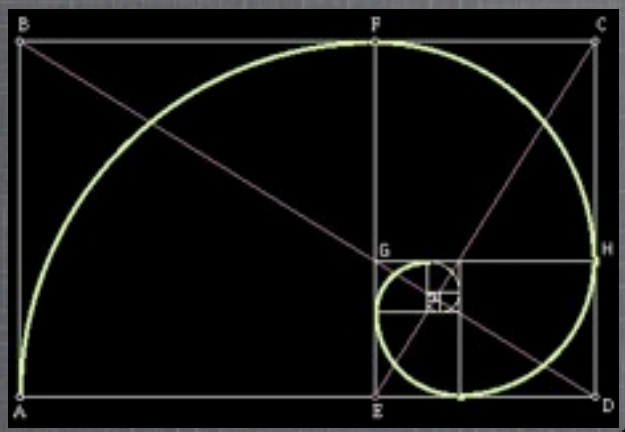
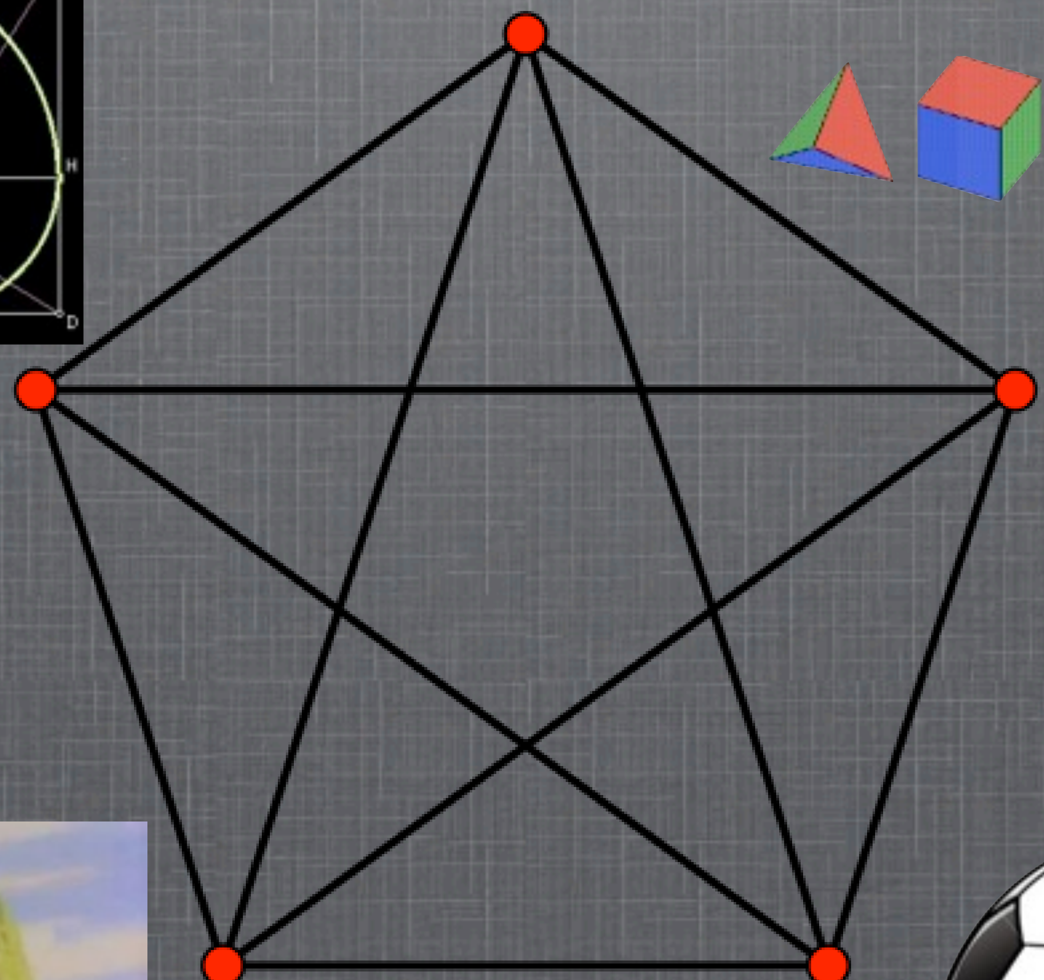
GOLDEN ROOTS OF CHROMATIC POLYNOMIALS

by Troy Parkinson

Color





WHO, WHERE & WHEN

- Ruth Bari - Johns Hopkins Univ., 1966

- William Tutte - Univ. of Waterloo, Canada, 1968

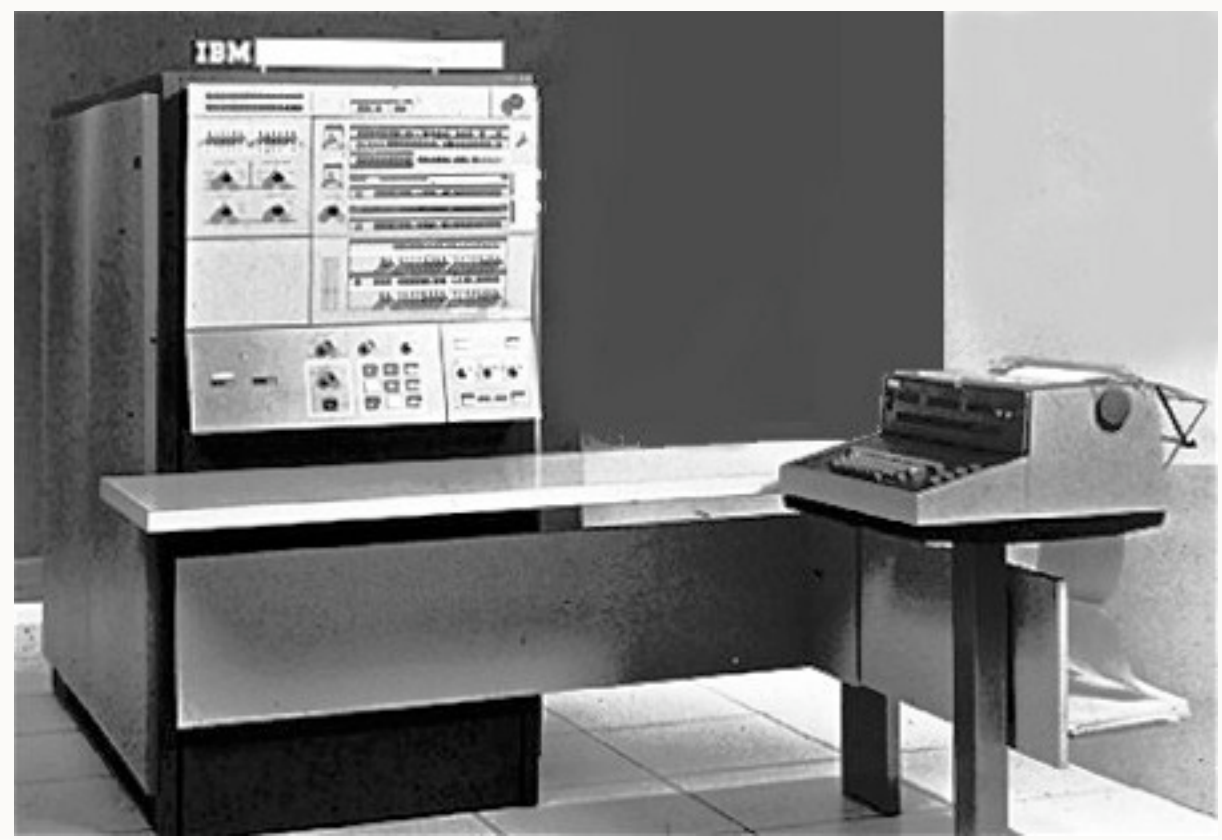
- Saaty & Kainen - The Four-Color Problem, 1977

WHAT & WHY - BARI

- Absolute Reducibility of Maps of at Most 19 Regions
- All maps with < 20 regions determined up to homeomorphisms

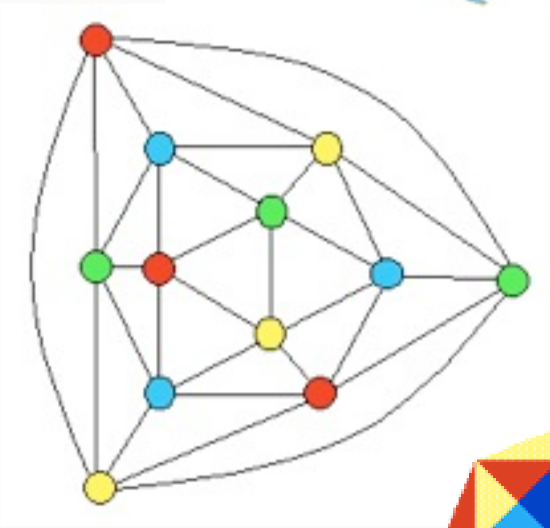
WHAT & WHY - TUTTE

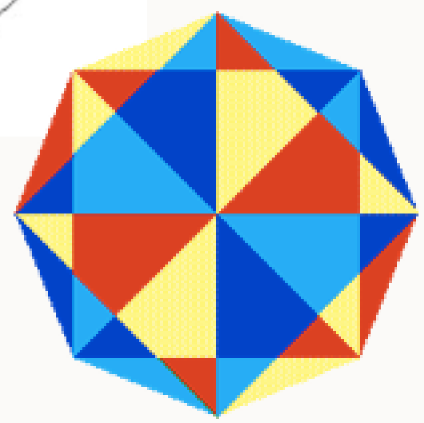
- 1 negative, non-integral root, $u = (-3 + \sqrt{5})/2 = -0.38196601...$
- $u + 2 = \phi = 1.618033988...$
- $(3 + \sqrt{5})/2 = 2.618033988...$ = Golden Root ➔ $\phi + 1$



WHAT & WHY - SAATY & KAINEN

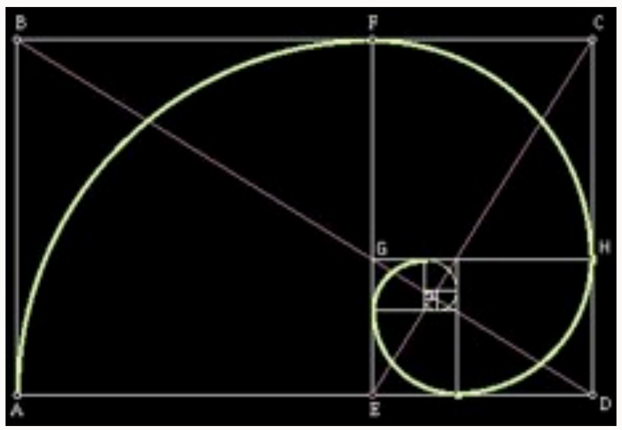
- Four-Color Theorem - proved in 1976 by Appel & Haken
- 1st major theorem to be proved using a computer
- Four-Color Problem - Assaults & conquest.
- ϕ + 1 is never a root of a chromatic polynomial





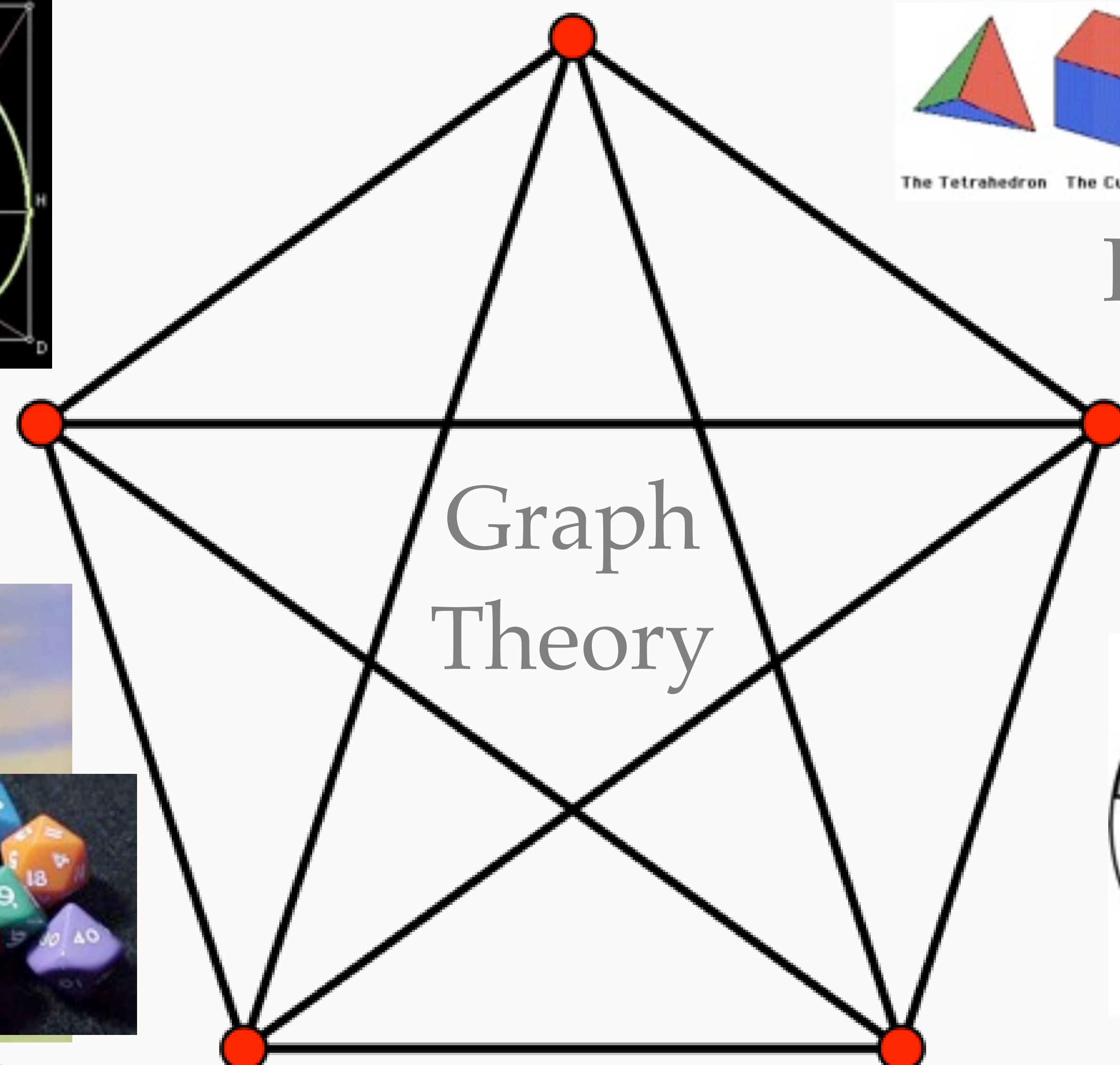
WHAT & WHY - PARKINSON
ACADEMIC MOTIVATION

Color



Platonic Solids

ϕ



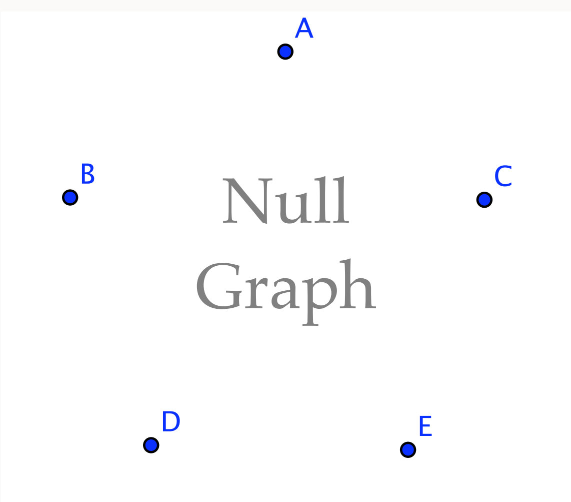
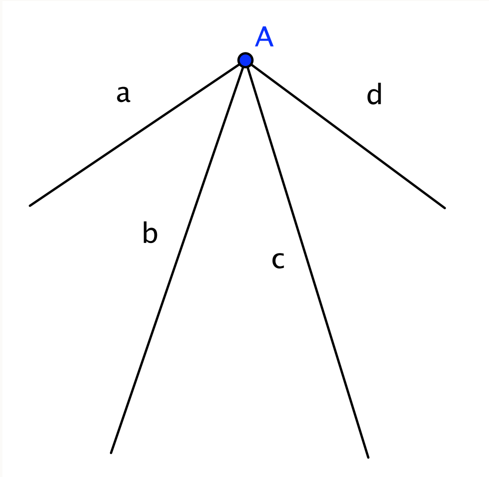
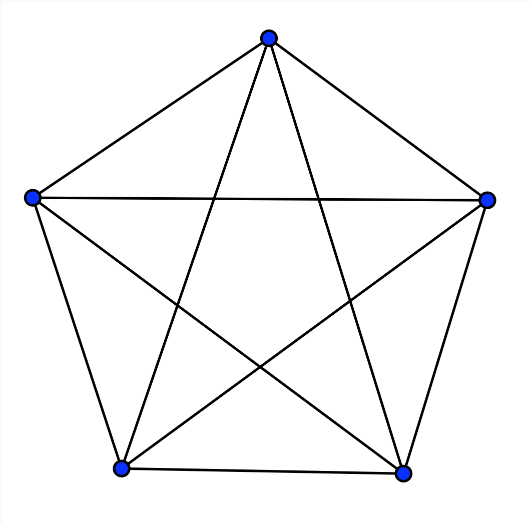
Graph Theory

Dungeons&Dragons

Soccer Ball

TERMINOLOGY

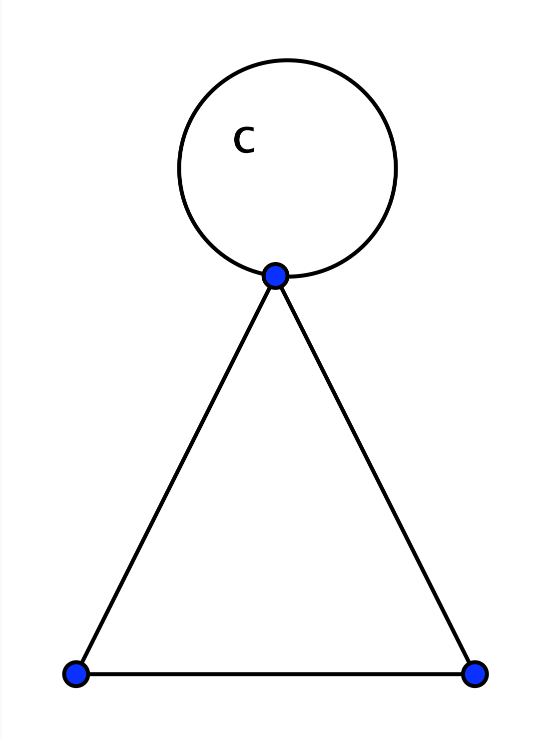
Graph, G:
{V, E, ~}

Vertices	Edges	Adjacency Relation
		
V={A,B,C,D,E} |V|= 5	E={a,b,c,d} |E|= 4	All $V_i \sim V_j$ |V|= 5, |E|= 10

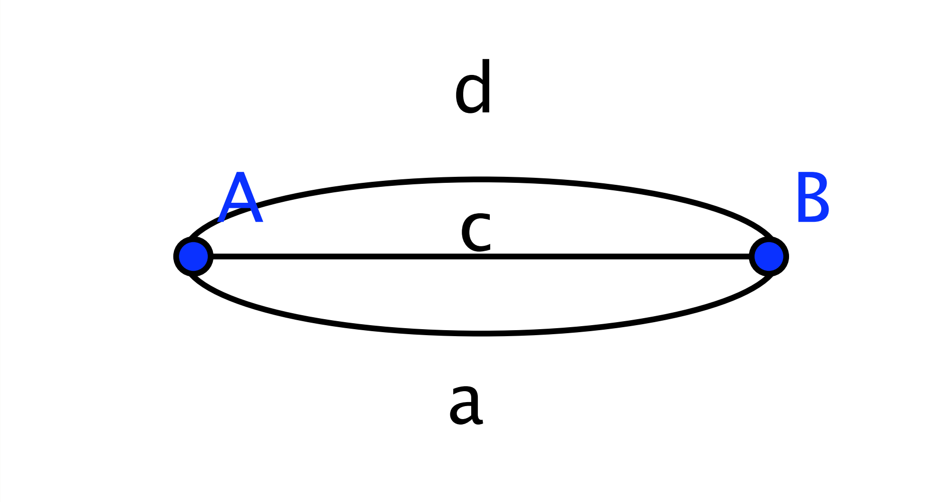
Vertex *Degree*: the number of edges at a vertex

SIMPLE GRAPHS

Loop



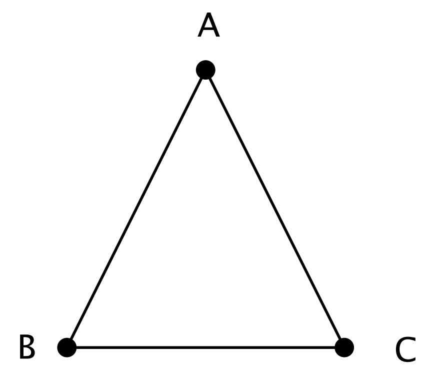
Multi-Edges

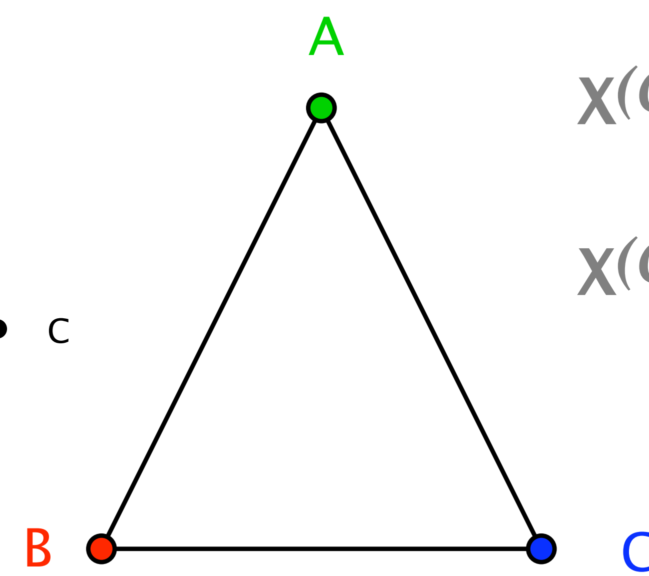


All graphs finite and loopless, but multi-edges have no effect on outcome.

VERTEX COLORING

- For a Graph, *G* and a positive integer, *k*

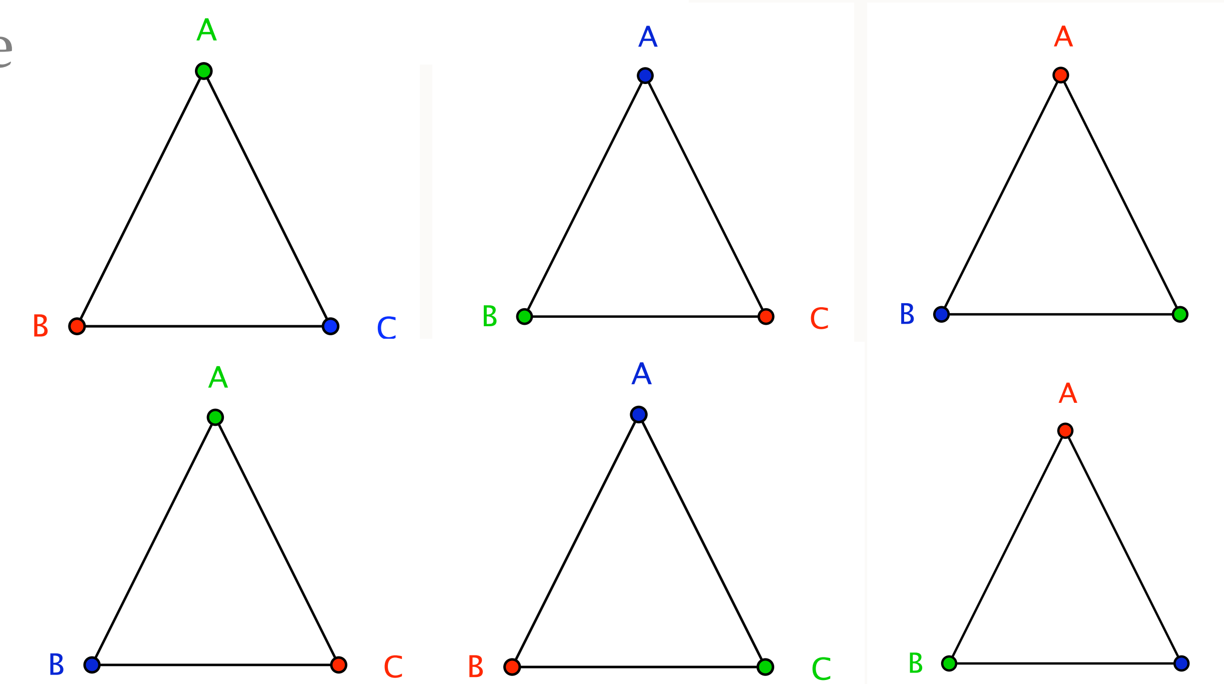




$\chi(G,0) = 0,\ \ \chi(G,1) = 0,$

$\chi(G,2) = 0,\ \ \chi(G,3) = 6$

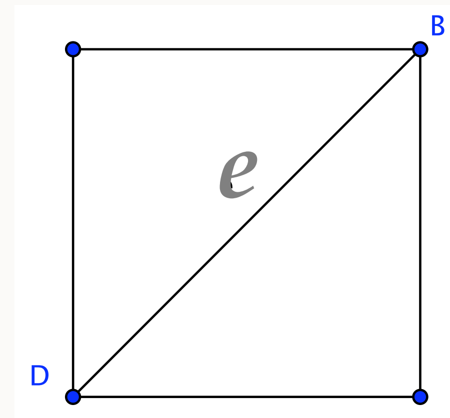
- $\chi(G,k)$: number of ways to color *G* with *k* given colors so that no edge has both of its ends the same color



- Not all *k* colors need be used, Permutations of *k* colors used gives a new coloring for *G*

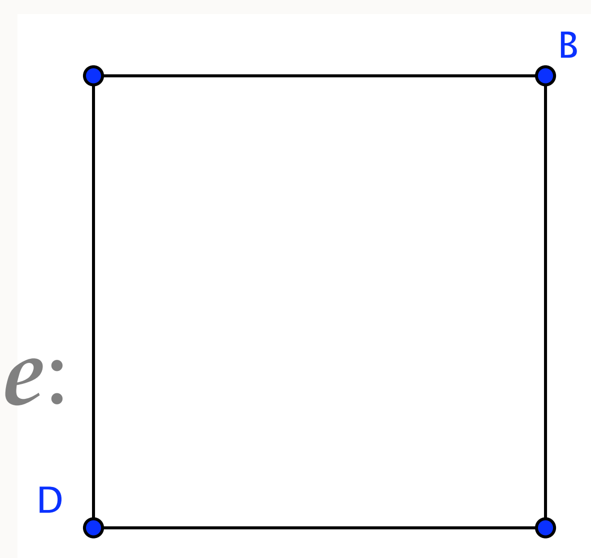
DELETION/ CONTRACTION

Let G:



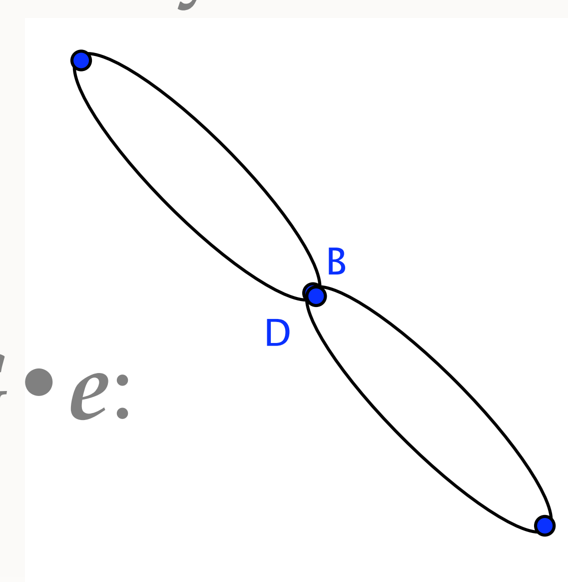
Deletion: remove an edge, keep its vertices

G-e:



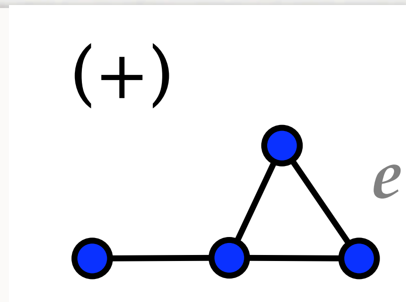
Contraction: remove an edge and identify their vertices

$G \bullet e$:

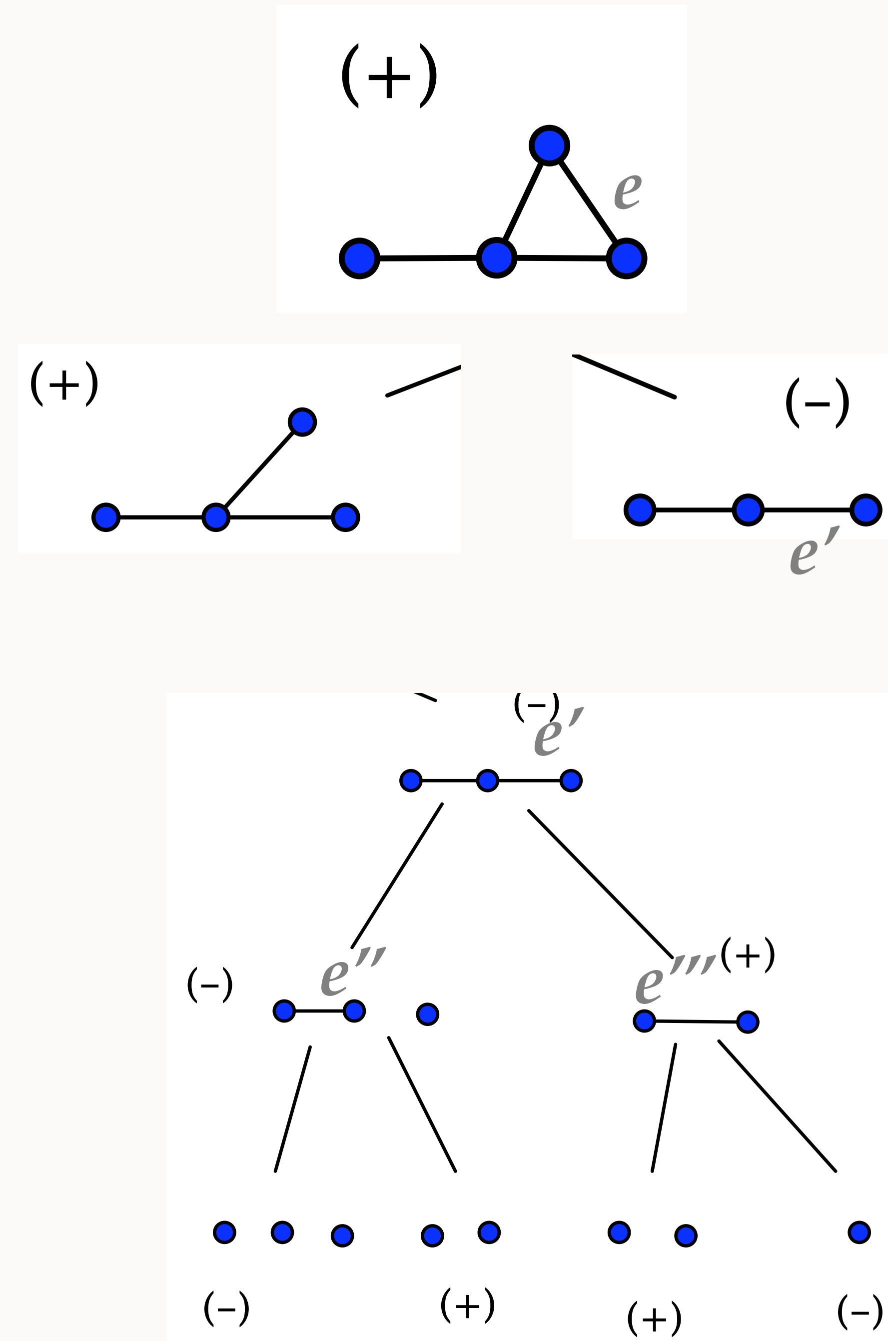


DELETION/CONTRACTION ALGORITHM

- Give ***G*** a *positive value*

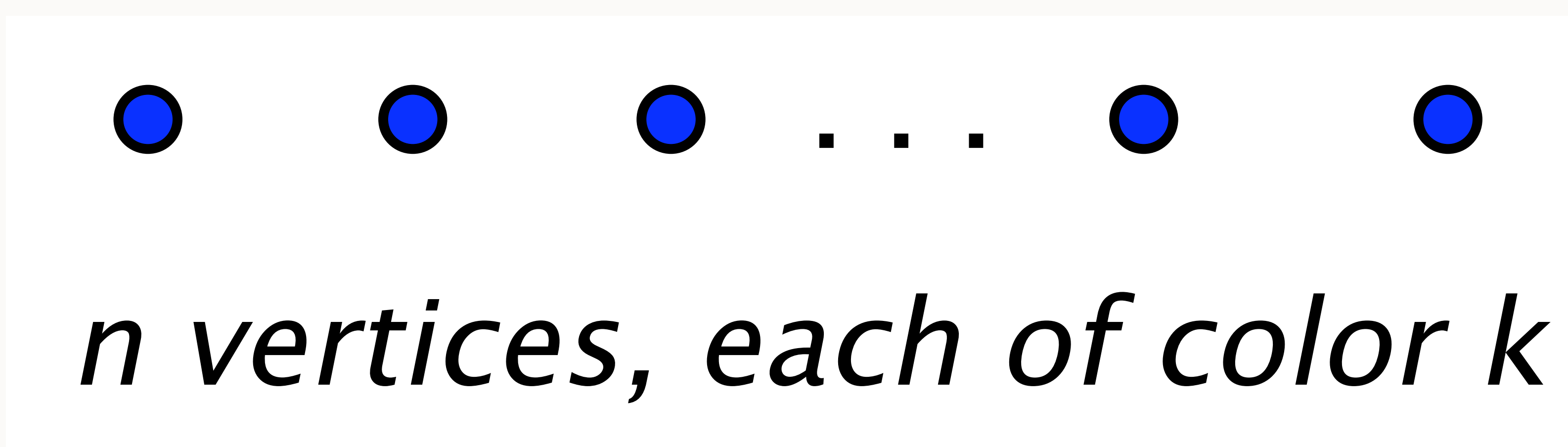


- While there is a *signed* graph, and an edge, ***e***, in the signed graph, Do:
  - Choose a non-null, signed graph and an edge, ***e***
  - Remove ***e*** from the graph, while **keeping** its sign if ***e*** was *deleted*, and **negating** its sign if ***e*** was *contracted*
- Sum up all **χ*(G,k)*** of null graphs with the appropriate signs



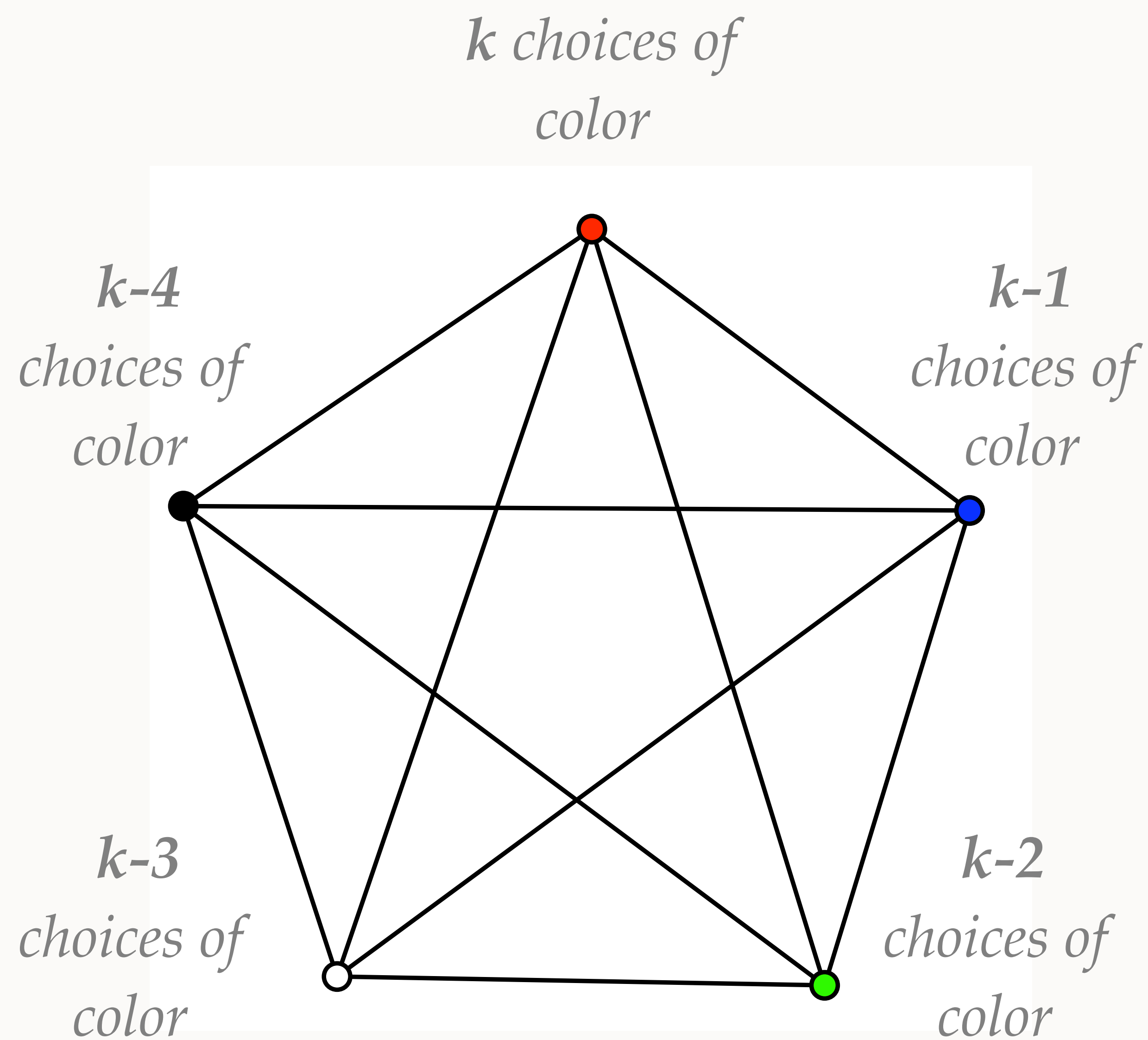
THEOREM 1: IF G HAS *n* >0 VERTICES AND NO EDGES, THEN $\chi(G, k) = k^n$

- Suppose G is a Null Graph.
- Assume ∃ *n* vertices, *k* colors
- No Edges ⇒ Vertices are not adjacent
- $\chi(G, n) = \overbrace{k \cdot k \cdot k \cdot \ldots \cdot k \cdot k}^{n \text{ times}} = k^n$



THEOREM 2: IF G IS A COMPLETE GRAPH WITH n>0 VERTICES, THEN $\chi(K_n, k) = k!/(k-n)!$

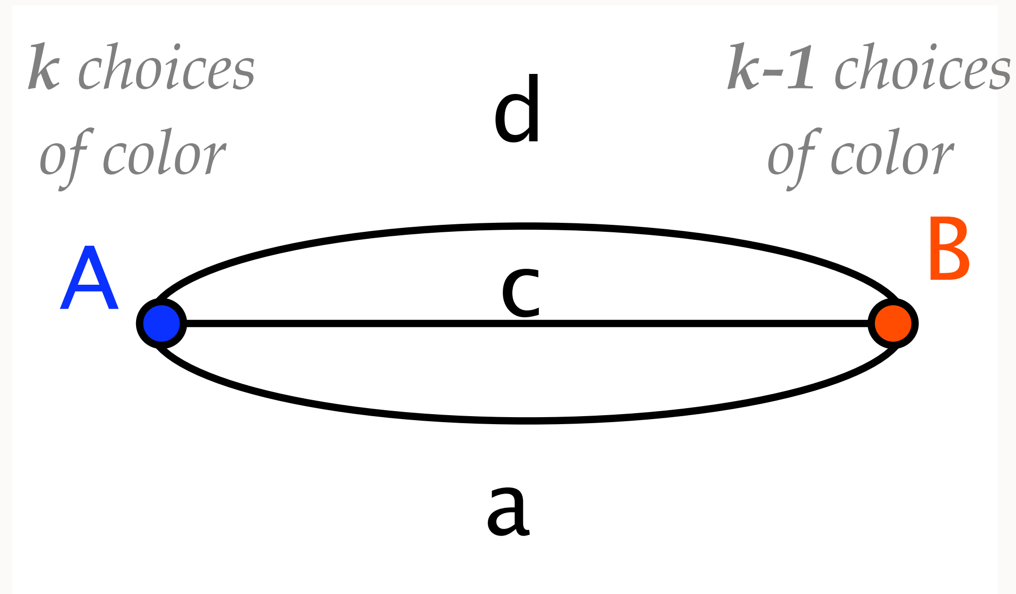
- Suppose G is a Complete Graph, K_n
- All vertices are adjacent to one another
- Choose any vertex and color it with one of *k* colors available.
- Next vertex has ***k-1*** remaining colors to choose from and so on.
- $\chi(K_n, k) = k!/(k-n)!$



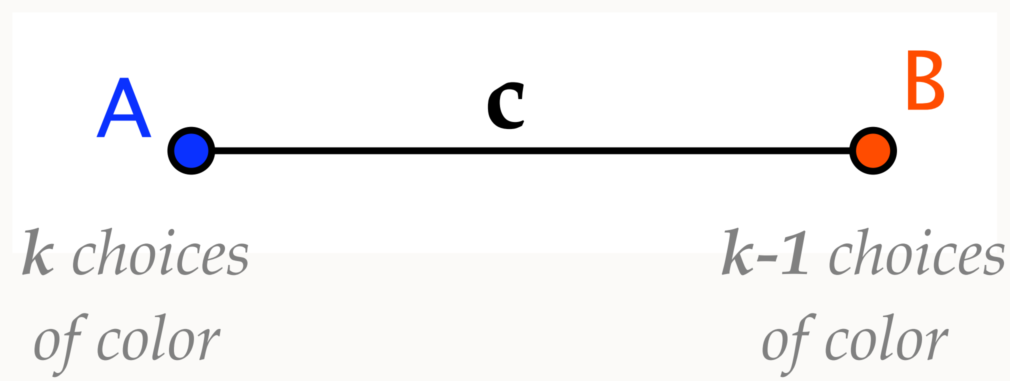
For n=5, $\chi(K_5, k) = k(k-1)(k-2)(k-3)(k-4)$

THEOREM 3: IF G HAS 2 EDGES WITH SAME PAIR OF VERTICES, THEN THE DELETION OF 1 OF THE EDGES DOES NOT AFFECT THE VALUE OF X(G, k)

- Suppose *G* contains multi-edges
- Delete *extra* edges
- More or less adjacent has no effect on colorings
- Unchanged **X**(G,k)



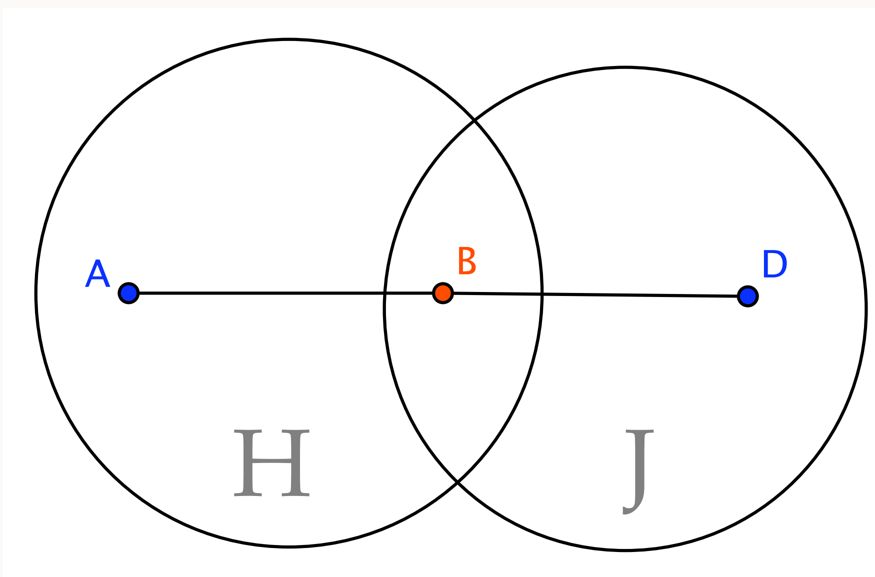
X(G,k) = k(k-1)

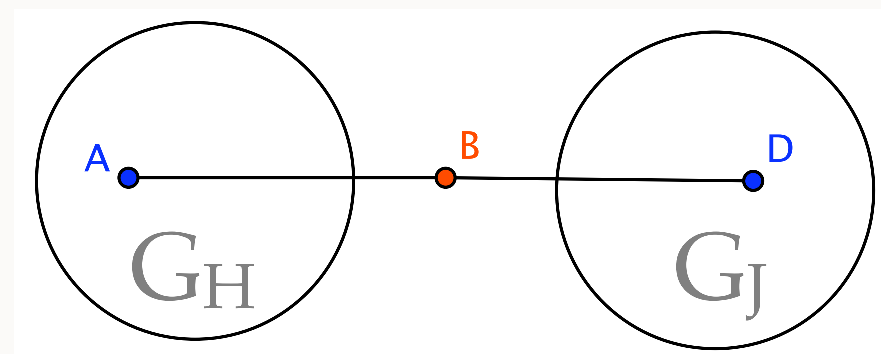


X(G,k) = k(k-1)

THEOREM 4: LET G BE THE UNION OF TWO SUBGRAPHS H AND J WHOSE INTERSECTION IS A COMPLETE GRAPH. THEN $X(G, k) \cdot X(H \cap J, k) = X(H, k) \cdot X(J, k)$

- Suppose Graph $G = H \cup J$ where $H \cap J = K_n$



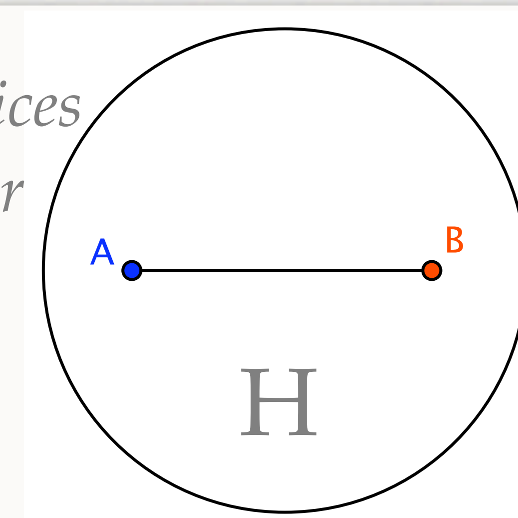


- ∃ 2 disjoint vertex sets, G_H and G_J, whose colorings not determined by other

(CONTINUED)THEOREM 4: LET G BE THE UNION OF TWO SUBGRAPHS H AND J WHOSE INTERSECTION IS A COMPLETE GRAPH. THEN $\mathbf{X}(G, k) \cdot \mathbf{X}(H \cap J, k) = \mathbf{X}(H, k) \cdot \mathbf{X}(J, k)$

- Hence,
  - $\mathbf{X}(H,k) = \mathbf{X}(K_n,k) \cdot \mathbf{X}(G_H,k) =$
  - $\mathbf{X}(H,k) = k(k-1) \cdots (k-r)(k-r-1)\mathbf{X}(G_H,k)$

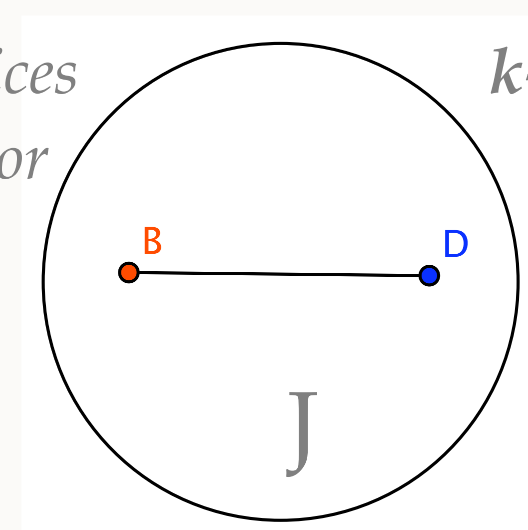
k-1 *choices of color*



k *choices of color*

- Similarly,
  - $\mathbf{X}(J,k) = \mathbf{X}(K_n,k) \cdot \mathbf{X}(G_J,k) =$
  - $\mathbf{X}(J,k) = k(k-1) \cdots (k-r)(k-r-1)\mathbf{X}(G_J,k)$

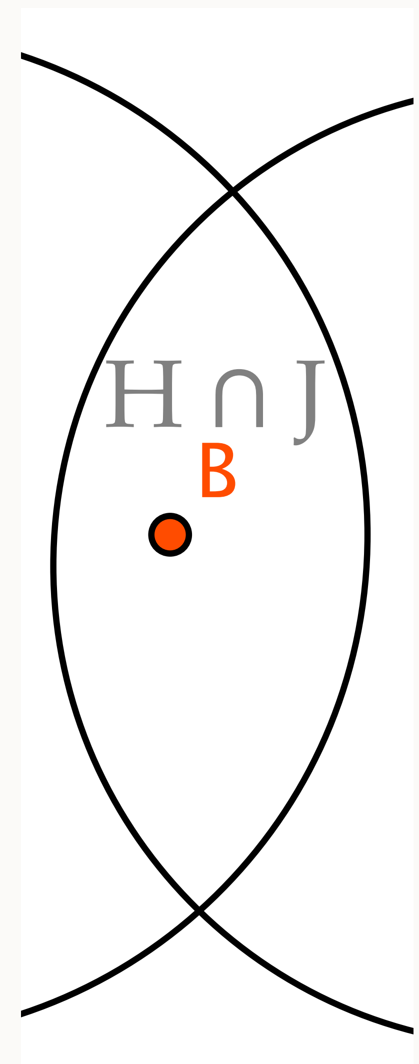
k *choices of color*



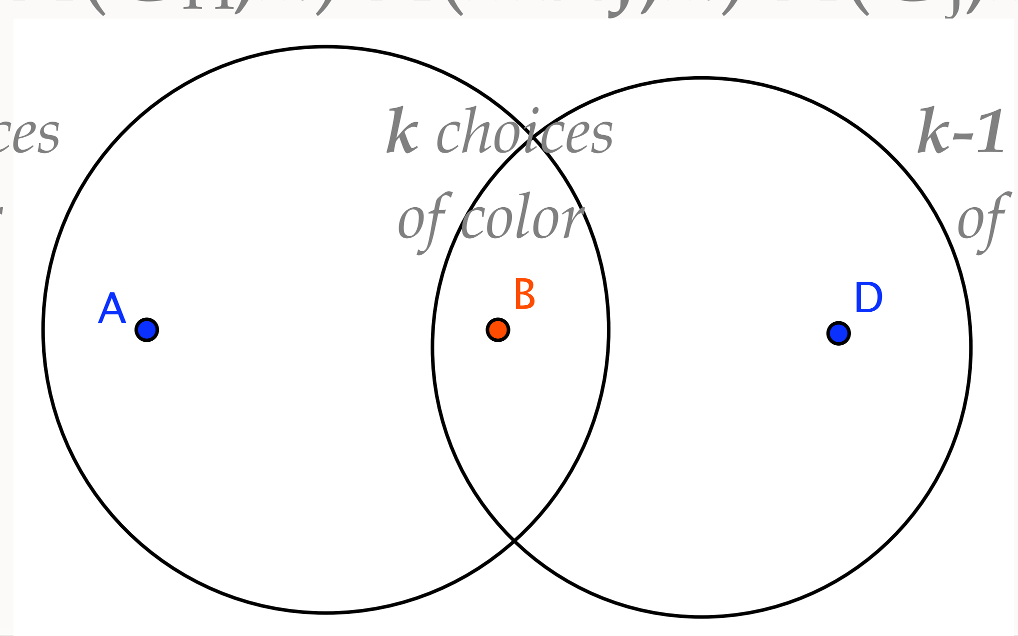
k-1 *choices of color*

(CONTINUED)THEOREM 4: LET G BE THE UNION OF TWO SUBGRAPHS H AND J WHOSE INTERSECTION IS A COMPLETE GRAPH. THEN $\mathbf{X}(G, k)\cdot \mathbf{X}(H \cap J, k) = \mathbf{X}(H, k)\cdot\mathbf{X}(J, k)$

- Vertex set of H ∩ J is complete, so disjoint from either G_H or G_J
- Thus, $\mathbf{X}(H\cap J,k) = \mathbf{X}(K_n,k)$ and
- $\mathbf{X}(G,k)= \mathbf{X}(G_H,k)\cdot\mathbf{X}(H\cap J,k)\cdot\mathbf{X}(G_J,k)$.



k-1 choices of color



k-1 choices of color

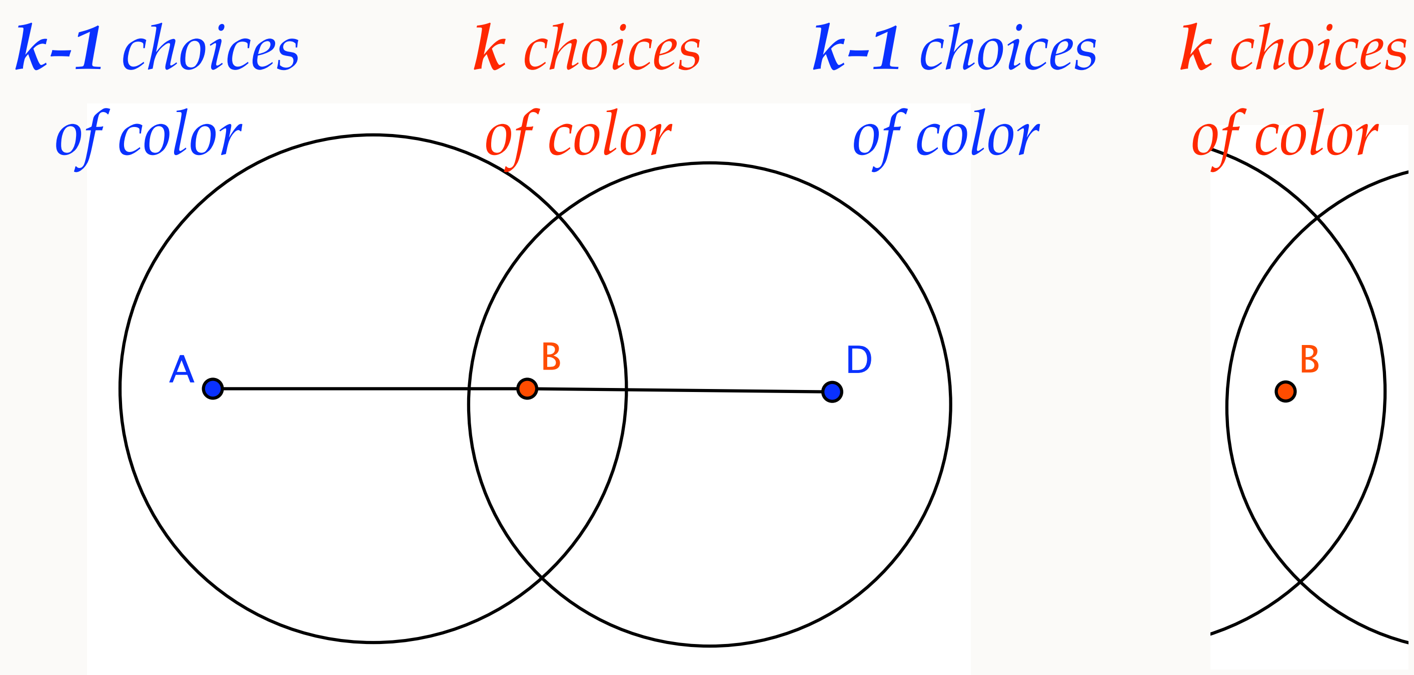
(CONCLUDED)THEOREM 4: LET G BE THE UNION OF TWO SUBGRAPHS H AND J WHOSE INTERSECTION IS A COMPLETE GRAPH. THEN $\mathbf{X}(G, k)\cdot \mathbf{X}(H\cap J, k) = \mathbf{X}(H, k)\cdot\mathbf{X}(J, k)$

- $\mathbf{X}(G,k)\cdot\mathbf{X}(H\cap J,k) =\{\mathbf{X}(G_H,k)\cdot\mathbf{X}(G_J,k)\cdot\mathbf{X}(H\cap J,k)\}\cdot\mathbf{X}(K_n,k)$

  (2 copies of K_n)

- $= \mathbf{X}(G_H,k)\cdot[k(k-1)\cdots(k-r)(k-r-1)]\ \mathbf{X}(G_J,k)\cdot[k(k-1)\cdots(k-r)(k-r-1)]$

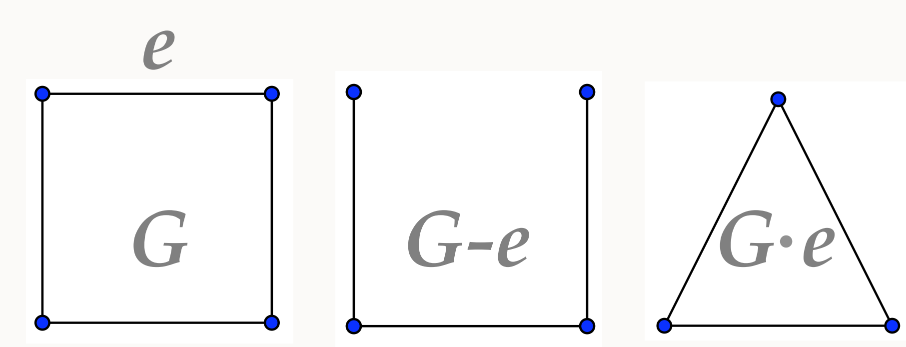
- $= \mathbf{X}(H,k)\cdot\ \mathbf{X}(J,k)$

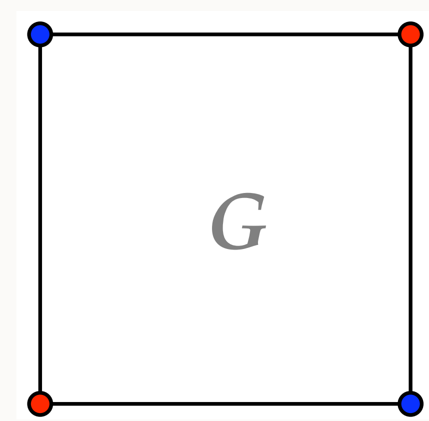


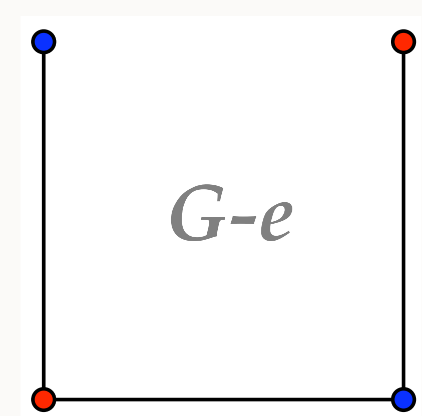
$\mathbf{X}(P_3,k)=k^2(k-1)^2/k$

THEOREM 5: DELETION-CONTRACTION $\chi(G,k)= \chi(G-e,k)-\chi(G\cdot e,k)$

- Let∃ *e* in *G*, *G-e* and *G·e*
- k-colorings of *G* are k-colorings of *G-e* ⇔ k-coloring gives distinct colors to vertices of edge *e*
- Subtract from $\chi(G-e,k)$ number of k-colorings of *G-e* that give endpoint vertices the same color
- These colorings correspond to *G·e*'s

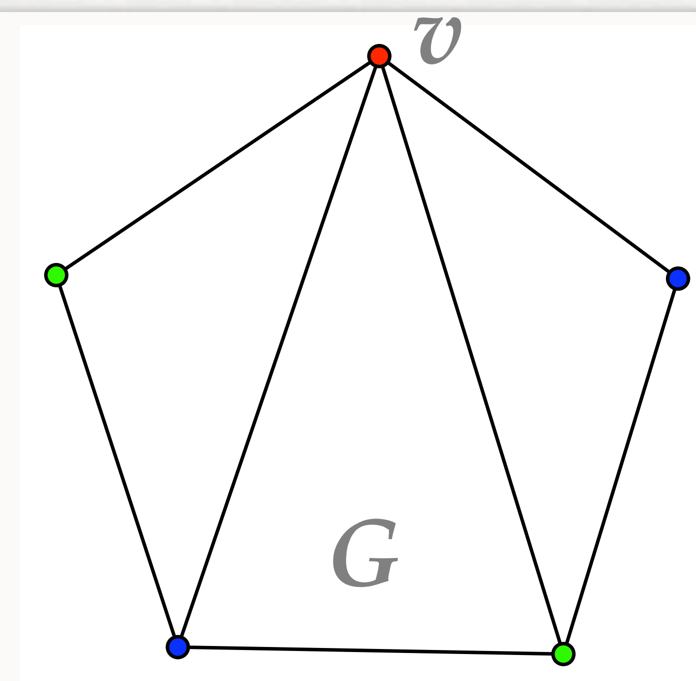




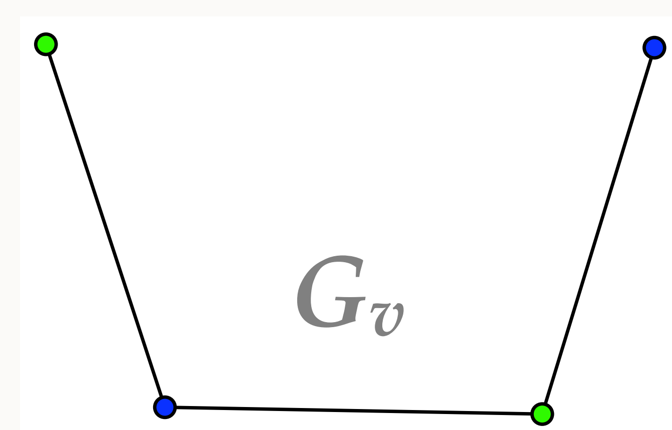


THEOREM 6: LET V BE A VERTEX THAT IS ADJACENT TO EVERY OTHER VERTEX IN A GRAPH, G, THEN $X(G, k) = k\, X(G_v, k-1)$

- v adjacent to all other vertices, no vertex has same color as v
- $X(G,k)$ an arrangement with respect to k colors, we selected 1 of k colors.
- Rest of arrangement determined by colorings of G_v without use of first k color chosen

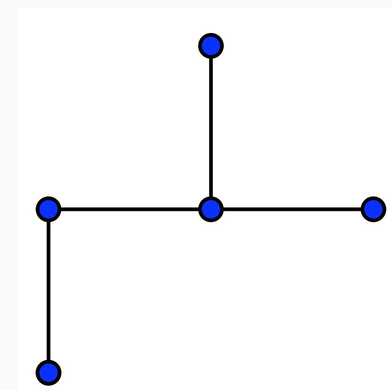
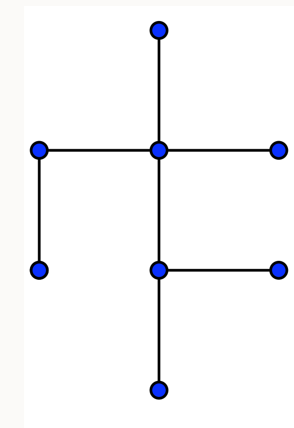
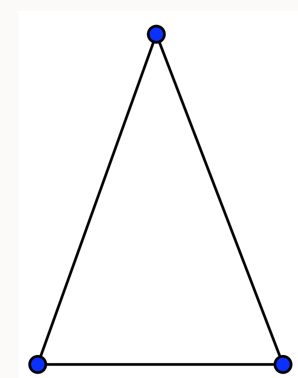
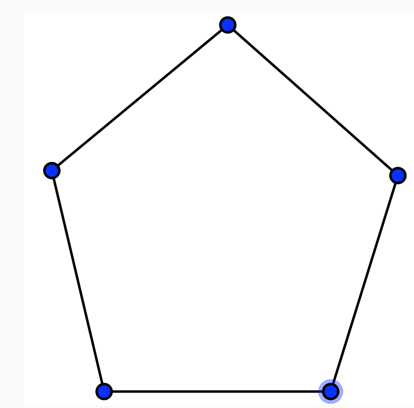
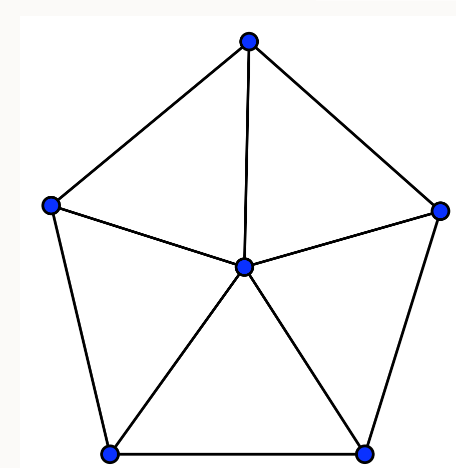


$X(G, K) = K \cdot X(G_v, K-1)$



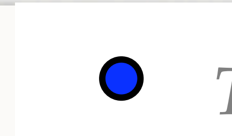
SPECIFIC GRAPH EXAMPLES & X(G,*k*)

- ***Tree, T_n***: A connected graph that contains no cycles
- ***Cycle, C_n***: A closed path in a graph
- ***Wheel, W_n***: A graph that has a single vertex adjacent to every other ***n*** vertices of a cycle

Theorem 7: Let Tn be a tree with n vertices, then

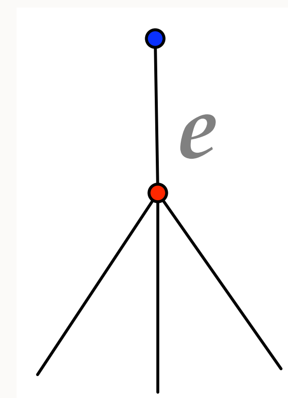
$$\mathbf{X}(T_n, k) = k\cdot(k-1)^{n-1}$$



T_n n=1 k colors

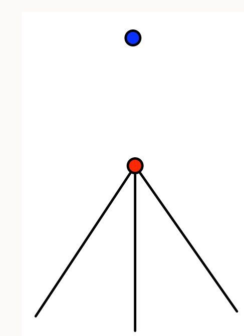
$\mathbf{X}(T_n,k)=k$

- By induction, Base Case for n=1 true
- Suppose IndHyp. true for some T_{n-1} and $\exists\ e$ adjacent to leaf



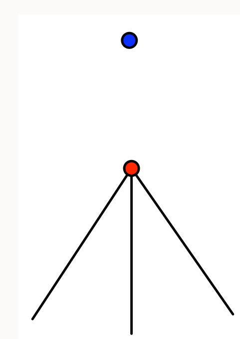
T_{n-1} n-1 verts

- $T\text{-}e$ has two components (note: $T_{n-2} = T\cdot e$)



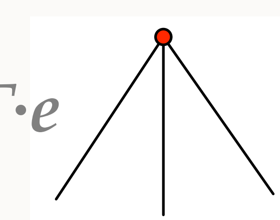
$T\text{-}e$

- $\mathbf{X}(T\text{-}e,k)=k\cdot\mathbf{X}(T\cdot e,k)$ by IndHyp.,
  $\mathbf{X}(T_n,k)= \mathbf{X}(T\text{-}e,k)-\mathbf{X}(T\cdot e,k)$ by Theorem 5
- $\mathbf{X}(T_n,k)=k\cdot\mathbf{X}(T\cdot e,k)-k(k-1)^{n-2}$
  $=k\cdot(k-1)^{n-2}\cdot[k-1]$
  $=k\cdot(k-1)^{n-1}$



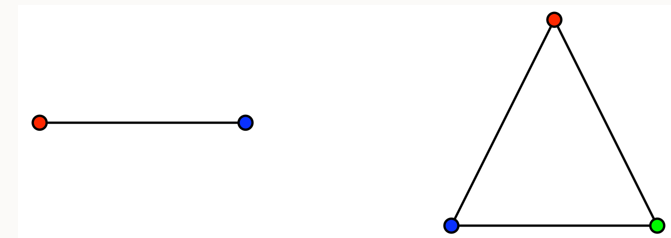
T_1

T_{n-2} $= T\cdot e$

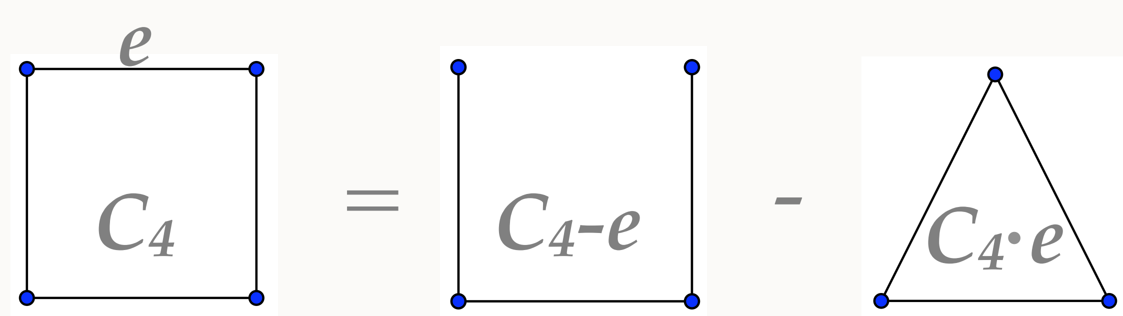


Theorem 8: Let C_n be a cycle with n vertices, then

$\chi(C_n, k) = (k-1)^{n-1} + (-1)^n (k-1)$



- $\chi(C_2,k) = \chi(K_2,k) = \chi(T_2,k) = k\cdot(k-1)$,
  $\chi(C_3,k) = \chi(K_3,k) = k\cdot(k-1)(k-2)$



- C_n **-** e is a tree on n vertices, so $\chi(C_n - e,k) = k\cdot(k-1)^{n-1}$
- $C_n \cdot e$ is a cycle graph with $n-1$ vertices, so Del/Con Alg ⇒

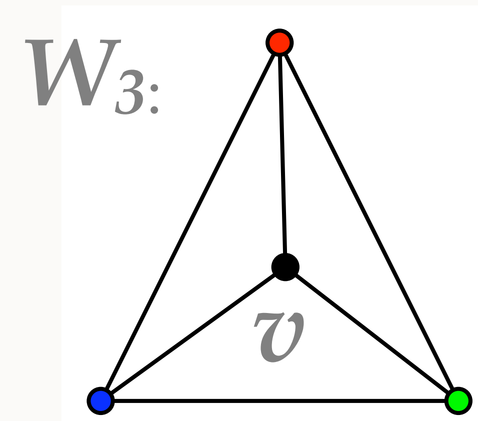
  $\chi(C_n,k) = k\cdot(k-1)^{n-1} - \chi(C_n \cdot e,k)$

- Using $\chi(C_3,k) = (k-1)^3 - (k-1)$ as initial condition to solve recursion or using characteristic equation

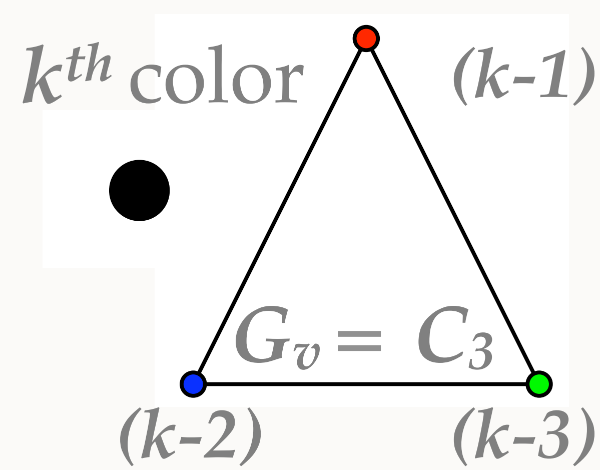
$\chi(C_n, K) = (K-1)^{n-1} + (-1)^n (K-1)$

Theorem 9: Let W_n be a wheel with n spokes, then

$$\mathrm{X}(W_n,\ k) = k(k\text{-}2)^n + (\text{-}1)^n (k\text{-}2)$$



- ∃ cycle, C_n surrounding ***hub*** vertex, v.
- Theorem 6 ⇒ $\mathrm{X}(W_n, k) = k \cdot \mathrm{X}(G_v,\ k\text{-}1)$
- Assuming v is wheel center, $G_v = C_n$ and by Theorem 8, $\mathrm{X}(Cn,\ k) = (k\text{-}1)^{n\text{-}1} + (\text{-}1)^n\ (k\text{-}1)$
- Substituting ***(n-1)*** for n gives us the result

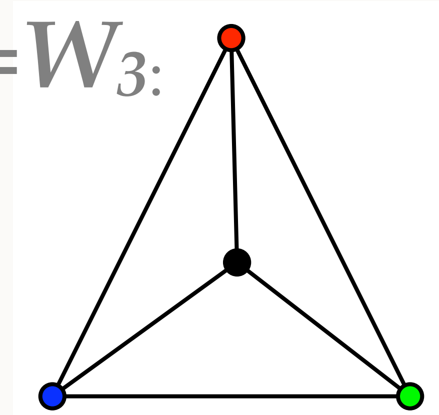


$\mathrm{X}(W_3, k) = k(k\text{-}1)(k\text{-}2)(k\text{-}3)$

CHROMATIC POLYNOMIAL, X(G,k)

- $X(G,k)$ is function counting distinct ways to color G, with k or fewer colors, and where permutations of colors are also distinct
- For G with n vertices, m edges, and t components, $G_1, G_2, \ldots, G_t$:
  - Coefficient of k^n in $X(G,k)$ is 1
  - Coefficient of k^{n-1} in $X(G,k)$ is $-m$
  - Coefficients of k^0, k^1, k^{t-1} are all 0
  - Coefficient of k^t is nonzero
  - $X(G,k) = X(G_1)\, X(G_2) \ldots X(G_t)$
  - Coefficients of *every* $X(G,k)$ alternate in signs

$K_4 = W_3$:

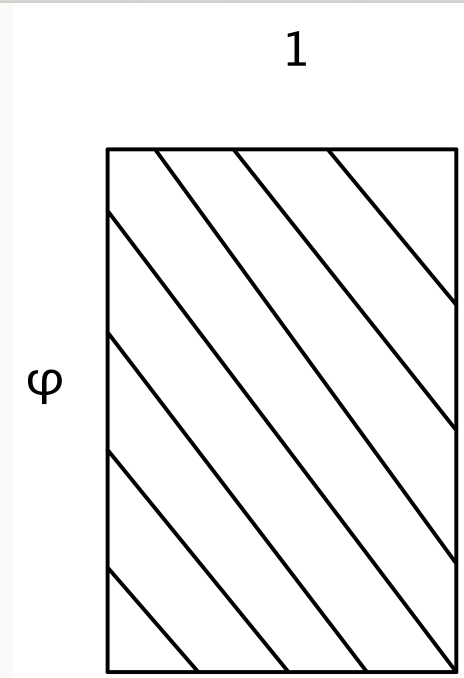


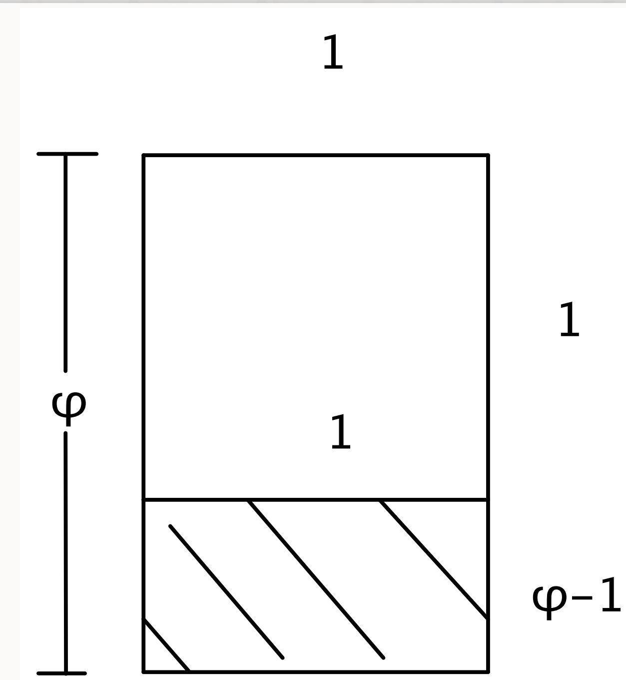
$X(K_4,k) = k(k-1)(k-2)(k-3)$

$X(K_4,k) = k^4 - 6k^3 + 11k^2 - 6k$

Φ, THE GOLDEN RATIO

- Given a rectangle having sides in the ratio 1:Φ
- Φ is defined as unique number such that partitioning the original rectangle into a square and new rectangle which also has sides in ratio 1:Φ





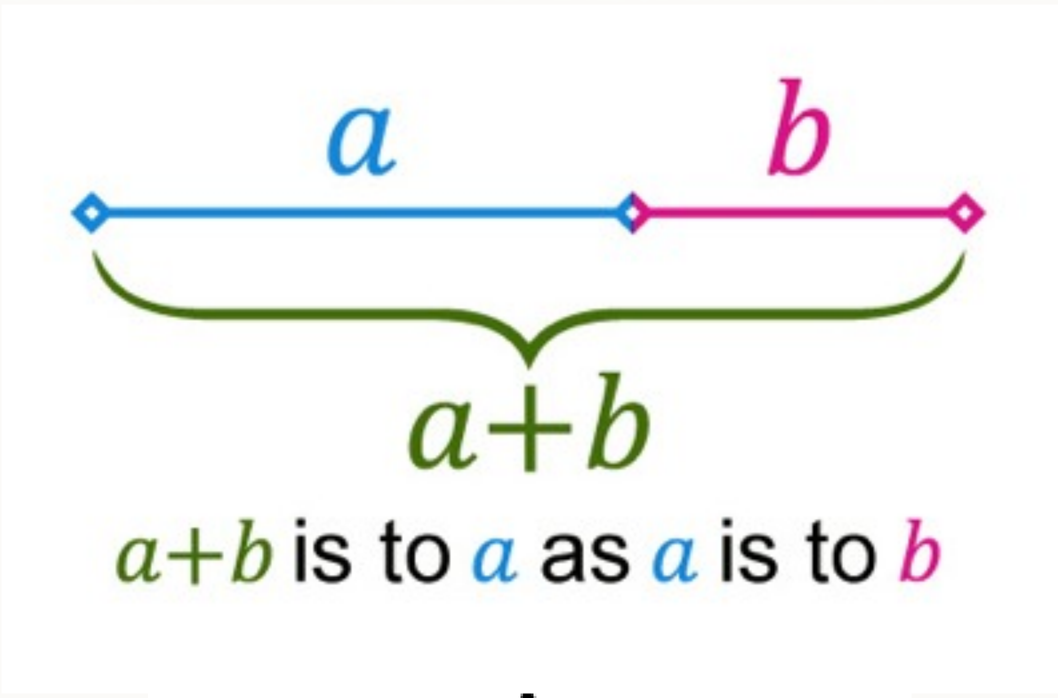
$$\frac{\phi}{1} = \frac{1}{\phi - 1} \qquad \phi^2 - \phi - 1 = 0.$$

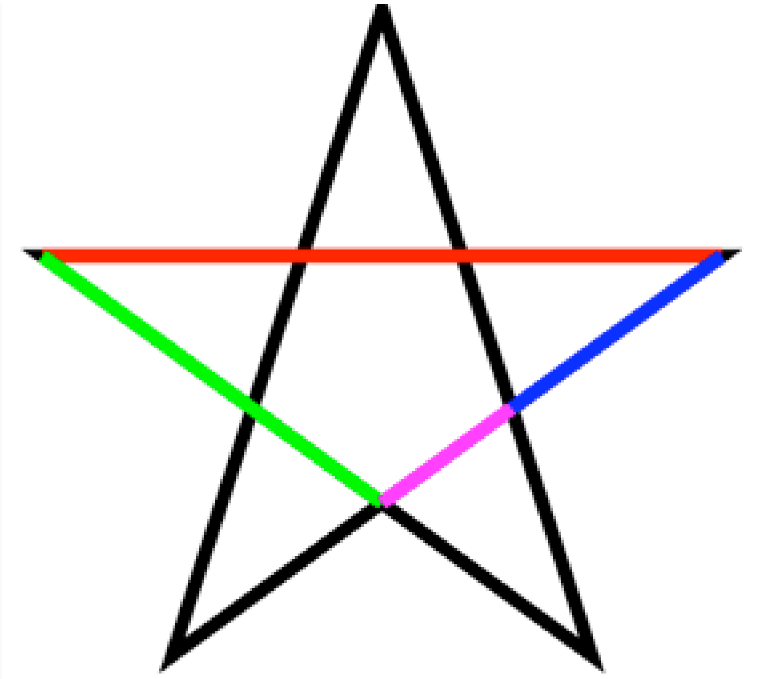
$$\phi = 1/2 + \sqrt{5}/2$$

$$= 1.6180339887498948482045868343...$$

Φ, THE GOLDEN RATIO

- If $\Phi^2 - \Phi - 1 = 0$, then $\Phi^2 = \Phi + 1$
- If $k = \Phi + 1$
- Then $k = \Phi^2$, $k - 1 = \Phi$, and $k - 2 = \Phi^{-1}$





THEOREM 10: LET G BE THE UNION OF TWO SUBGRAPHS H AND J WHOSE INTERSECTION IS A COMPLETE GRAPH WITH $0 \leq n_k \leq 3$ VERTICES. THEN $X(G, 1 + \Phi) = \Phi^{-\theta} X(H, 1 + \Phi) \cdot X(J, 1 + \Phi)$

- Where $\theta = 0, 2, 3$ or 2, respectively to $n_k = 0, 1, 2, 3$
- Proof consists of 4 cases
- Case1: $\theta = 0$, $n_k = 0 \Rightarrow$ The intersection of *H* and *J* is a null graph (complete 0-graph), $X(\varnothing,k)=1$ by Theorem 1 and $X(H \cap J,k)=1$
- Theorem 4 $\Rightarrow X(G, 1 + \Phi) \cdot 1 = X(H, 1 + \Phi) \cdot X(J, 1 + \Phi)$
- $X(G, 1 + \Phi) = \Phi^0 X(H, 1 + \Phi) \cdot X(J, 1 + \Phi)$

(CONCLUDED)THEOREM 10: LET $G=H\cup J$, WHERE $H\cap J=K_n$ AND WITH $0 \le n_k \le 3$ VERTICES. THEN $X(G, 1+\Phi) = \Phi^{-\theta} X(H, 1+\Phi)\cdot X(J, 1+\Phi)$

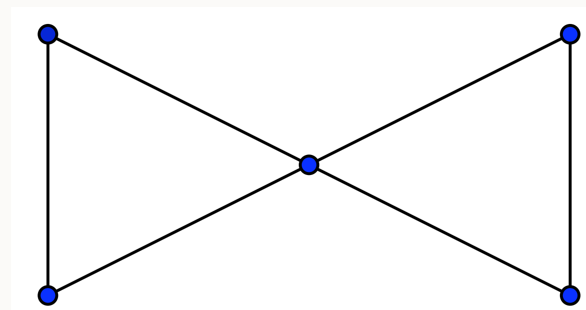
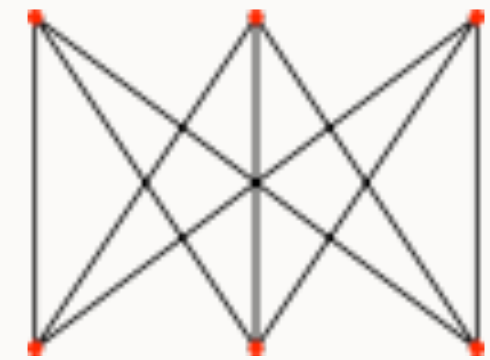
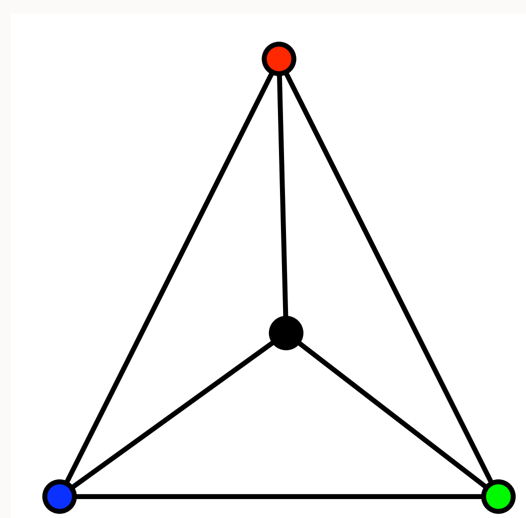
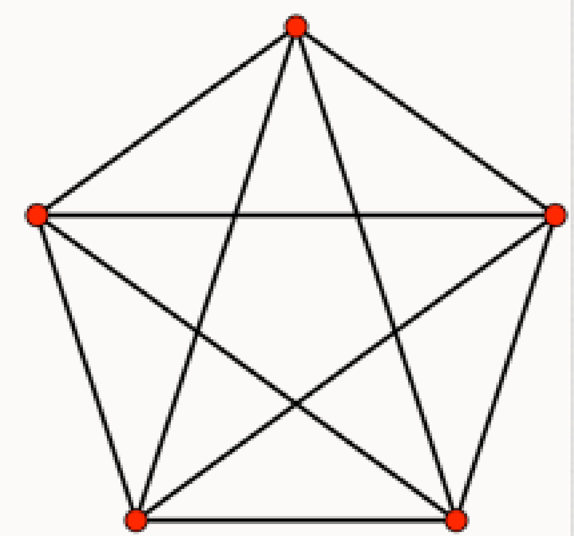
- Case 4: $\theta = 2$, $n_k = 3 \Rightarrow H \cap J = K_3$, and by Theorem 2 $X(H \cap J, k) = k(k-1)(k-2)$

- $X(K_3, 1+\Phi) \Rightarrow X(K_3, 1+\Phi = \Phi^2) \cdot \Phi^2((\Phi+1)-1)((\Phi+1)-2) = \Phi^2(\Phi)(\Phi-1) = \Phi^2(\Phi)(\Phi^{-1}) = \Phi^2$

- Theorem 4 $\Rightarrow X(G, 1+\Phi)\cdot\Phi^2 = X(H, 1+\Phi)\cdot X(J, 1+\Phi)$ $\Rightarrow X(G, 1+\Phi) = \Phi^{-2}\cdot X(H, 1+\Phi)\cdot X(J, 1+\Phi)$

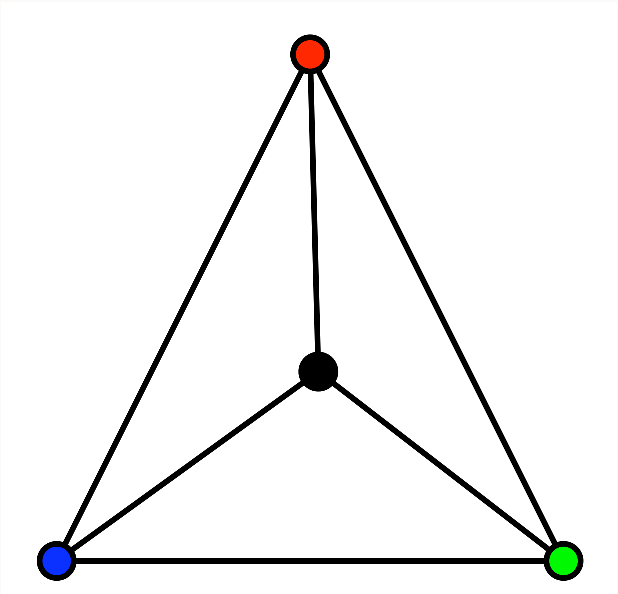
PLANAR GRAPHS: A GRAPH THAT CAN BE DRAWN ON THE PLANE IN SUCH A WAY THAT ITS EDGES INTERSECT ONLY AT THEIR ENDPOINTS

- Handshaking Lemma
- Euler's Formula
- $m \leq 3n - 6$
- no triangles, then $m \leq 2n-4$
- if planar, then $\exists$ at least one vertex of degree 5

Planar	*Non-Planar*
Bowtie	$K_{3,3}$
	
K_4	K_5
	

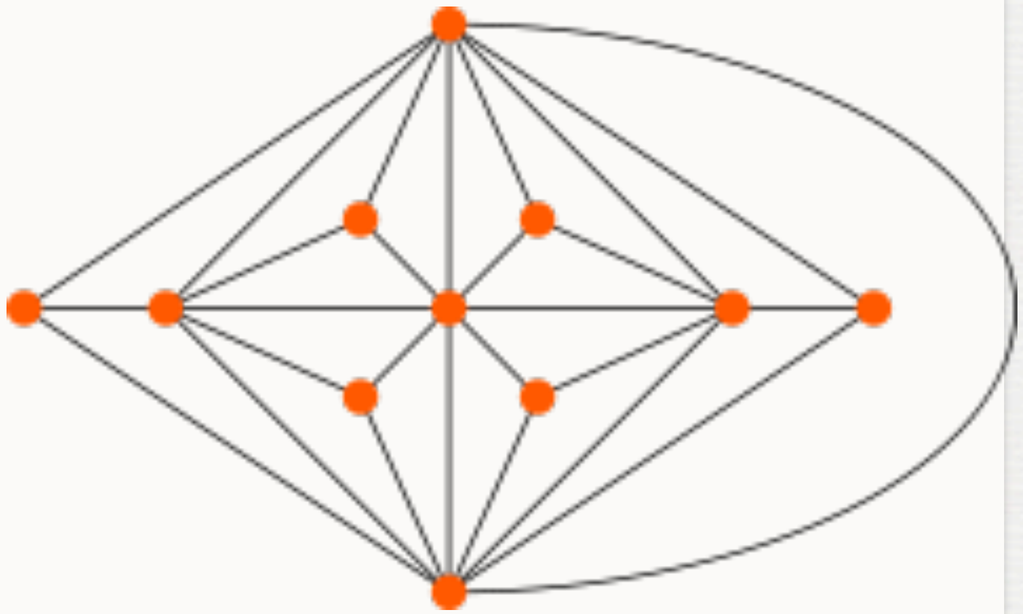
G PLANAR, CONNECTED⇒G HAS AT LEAST ONE VERTEX OF DEGREE < 6

- By contradiction, let ***every*** vertex in ***G*** be of degree ≥ 6



- $\sum \boldsymbol{\delta}(G) = 2 \cdot e \leq 2(3n\text{-}6) = 6n\text{-}12$, so degree sum at most $6n\text{-}12$
- average vertex degree

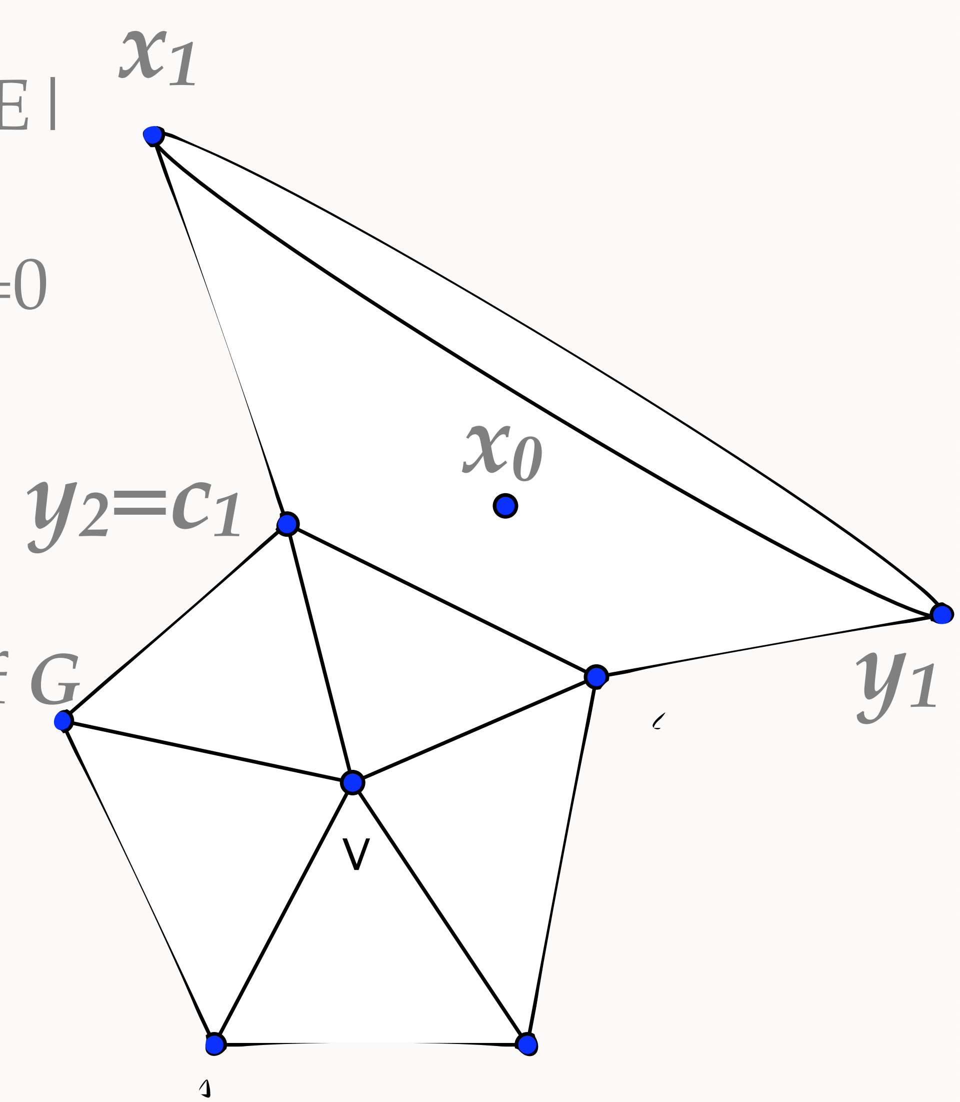
$$\frac{6n-12}{n} = 6 - \frac{12}{n} < 6$$



- ✽ Contradiction since every vertex had degree ≥ 6

THEOREM 11: vertex v enclosed by cycle C_n in planar graph, $G \Rightarrow \chi(G, 1+\Phi)=(-1)^n \Phi^{1-n} \chi(G_v, 1+\Phi)$

- Cycle enclosing v, induct on $|G| = |V| + |E|$
- Base Case true since $|G| = 1 \Rightarrow \nexists$ cycle, $n=0$
- 3 subcases
  - x_0 is not adjacent to ***any*** other vertex of G
  - x_1 adjacent to vertex $y_1 \neq v$, but not a vertex C_i in cycle C
  - x joined to some vertex $y_2 = C_i$ in cycle C

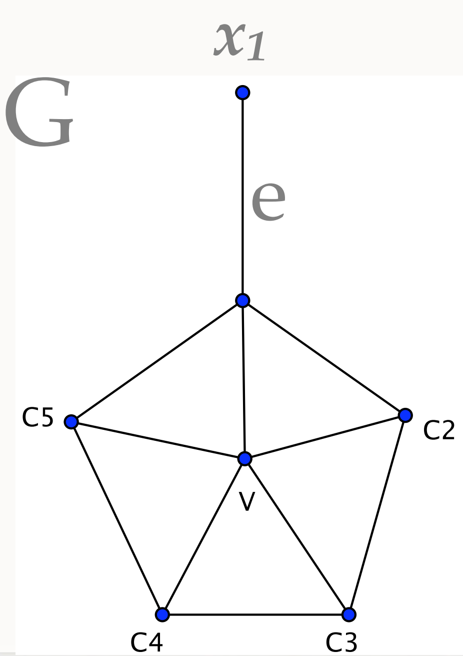


THEOREM 11: SUBCASE A-

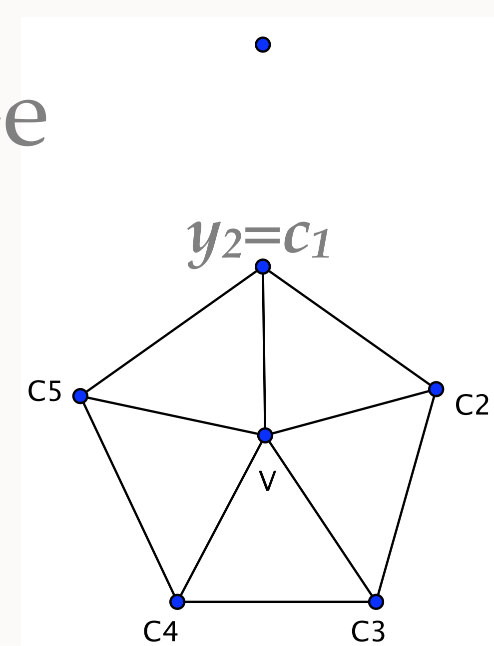
$\chi(G, 1+\Phi)=(-1)^n \Phi^{1-n} \chi(G_v, 1+\Phi)$

- *G-e* and *G·e* retain planarity since invariant WRT deletions/contractions
- $|G\cdot e| < |G-e| < |G|$, $(G_v)-e = (G-e)_v$ and $(G_v)\cdot e = (G\cdot e)_v$
- Del/Con ⇒ $\chi(G, 1+\Phi) = \chi(G-e, 1+\Phi) - \chi(G\cdot e, 1+\Phi)$
- Ind.Hyp ⇒ $(-1)^n \Phi^{1-n} \cdot \chi((G-e)_v, 1+\Phi) - \chi((G\cdot e)_v, 1+\Phi) = (-1)^n \Phi^{1-n} \cdot \chi(G_v, 1+\Phi)$

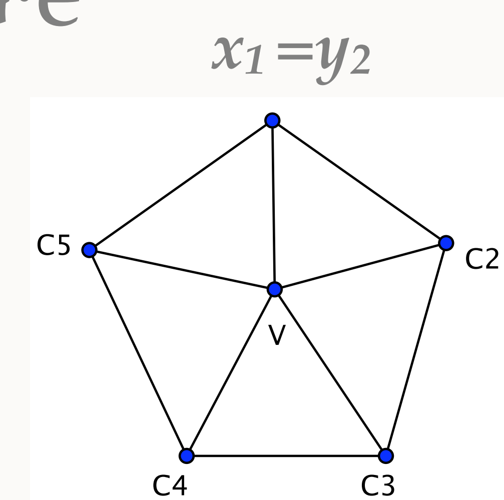
G



G-e



G·e



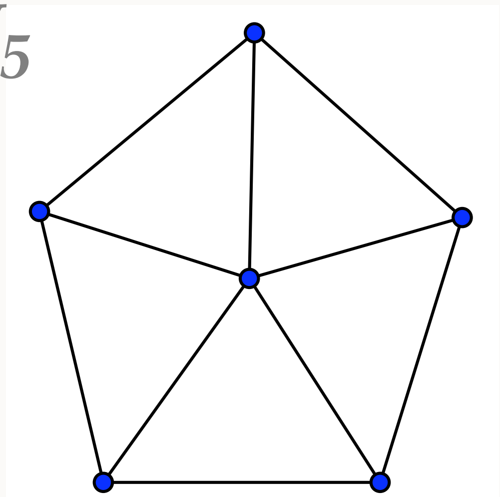
THEOREM 11: CASE B- Assume that G has no other vertices other than type v and C_i

$$\chi(G, 1+\Phi) = (-1)^n \Phi^{1-n} \chi(G_v, 1+\Phi)$$

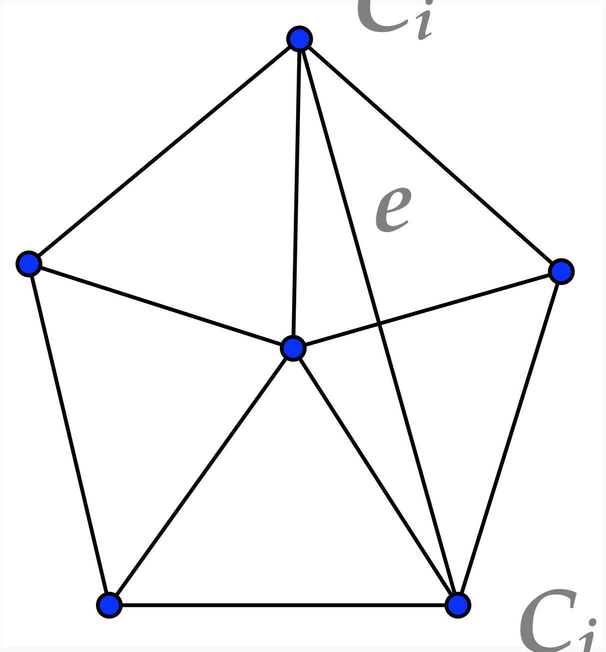
- 2 subcases
  - G is equivalent to a ***wheel*** of n spokes and n enclosing cycle edges
  - $\exists$ edge e whose endpoints

    are non-consecutive vertices C_i and C_j of cycle C

W_5



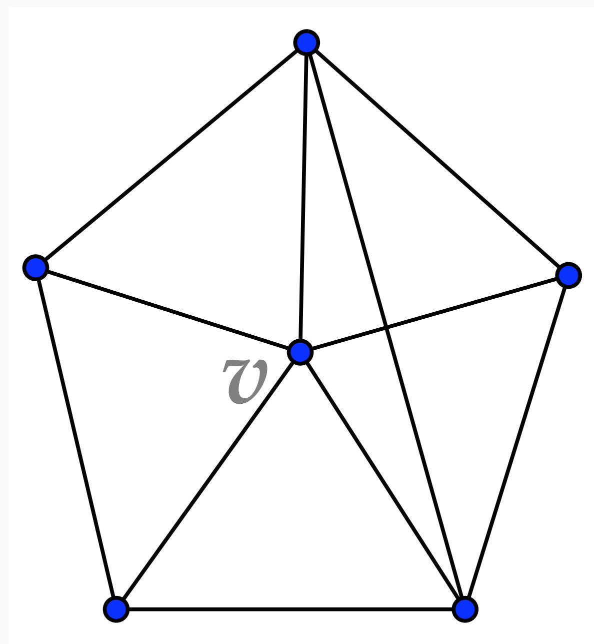
C_i

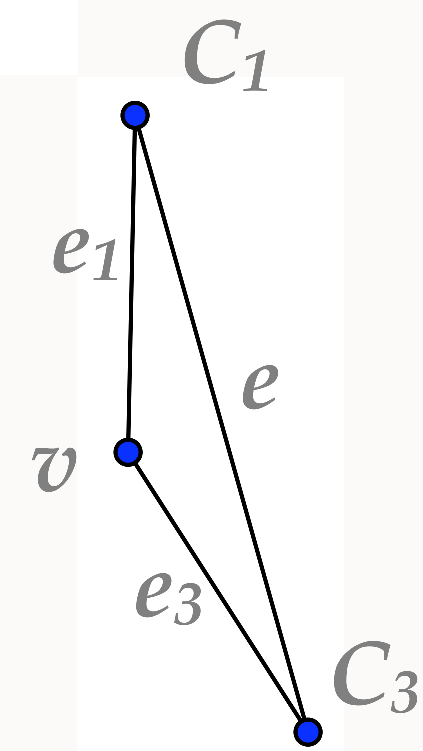


C_j

THEOREM 11: SUBCASE B-
$\chi(G, 1+\Phi) = (-1)^n \Phi^{1-n} \chi(G_v, 1+\Phi)$

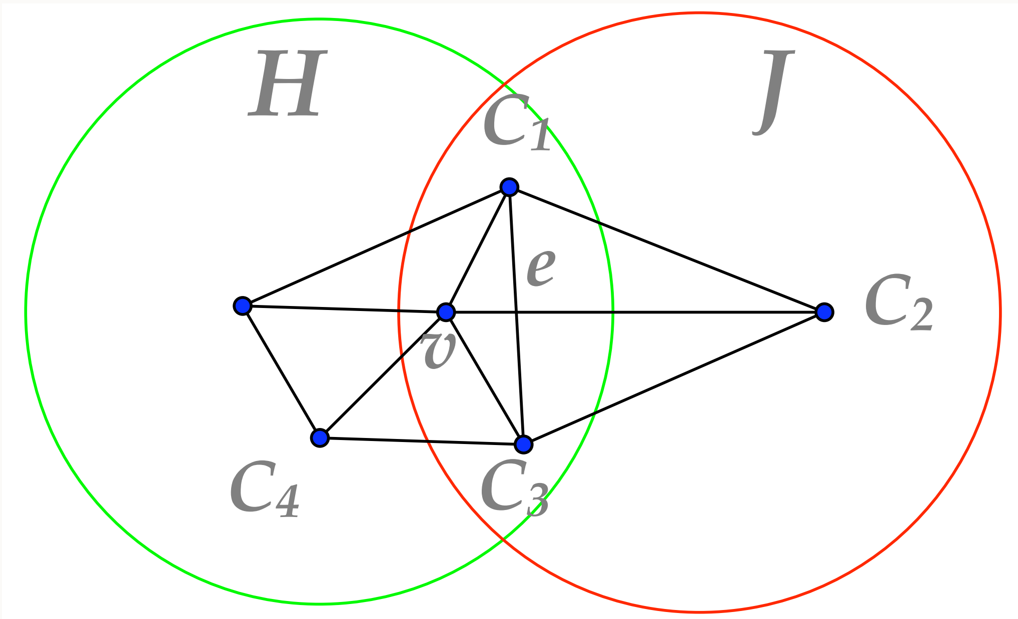
- $\exists$ edges e_1 and e_j joining v to C_1 and C_j, respectively
- Consider complete 3-graph formed by edges e, e_1 and e_j



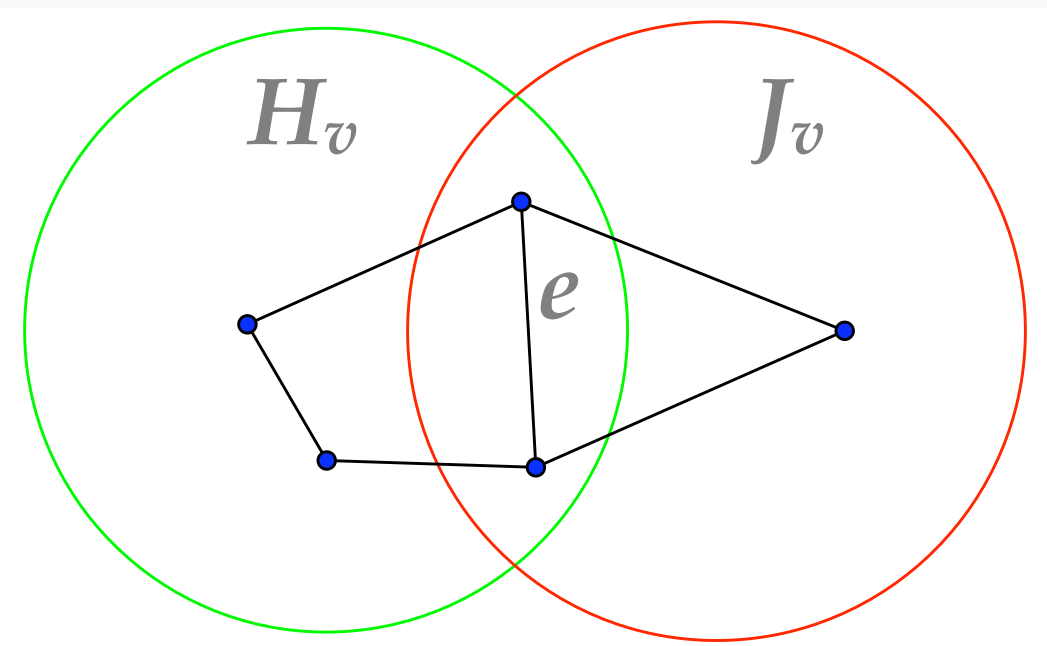


(concluded) THEOREM 11: SUBCASE B- $X(G, 1+\Phi)=(-1)^n \Phi^{1-n} X(G_v, 1+\Phi)$

- Planar $G = H \cup J$, where $H \cap J$ is cycle $v \to C_1 \to C_3 = C_j$
- H includes C_{j+1} not C_2, J includes C_2 not C_{j+1}

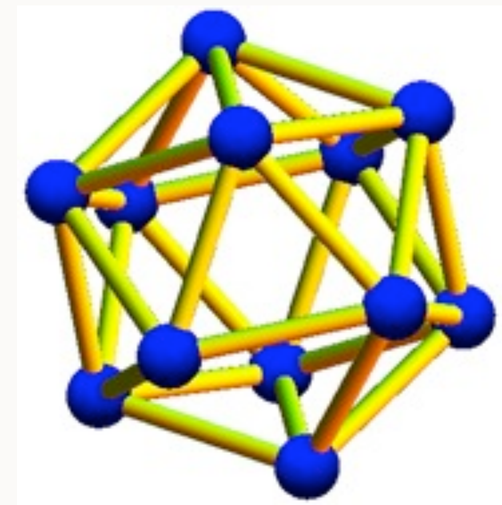


- Planar $G_v = H_v \cup J_v$, where $H_v \cap J_v$ is K_2 using e
- So Ind.Hyp, Thm 10 $\Rightarrow X(G, 1+\Phi) = (-1)^n \Phi^{-2-n} X(H_v, 1+\Phi) \cdot X(J_v, 1+\Phi) = (-1)^n \Phi^{1-n} X(G_v, 1+\Phi)$

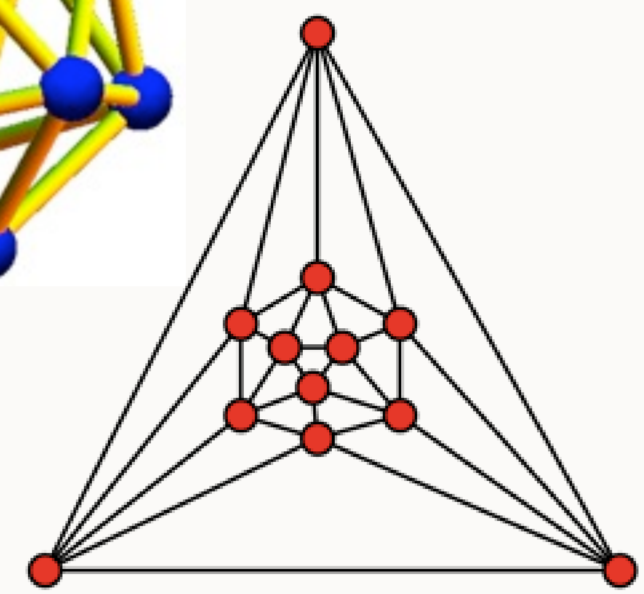


FINAL SETUP FOR MAIN THEOREM

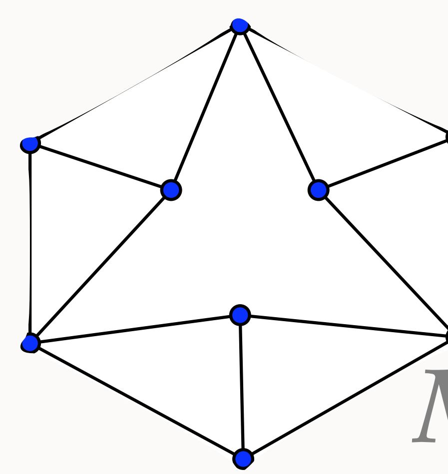
- Let *G* be in 2-sphere or closed plane
- Regions bounded by cycle in *G*, regions are ***faces*** of a ***plane map M*** of which *G* is the 1-section (the face that bounds all other regions)
- Faces are *m*-gons
- Edge or vertex incident with face if it belongs to bounding cycle of that face
- $X(G, k) = X(M_n, k)$
- Planar Triangulations, ***Z(n), Z(n, m)***



M_{12}



3-gons



$M_{9,6}$

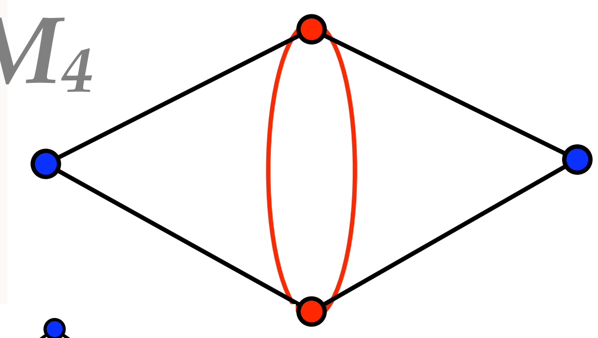
MAIN THEOREM:

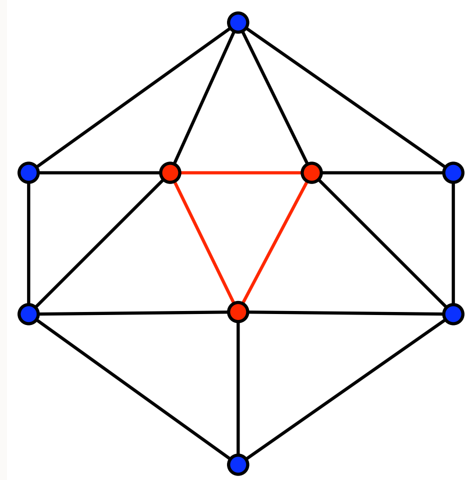
(i) If $M \in Z(n)$, then $|X(M_n, 1+\Phi)| \leq \Phi^{5-n}$

(ii) If $M \in Z(n,m)$, then $|X(M_{n,m}, 1+\Phi)| \leq \Phi^{3+m-n}$

- Induct on $|M|$: Base case true for n=1, m=0, and for null graph $X(M_1,k)=1$
- $|X(M_1, 1+\Phi)| = |1| \leq \Phi^{5-n} = \Phi^{5-1} = \Phi^4$
- $|X(M_{1,0}, 1+\Phi)| = |1| \leq \Phi^{3+m-n} = \Phi^{3+0-1} = \Phi^2$
- 3 Subcases for 1-section G of M
  - G has a cycle that is a 2-gon
  - G may be a complete 3-graph
  - G wheel-like at v (note M_{20} does not work)

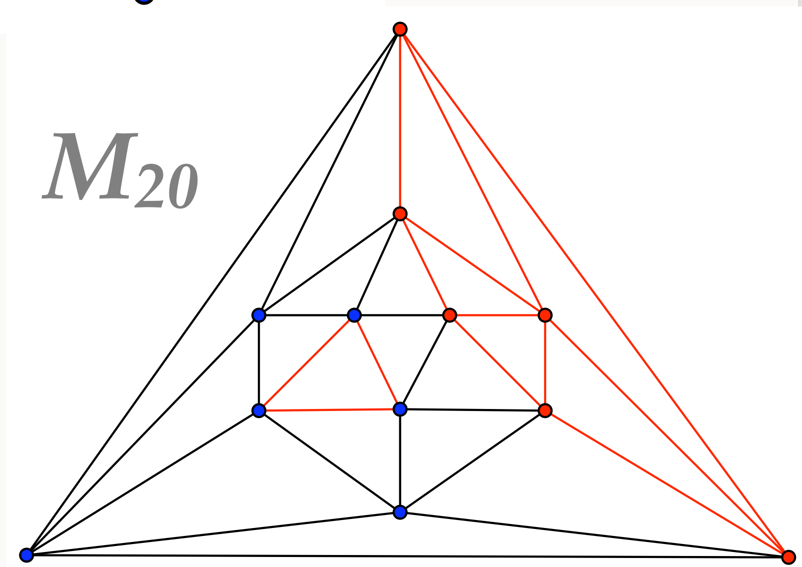
M_4





M_9

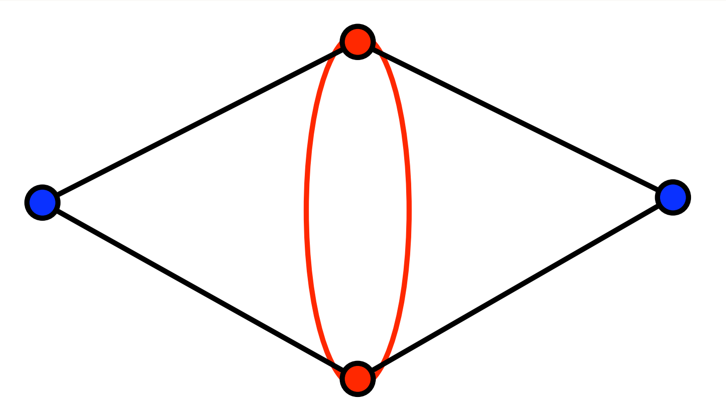
M_{20}



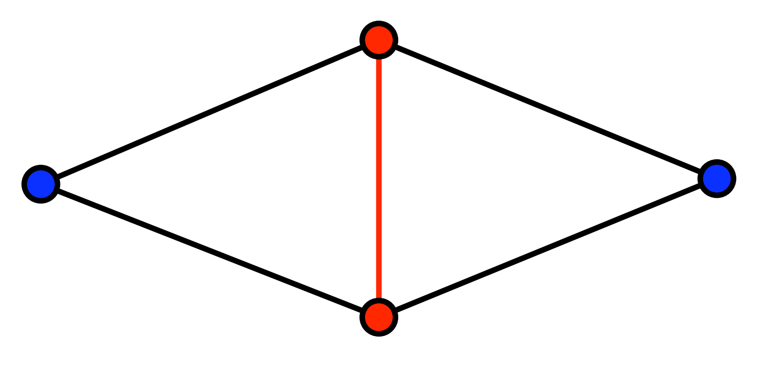
MAIN THEOREM:

(ii)If $M \in Z(n,m)$, then $|\chi(M_{n,m}, 1+\Phi)| \leq \Phi^{3+m-n}$

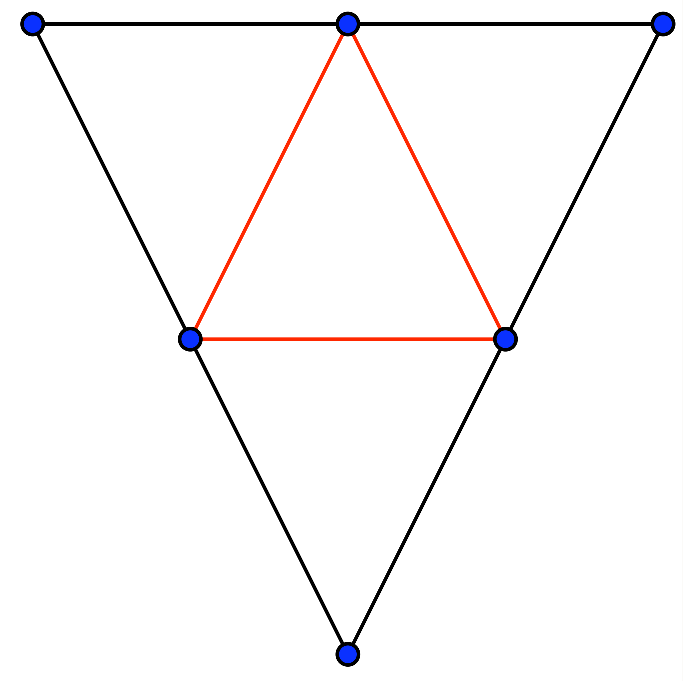
- 4 subcases for $M \in Z(n,m)$



- $m=2 \Rightarrow$ convert by deleting edge of the 2-gon



- $m=3 \Rightarrow M$ in $Z(n)$



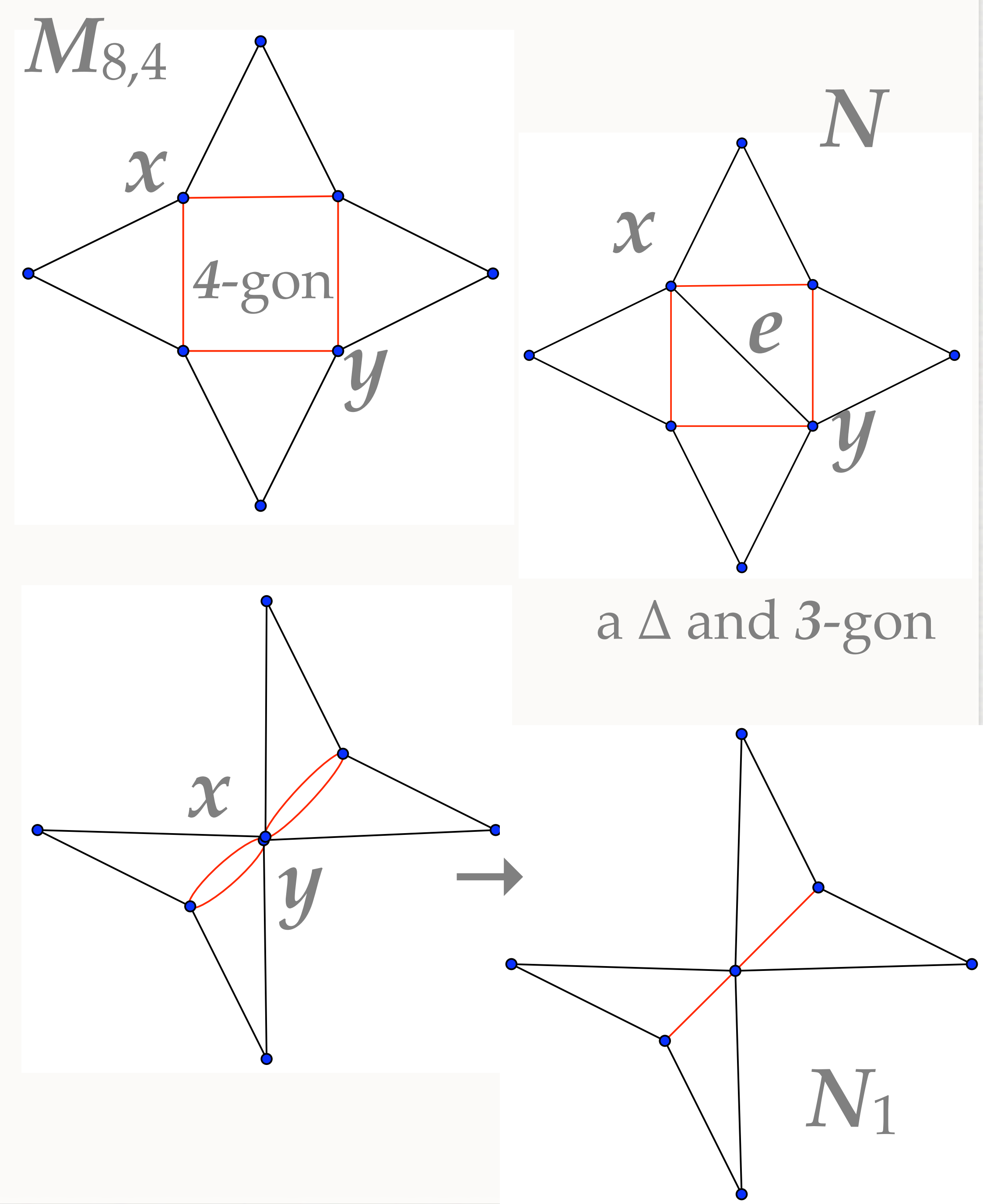
MAIN THEOREM: SUBCASES

(ii)If $M \in Z(n,m)$, then $|\chi(M_{n,m}, 1+\Phi)| \leq \Phi^{3+m-n}$

- $m = 4 \Rightarrow \exists$ 2 non-consecutive vertices
- Add edge to subdivide m-gon into a triangle and an (m-1)-gon - - Call it N
- Identify vertices x and y - Call it N_1
- $\chi(M, 1+\Phi) = \chi(N, 1+\Phi) - \chi(N_1, 1+\Phi) \Rightarrow$

  $|\chi(M, 1+\Phi)| \leq |\chi(N, 1+\Phi)| + |\chi(N_1, 1+\Phi)|$
- For m=4, $N \in Z(q)$ and $N_1 \in Z(q-1,2) \Rightarrow$

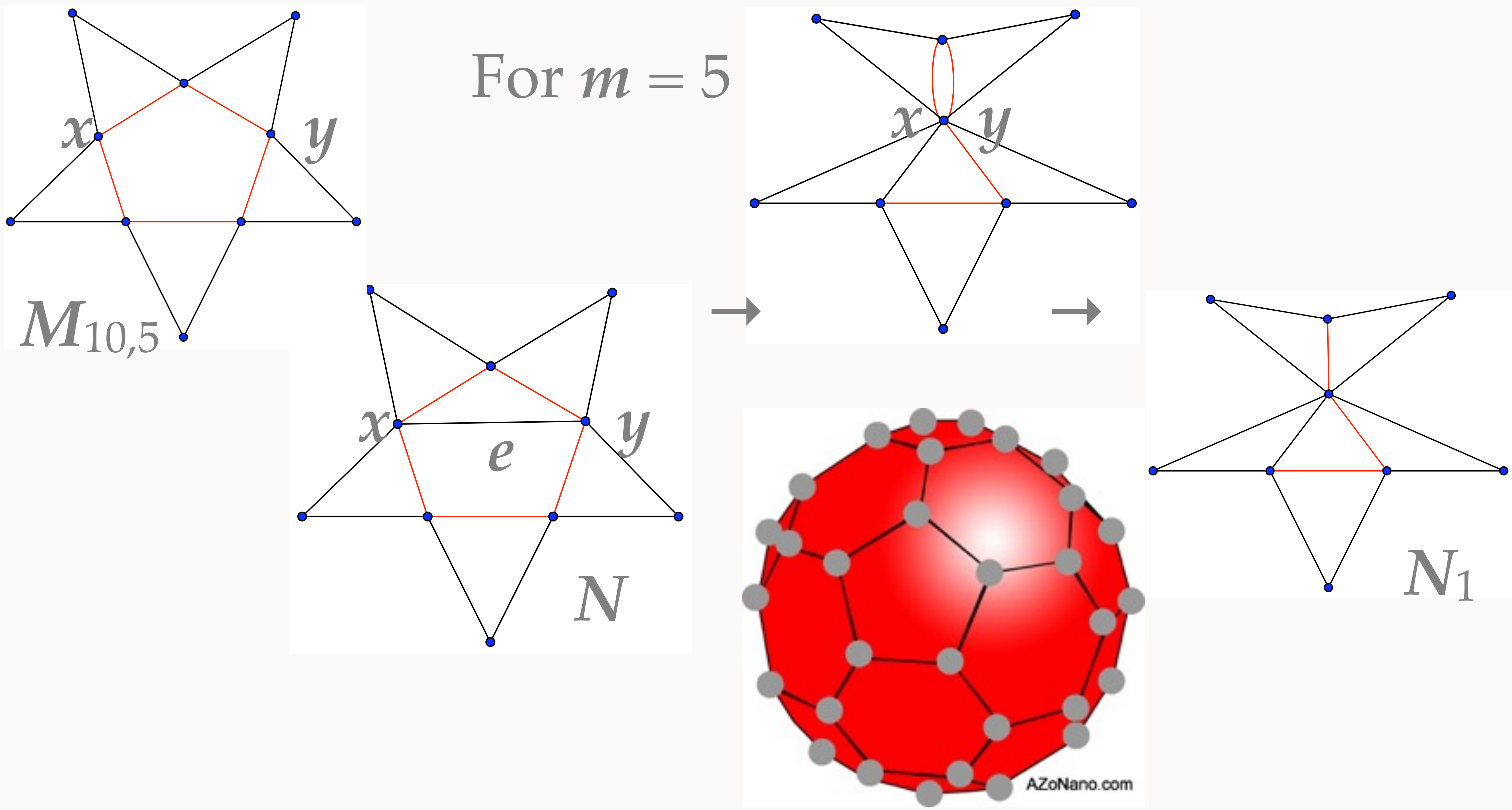
  $|\chi(M,1+\Phi)| \leq \Phi^{5-q} + \Phi^{5-(q-1)} = \Phi^{7-q} = \Phi^{3+m-q}$ since m=4



MAIN THEOREM: SUBCASES

(ii)If $M \in Z(n,m)$, then $|\chi(M_{n,m}, 1+\Phi)| \leq \Phi^{3+m-n}$

For $m = 5$



EPILOGUE: $1+\Phi$ never a root of a Chromatic Polynomial

- $\chi(G, k) > 0$ if $|V|$ is even, $\chi(G, k) < 0$ if $|V|$ is odd
- For *K* components, $\exists\ \sigma(G, k) \ni \chi(G, k) = k^{|K|} \cdot \sigma(G, k)$
- $(-1)^{|V|+K} \cdot \sigma(G, k) > 0$, for $0<k<1$
- $(-1)^{|V|} \cdot \chi(G, k) < 0$, for $0<k<1$
- $1+\Phi=(3+\sqrt{5})/2$ is root$\Leftrightarrow$

  $(3-\sqrt{5})/2$ is root
- ✾ since $0< (3-\sqrt{5})/2 <1$