissect the surface into simply connected regions each bounded by a circuit of G. For any natural number n, an n-sphere of radius r is defined as the set of points in (n + 1)-dimensional Euclidean space which are at distance r from a central point, where the radius r may be any positive real number. These regions are the *faces* of a *plane map* M of which G is the *1-section*, the face that bounds all other regions. A face is called a *m-gon* if its bounding circuit has exactly m edges. Evidently each face is at least a 2-gon.

The edges and vertices of G are called the *edges* and *vertices* of M, respectively. An edge or vertex is *incident* with a face F if it belongs to the bounding circuit of that face. The chromatic polynomial $P(G, n)$ of G is also called the chromatic polynomial of M, and accordingly may be written as $P(M, n)$.

The map M is called a *triangulation* of the 2-sphere or closed plane, or a *planar triangulation*, if its faces are all triangulations, that is, 3-gons. The class of all planar triangulations with exactly k vertices will be denoted by *Z(k)*. We write *Z(k, m)*, where m is a positive integer, for the class of all plane maps of k vertices in which one face is an m-gon and the others are all triangles.

11 Main Theorem

We now state and prove the main theorem of the paper.

Theorem 12 *If $M \in Z(k)$, then*

$$|P(M, 1+\phi)| \leq \phi^{5-k}$$

If $M \in Z(k,m)$, where $2 \leq m \leq 5$, then

$$|P(M, 1+\phi)| \leq \phi^{3+m-k}.$$

Proof. Proceed by induction on $\alpha_v(G)$. The theorem is trivially true when $\alpha_v(G) = 0$. Since there are no vertices, $Z(0)$ and $Z(m, 0)$ would surely be null. Indeed, evaluating any of the two absolute value inequalities with $P(m, n) = 1$ implies

$|1| \leq \phi^5$ *and* $|1| \leq \phi^{3+m}$ for any m >0

Assume as an inductive hypothesis that it is true whenever $\alpha_v(G)$ is less than some positive integer q, and consider the case $\alpha_v(G) = q$. Observe that we are working with q vertices now.

Suppose first that $M \in Z(q)$. It may happen that the 1-section G of M has a circuit C of two edges, i.e. a 2-gon or line. Then there exist positive integers q_1 and q_2 such that M can be formed from a member M_1 of $Z(q_1, z)$ and a member of M_2 of $Z(q_2, z)$ by deleting the 2-gons and identifying their bounding circuits. In other words, there exist numbers of vertices where our triangulation can be formed from a plane map on this number of vertices (i.e. q_1 and q_2) and one face is a 2-gon. We form M by deleting the 2-gons and identifying bounding circuits.

Evidently, q_1 and q_2 are both less than q and their sum is $q + 2$. This is because the whole number of vertices in M is q. We have two subscripted *M's* - namely, M_1 and M_2 that live within M and therefore must have a smaller amount of vertices than q. Now let H_1 and H_2 be the 1-sections of M_1 and M_2, respectively. These two subgraphs of G have G as their union and C as their intersection. Thus,

$|P(M, 1 + \phi)| = \phi^{-3} \cdot |P(M_1, 1 + \phi)| \cdot |P(M_2, 1 + \phi)|$ (By Theorem 2 and Theorem 10 since "k" $= 2 \Rightarrow \theta = 3$)

$|P(M, 1 + \phi)| \leq \phi^{-3} \cdot \phi^{5-q_1} \cdot \phi^{5-q_2}$ (By our inductive hypothesis)

$|P(M, 1 + \phi)| \leq \phi^{-3} \cdot \phi^{10} \cdot \phi^{-q_1} \cdot \phi^{-q_2}$

$|P(M, 1 + \phi)| \leq \phi^{7} \cdot \phi^{-q_1 \cdot -q_2}$

$|P(M, 1 + \phi)| \leq \phi^{7} \cdot \phi^{-q-2}$

Resulting in $|P(M, 1 + \phi)| \leq \phi^{5-q}$.

We may now suppose G to have no circuit such as C.
It may happen that G is a complete 3-graph. Then M has exactly two faces, with the same bounding circuit G. In this case,

$|P(M, 1 + \phi)| = \phi^2 = \phi^{5-q}$ (by Theorem 2)

$|P(G, k)| = k(k-1)(k-2)$ where $k = 3 \Rightarrow \theta = 2$

$|P(G, 1 + \phi)| = \phi^{-2} \cdot k[(k-1)(k-2)] = \phi^{-2} \cdot \phi^2[\phi \cdot (\phi - 1)] = \phi(\phi) = \phi^2$

or

$q = 3 \Rightarrow P(M, 1 + \phi) = \phi^{5-q} = \phi^{5-3} = \phi^2$

and the theorem is satisfied.

It may also occur that G is wheel-like at v, and v is enclosed by a circuit in G of m edges, where m is the valency of v in G. By deleting v and its incident edges and fusing the triangles of M incident with v into a single new face, we derive from M a member M_v of $Z(q-1, m)$. The 1-section of M_v is G_v. Then

$|P(M, 1 + \phi)| = \phi^{1-m} \cdot |P(M_v, 1 + \phi)|$ (by Theorem 11)

$|P(M, 1 + \phi)| = \phi^{1-m} \cdot |P(M_v, 1 + \phi)| \leq \phi^{1-m} \cdot \phi^{3+m-q}$ (by induction hypothesis)

But we deleted v so we now have $q - 1$ vertices and thus

$|P(M, 1 + \phi)| \leq \phi^{1-m} \cdot \phi^{3+m-q} = \phi^{1-m} \cdot \phi^{3+m-(q-1)} = \phi^{1-m} \cdot \phi^{4+m-q}$

$|P(M, 1 + \phi)| \leq \phi^{5-q}$ and the theorem is satisfied.

We have now shown that M satisfies the theorem if it belongs to $Z(q)$.

Suppose that $M \in Z(q,m)$ where $m = 2, 3, 4\, or\, 5$.

For $m = 2$, we can convert M into a planar triangulation N by deleting one edge of the 2-gon. The deletion does not alter the chromatic polynomial by Theorem 3. So, by the result already proven

$|P(M, 1+\phi)| = |P(N, 1+\phi) \leq \phi^{5-q} = \phi^{3+m-q}|$ since $m = 2$.

For $m = 3$, the M is a planar triangulation as all faces obviously would be triangles. Again, by the result already proven

$|P(M, 1+\phi)| \leq \phi^{5-q} < \phi^{3+m-q}|$ since $m = 3$.

In the remaining case, $m = 4$ or $m = 5$, we can find two non-consecutive vertices x and y in the bounding circuit of the m-gon such that no edge of G joins x and y. We can add a new edge A joining x and y so as to subdivide the m-gon into a triangle and an $(m-1)-gon$. (i.e. a $\triangle$ and another $\triangle$ (3-gon) or a $\triangle$ and a $\square$ (4-gon)). Let us denote the resulting map by N and its 1-section by J. Instead of adding A, we can identify x and y and delete one edge incident with a resulting 2-gon. This procedure yields a map $N_1 \in Z(q-1, m-2)$ (note that $Z(q-1, m-2)$ has one less vertex and 2 less edges). Let us denote the 1-section of N_1 by H. Then $G = J'_A$ (J minus edge A yet keeping x and y) and H have the same chromatic polynomial as J''_A (contracting and identifying$x$ and y) by Theorem 3. Using Theorem 5 we find that

$P(G, n) = P(G'_A, n) - P(G''_A, n)$

$P(M, 1+\phi) = P(N, 1+\phi) - P(N_1, 1+\phi)$

$|P(M, 1+\phi)| \leq |P(N, 1+\phi)| + |P(N_1, 1+\phi)|$

If $m = 4$, we have $N \in Z(q)$ (since it is a obvious triangulation) and $N_1 \in Z(q-1, 2)$. Hence, by the inductive hypothesis and the results already proven we have

$|P(M, 1+\phi)| \leq |P(N, 1+\phi)| + |P(N_1, 1+\phi)| = \phi^{5-q} + \phi^{5-(q-1)} = \phi^{5-q} + \phi^2 = \phi^{7-q} = \phi^{3+m-q}$ since $m = 4$

If $m = 5$, we have $N \in Z(q, 4)$ and $N_1 \in Z(q-1)$. Hence, by the inductive hypothesis and the result of the previous paragraph we have

$|P(M, 1+\phi)| \leq |P(N, 1+\phi)| + |P(N_1, 1+\phi)| = \phi^{7-q} + \phi^{5-(q-1)} = \phi^{7-q} + \phi^{5-q} \cdot \phi = \phi^{8-q} = \phi^{3+m-q}$ since $m = 5$

By induction, the theorem is now proven.

$\square$

12 Epilogue

It turns out that the Four Color Theorem is equivalent to stating that the vertices of any planar graph can be 4-colored. We just have to look at the dual graph - a graph where two vertices are connected by an edge if the corresponding faces in the original graph have a boundary edge in common. It is obvious that the vertices of the original graph are in a one-to-one correspondence with the edges of the dual graph. This tells us that another way to state the 4-color theorem is that for no planar graph does the chromatic polynomial $P(G, n)$ have a root at $n = 4$. The *chromatic number* of a graph is the smallest positive integer such that the chromatic polynomial is greater than zero.

Conjecture: ϕ+1 is never a root of the chromatic polynomial of a graph.

But we must consider the behavior of $P(G, n)$ on the real number line by introducing two theorems, a lemma and a corollary...

Theorem 13 *If $n < 0$ and G is not the empty graph, then for G connected*

$$P(G, n) = \begin{Bmatrix} >0 & if\ \alpha_v(G) = order\ of\ G\ is\ even \\ <0 & if\ \alpha_v(G) = order\ of\ G\ is\ odd \end{Bmatrix}.$$

Proof. Suppose

$$P(G, n) = n^{\alpha_v(G)} - \ An^{\alpha_v(G)-1} + . \ . \ . + (-1)^{\alpha_v(G)-1} \ A_{\alpha_v(G)-1} n.$$

Then for $n > 0$,

$$P(G, n) = (-1)^{\alpha_v(G)} (|n|^{\alpha_v(G)} + \ A_1 |n|^{\alpha_v(G)-1} + \ . \ . \ . \ + A_{\alpha_v(G)} |n|),$$

where $|n|$ denotes the absolute value of n. The theorem follows immediately upon a parity inspection of the leading $(-1)^{\alpha_v(G)}$ term and the result is proven.

□

Lemma 4 *Let $k(G)$ denote the number of components of G. There is a polynomial $S(G,n)$ such that $P(G,n) = n^{k(G)} \cdot S(G,n)$*

Proof. Let $k = k(G)$ and let $G_1, \ldots, G_k$ be the components of G. Then

$P(G,n) = \prod_{i=1}^{k} P(G_i,n)$ (by theorem 5).

Notice that if G is not the empty graph and has order n, then $\alpha_n(G) = 0$. This is because $P(G,0) = \alpha_n(G)$, but $P(G,0) = 0$ unless G is the empty graph. Now we can immediately see that each factor $P(G_i,n)$ is divisible by n and so $P(G,n)$ is divisible by n^k. Therefore, $P(G,n) = n^{k(G)} \cdot S(G,n)$ and the result is proven.

□

Note that if G has a loop, one writes $S(G,n) = 0$.

Theorem 14 *If G has no loops and G is not the empty graph, then $(-1)^{\alpha_v(G)+k} \cdot S(G,n) > 0$, where $\alpha_v(G)$ is the order of, the number of components, $k =$ k(G), and with $0 < n < 1$.*

Proof. Suppose the theorem is false and let G be a couter-example with a minimum number of m edges. Certainly, $m > 0$ since a graph H without edges has $\alpha_v(G) = k$ and $S(H,n) = 1$. Also, G cannot be a forest (a graph each of whose components is a tree), for if G were a forest, then $P(G,n) = n^k(n-1)^{\alpha_v(G)-k}$ and so $S(G,n) = (n-1)^{\alpha_v(G)-k}$. Since $n < 1$, $n-1 < 0$ and so $(-1)^{n+k}(n-1)^{\alpha_v(G)-k} > 0$.
Since G is not a forest, it must have an edge A which is not a bridge (an edge whose removal disconnects the graph). Hence, $k(G) = k(G'_A)$ and, for any edge B, $k(G) = k(G''_B)$.
Now, for any graph G of order $\alpha_v(G)$, let us write $T(G,n) = (-1)^{\alpha_v(G)+k}S(G,n)$ where $k = k(G)$. Since $P(G,n) = P(G'_A,n) - P(G''_A,n)$ by Theorem 6 and $k(G) = k(G'_A) = k(G''_A)$, $S(G,n) = S(G'_A,n) - S(G''_A,n)$. But G and G'_A have order $\alpha_v(G)$ while G''_A has order $\alpha_v(G) - 1$. Hence,

$$T(G,n) = T(G'_A,n) + T(G''_A,n)$$

Since G is loopless, G'_A is loopless with fewer edges than G so $T(G'_A,n) > 0$ for $n < 1$. Also, $T(G''_A,n) > 0$ for $n < 1$. Therefore, $T(G,n) > 0$ for $n < 1$ and, indeed, $(-1)^{\alpha_v(G)+k} \cdot S(G,n) > 0$ and the result is proven.

□

Corollary 4 *For $0 < n < 1$, and if G connected and loopless of order $\alpha_v(G)$, then*

$$(-1)^{\alpha_v(G)} P(G,n) < 0.$$

Proof. Write $P(G,n) = nS(G,n)$. Now $(-1)^{\alpha_v(G)+1} S(G,n) > 0$ by Theorem 14. Hence,

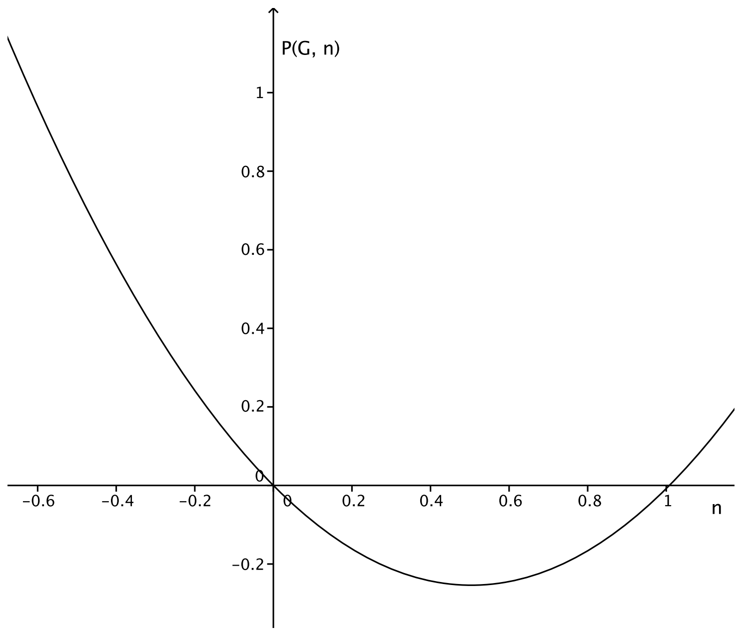
$$(-1)^{\alpha_v(G)} P(G,n) = (-1)^{\alpha_v(G)} nS(G,n)$$
$$(-1)^{\alpha_v(G)} nS(G,n) = (-n)(-1)^{\alpha_v(G)+1} S(G,n)$$
$$(-n)(-1)^{\alpha_v(G)+1} S(G,n) < 0$$

and we are done.

□

Observe that connectivity is required in the hypothesis of the previous corollary. Else, the graph of $G = K_2$, the disjoint union of two copies of K_2 results in chromatic polynomial $P(G,n) = n^2(n-1)^2$. Clearly, $P(G,n) > 0$ and, in fact, $(-1)^4 P(G,n) > 0$ for $0 < n < 1$.

Tying together the information we have about the golden ratio, its occurrence in the chromatic polynomials of a specific family of graphs, the real roots of those chromatic polynomials, we are able to obtain a picture of the behavior of $P(G,n)$ for $n \leq 1$ for a graph that is loopless, connected, nontrivial graph. Without loss of generality, assume that the order $\alpha_v(G)$ of G is even. Then, we can graph $P(G,,n)$:



And finally, with the completed proof of the Four Color Theorem, Saaty and Kainen have shown that:

Theorem 15 $\phi + 1$ *is never a root of the chromatic polynomial of a graph.*

Proof. Suppose $(3 + \sqrt{5})/2 = \phi + 1$ is a root of $P(G, n)$ for some loopless graph. Then, since $P(G, n)$ is a polynomial with integral coefficients, $(3 - \sqrt{5})/2$ must also be a root. But $0 < (3 - \sqrt{5})/2 < 1$, contradicting Corollary 4 and we are done.

□