FRACTIONAL COLORINGS AND THE MYCIELSKI GRAPHS

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Chapter 1

Overview

In this paper, we discuss a result about graph colorings from 1995. The paper we will be investigating is "The Fractional Chromatic Number of Mycielski's Graphs," by Michael Larsen, James Propp and Daniel Ullman [3].

We will begin with some preliminary definitions, examples, and results about graph colorings. Then we will define *fractional colorings* and the *fractional chromatic number*, which are the focus of Larsen, Propp and Ullman's paper. We will define fractional colorings in two different ways: first in a fairly intuitive, combinatorial manner that is characterized in terms of *graph homomorphisms*, and then in terms of independent sets, which as we shall see, lends itself to calculation by means of linear programming. In this second context, we shall also define *fractional cliques*, and see how they relate to fractional colorings. This connection between fractional colorings and fractional cliques is the key to Larsen, Propp and Ullman's proof.

Chapter 2

Preliminaries

2.1 Graphs and graph colorings

The material in this section is based on the exposition found in chapters 1 and 7 in the excellent book by Godsil and Royle [2].

2.1.1 Basic definitions

A graph is defined as a set of vertices and a set of edges joining pairs of vertices. The precise definition of a graph varies from author to author; in this paper, we will consider only finite, simple graphs, and shall tailor our definition accordingly.

A **graph** G is an ordered pair $(V(G), E(G))$, consisting of a vertex set, $V(G)$, and an edge set, $E(G)$. The vertex set can be any finite set, as we are considering only finite graphs. Since we are only considering simple graphs, and excluding loops and multiple edges, we can define E(G) as a subset of the set of all *unordered* pairs of *distinct* elements of $V(G)$.

2.1.2 Independent sets and cliques

If u and v are elements of $V(G)$, and $\{u, v\} \in E(G)$, then we say that u and v are **adjacent**, denoted $u \sim v$. Adjacency is a symmetric relation, and in the case of simple graphs, anti-reflexive. A set of pairwise adjacent vertices in a graph is called a **clique** and a set of pairwise non-adjacent vertices is called an **independent set**.

For any graph G, we define two parameters: $\alpha(G)$, the **independence number**, and $\omega(G)$, the **clique number**. The independence number is the size of the largest independent set in $V(G)$, and the clique number is the size of the largest clique.

2.1.3 Examples

As examples, we define two families of graphs, the cycles and the complete graphs.

The **cycle on n vertices** ($n > 1$), denoted C_n, is a graph with $V(C_n) = \{1, \ldots, n\}$ and $x \sim y$ in C_n if and only if $x - y \equiv \pm 1 \pmod{n}$. We often depict C_n as a regular n-gon. The independence and clique numbers are easy to calculate: we have $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$ and $\omega(C_n) = 2$ (except for C_3, which has a clique number of 3).

The **complete graph on n vertices**, K_n, is a graph with $V(K_n) = \{1, \ldots, n\}$ and $x \sim y$ in $V(K_n)$ for all $x \neq y$. It is immediate that $\alpha(K_n) = 1$ and $\omega(K_n) = n$.

The graphs C_5 and K_5 are shown in Figure 1.

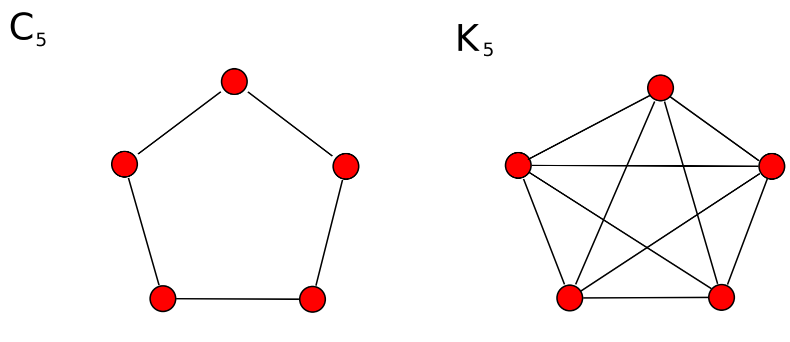

Figure 1: Cycle and complete graph on five vertices

2.1.4 Graph colorings and graph homomorphisms

A **proper n-coloring** (or simply a **proper coloring**) of a graph G can be thought of as a way of assigning, from a set of n "colors", one color to each vertex, in such a way that no adjacent vertices have the same color. A more formal definition of a proper coloring relies on the idea of *graph homomorphisms.*

If G and H are graphs, a **graph homomorphism** from G to H is a mapping $\phi : V(G) \to V(H)$ such that $u \sim v$ in G implies $\phi(u) \sim \phi(v)$ in H. A bijective graph homomorphism whose inverse is also a graph homomorphism is called a graph **isomorphism**.

Now we may define a proper n-coloring of a graph G as a graph homomorphism from G to K_n. This is equivalent to our previous, informal definition, which can be seen as follows. Given a "color" for each vertex in G, with adjacent vertices always having different colors, we may define a homomorphism that sends all the vertices of the same color to the same vertex in K_n. Since adjacent vertices have different colors assigned to them, they will be mapped to different vertices in K_n, which are adjacent. Conversely, any homomorphism from G to K_n assigns to each vertex of G an element of $\{1, 2, \ldots, n\}$, which may be viewed as colors. Since no vertex in K_n is adjacent to itself, no adjacent vertices in G will be assigned the same color.

In a proper coloring, if we consider the inverse image of a single vertex in K_n, i.e., the set of all vertices in G with a certain color, it will always be an independent set. This independent set is called a **color class** associated with the proper coloring. Thus, a proper n-coloring of a graph G can be thought of as a covering of the vertex set of G with independent sets.

We define a graph parameter $\chi(G)$, the **chromatic number** of G, as the smallest positive integer n such that there exists a proper n-coloring of G. Equivalently, the chromatic number is the smallest number of independent sets required to cover $V(G)$. Any finite graph with k vertices can certainly be colored with k colors, so we see that $\chi(G)$ is well-defined for a finite graph G, and bounded from above by $|V(G)|$. It is also clear that, if we have a proper n-coloring of G, then $\chi(G) \leq n$.

We can establish some inequalities relating the chromatic number to the other parameters we have defined. First, $\omega(G) \leq \chi(G)$, since all the vertices in a clique must be different colors. Also, since each color class is an independent set, we have $\frac{|V(G)|}{\alpha(G)} \leq \chi(G)$, where equality is attained if and only if each color class in an optimal coloring is the size of the largest independent set.

We can calculate the chromatic number for our examples. For the complete graphs, we have $\chi(K_n) = n$, and for the cycles we have $\chi(C_n) = 2$ for n even and 3 for n odd. In Figure 2, we see C_5 and K_5 colored with three and five colors, respectively.

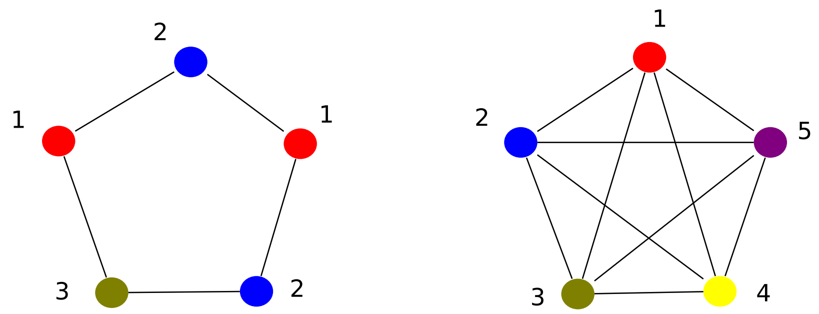

Figure 2: The graphs C_5 and K_5 with proper 3- and 5-colorings, respectively

2.2 Fractional colorings and fractional cliques

2.2.1 Fractional colorings

We now generalize the idea of a proper coloring to that of a *fractional coloring* (or a *set coloring*), which allows us to define a graph's *fractional chromatic number*, denoted $\chi_F(G)$, which can assume non-integer values.

Given a graph, integers $0 < b \leq a$, and a set of a colors, a **proper** a/b**-coloring** is a function that assigns to each vertex a set of b distinct colors, in such a way that adjacent vertices are assigned disjoint sets. Thus, a proper n-coloring is equivalent to a proper $n/1$-coloring.

The definition of a fractional coloring can also be formalized by using graph homomorphisms. To this end, we define another family of graphs, the *Kneser graphs*. For each ordered pair of positive integers (a, b) with $a \geq b$, we define a graph $K_{a:b}$. As the vertex set of $K_{a:b}$, we take the set of all b-element subsets of the set $\{1, \ldots, a\}$. Two such subsets are adjacent in $K_{a:b}$ if and only if they are disjoint. Note that $K_{a:b}$ is an empty graph (i.e., its edge set is empty) unless $a \geq 2b$.

Just as a proper n-coloring of a graph G can be seen as a graph homomorphism from G to the graph K_n, so a proper a/b-coloring of G can be seen as a graph homomorphism from G to $K_{a:b}$.

The **fractional chromatic number** of a graph, $\chi_F(G)$, is the infimum of all rational numbers a/b such that there exists a proper a/b-coloring of G. From this definition, it is not immediately clear that $\chi_F(G)$ must be a rational number for an arbitrary graph. In order to show that it is, we will use a different definition of fractional coloring, but first, we establish some bounds for $\chi_F(G)$ based on our current definition.

We can get an upper bound on the fractional chromatic number using the chromatic number. If we have a proper n-coloring of G, we can obtain a proper $\frac{nb}{b}$ coloring for any positive integer b by replacing each individual color with b different colors. Thus, we have $\chi_F(G) \leq \chi(G)$, or in terms of homomorphisms, we can simply note the existence of a homomorphism from K_n to $K_{nb:b}$ (namely, map i to the set of $j \equiv i \pmod{n}$).

To obtain one lower bound on the fractional chromatic number, we note that a graph containing an n-clique has a fractional coloring with b colors on each vertex only if we have at least $n \cdot b$ colors to choose from; in other words, $\omega(G) \leq \chi_F(G)$.

Just as with proper colorings, we can obtain another lower bound from the independence number. Since each color in a fractional coloring is assigned to an independent set of vertices (the fractional color class), we have $|V(G)| \cdot b \leq \alpha(G) \cdot a$, or $\frac{|V(G)|}{\alpha(G)} \leq \chi_F(G)$.

Another inequality, which will come in handy later, regards fractional colorings of subgraphs. A graph H is said to be a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Notice that if H is a subgraph of G, then any proper a/b-coloring of G, restricted to $V(H)$, is a proper a/b-coloring of H. This tells us that $\chi_F(H) \leq \chi_F(G)$.

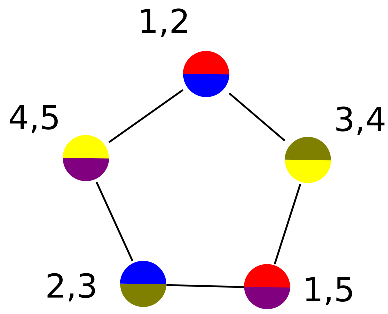

Figure 3: The graph C_5 with a proper $5/2$-coloring

2.2.2 Fractional colorings in terms of independent sets

Now we introduce an alternative definition of fractional colorings, one expressed in terms of independent sets of vertices. This definition is somewhat more general than the previous one, and we will see how fractional colorings, understood as homomorphisms, can be consistently reinterpreted in terms of independent sets. This new characterization of fractional colorings will lead us to some methods of computing a graph's fractional chromatic number.

Let $\mathcal{I}(G)$ denote the set of all independent sets of vertices in $V(G)$, and for $u \in V(G)$, we let $\mathcal{I}(G, u)$ denote the set of all independent sets of vertices containing u. In this context, a **fractional coloring** is a mapping $f : \mathcal{I}(G) \to [0, 1]$ with the property that, for every vertex u, $\sum_{J \in \mathcal{I}(G,u)} f(J) \geq 1$. The sum of the function values over all independent sets is called the **weight** of the fractional coloring. The

fractional chromatic number of G is the infimum of the weights of all possible fractional colorings of G.

It may not be immediately clear that this definition has anything to do with our previous definition, in terms of graph homomorphisms. To see the connection, consider first a graph G with a proper coloring. Each color class is an independent set belonging to $\mathcal{I}(G)$. We define $f : \mathcal{I}(G) \to [0,1]$ mapping each color class to 1 and every other independent set to 0. Since each vertex falls in one color class, we obtain

$$\sum_{J \in \mathcal{I}(G,u)} f(J) = 1$$

for each vertex u. The weight of this fractional coloring is simply the number of colors.

Next, suppose we have a graph G with a proper a/b coloring as defined above, with a b-element set of colors associated with each vertex. Again, each color determines a color class, which is an independent set. If we define a function that sends each color class to $\frac{1}{b}$ and every other independent set to 0, then again, we have for each vertex u, $\sum_{J \in \mathcal{I}(G,u)} f(J) = 1$, so we have a fractional coloring by our new definition, with weight a/b.

Finally, let us consider translating from the new definition to the old one. Suppose we have a graph G and a function f mapping from $\mathcal{I}(G)$ to $[0,1] \cap Q$. (We will see below why we are justified in restricting our attention to rational valued functions.) Since the graph G is finite, the set $\mathcal{I}(G)$ is finite, and the image of the function f is a finite set of rational numbers. This set of numbers has a lowest common denominator, b. Now suppose we have an independent set I which is sent to the number m/b. Thus, we can choose m different colors, and let the set I be the color class for each of them. Proceeding in this manner, we will assign at least b different colors to each vertex, because of our condition that for all u $\sum_{J \in \mathcal{I}(G,u)} \geq 1$. If some vertices are assigned more than b colors, we can ignore all but b of them, and we have a fractional coloring according to our old definition - if the weight of f is a/b, and we do not ignore any colors completely, then we will have obtained a proper a/b coloring. If some colors are ignored, then we actually have a proper d/b fractional coloring, for some $d < a$.

2.2.3 Fractional chromatic number as a linear program

The usefulness of this new definition of fractional coloring and fractional chromatic number in terms of independent sets is that it leads us to a method of calculation using the tools of linear programming. To this end, we will construct a matrix representation of a fractional coloring.

For a graph G, define a matrix $A(G)$, with columns indexed by $V(G)$ and rows indexed by $\mathcal{I}(G)$. Each row is essentially the characteristic function of the corresponding independent set, with entries equal to 1 on columns corresponding to vertices in the independent set, and 0 otherwise.

Now let f be a fractional coloring of G and let $y(G)$ be the vector indexed by $\mathcal{I}(G)$ with entries given by f. With this notation, and letting 1 denote the all 1's vector, the inequality $y(G)^T A(G) \geq 1^T$ expresses the condition that

$$\sum_{J \in \mathcal{I}(G,u)} f(J) \geq 1$$

for all $u \in V(G)$.

In this algebraic representation of a fractional coloring, the determination of fractional chromatic number becomes a linear programming problem. The entries of the vector $y(G)$ are a set of variables, one for each independent set in $V(G)$, and our task is to minimize the sum of the variables (the weight of the fractional coloring), subject to the set of constraints that each entry in the vector $y(G)^T A(G)$ be greater than 1, and that each variable be in the interval $[0, 1]$. This amounts to minimizing a linear function within a convex polyhedral region in n-dimensional space defined by a finite number of linear inequalities, where $n = |\mathcal{I}(G)|$. This minimum must occur at a vertex of the region. Since each hyperplane forming a face of the region is determined by a linear equation with integer coefficients, then each vertex has rational coordinates, so our optimal fractional coloring will indeed take on rational values, as promised.

The regular, integer chromatic number, can be calculated with the same linear program by restricting the values in the vector $y(G)$ to 0 and 1. This is equivalent to covering the vertex set by independent sets that may only have weights of 1 or 0. Although polynomial time algorithms exist for calculating optimal solutions to linear programs, this is not the case for integer programs or 0-1 programs. In fact, many such problems have been shown to be NP-hard. In this respect, fractional chromatic numbers are easier to calculate than integer chromatic numbers.

2.2.4 Fractional cliques

The linear program that calculates a graph's fractional chromatic number is the *dual* of another linear program, in which we attempt to maximize the sum of elements in a vector $x(G)$, subject to the constraint $A(G)x(G) \leq 1$. We can pose this maximization problem as follows: we want to define a function $h : V(G) \to [0, 1]$, with the condition that, for each independent set in $\mathcal{I}(G)$, the sum of function values on the vertices in that set is no greater than 1. Such a function is called a **fractional clique**, the dual concept of a fractional coloring. As with fractional colorings, we define the **weight of a fractional clique** to be the sum of its values over its domain. The supremum of weights of fractional cliques defined for a graph is a parameter, $\omega_F(G)$, the **fractional clique number**.

Just as we saw a fractional coloring as a relaxation of the idea of an integer coloring, we would like to understand a fractional clique as a relaxation of the concept of a integer clique to the rationals (or reals). It is fairly straightforward to understand an ordinary clique as a fractional clique: we begin by considering a graph G, and a clique, $C \subseteq V(G)$. We can define a function $h : V(G) \to [0,1]$ that takes on the value 1 for each vertex in C and 0 elsewhere. This function satisfies the condition that its values sum to no more than 1 over each independent set, for no independent set may intersect the clique C in more than one vertex. Thus the function is a fractional clique, whose weight is the number of vertices in the clique.

Since an ordinary n-clique can be interpreted as a fractional clique of weight n, we can say that for any graph G, $\omega(G) \leq \omega_F(G)$.

2.2.5 Equality of χ_F and ω_F

The most important identity we will use to establish our main result is the equality of the fractional chromatic number and the fractional clique number. Since the linear programs which calculate these two parameters are dual to each other, we apply the Strong Duality Theorem of Linear Programming. We state the theorem in full. The reader is referred to [4] for more information about linear programming.

Suppose we have a primary linear program (LP) of the form:

$$\text{Maximize} \quad c^T x$$

$$\text{subject to} \;\; Ax \leq b$$

$$\text{and} \;\; x \geq 0$$

with its dual, of the form:

$$\text{Minimize} \;\; y^T b$$

$$\text{subject to} \;\; y^T A \geq c^T$$

$$\text{and} \;\; y \geq 0$$

If both LPs are feasible, i.e., have non-empty feasible regions, then both can be optimized, and the two objective functions have the same optimal value.

In the case of fractional chromatic number and fractional clique number, our primary LP is that which calculates the fractional clique of a graph G. The vector c determining the objective function is the all 1s vector, of dimension $|V(G)|$, and the constraint vector b is the all 1s vector, of dimension $|\mathcal{I}(G)|$. The matrix A is the matrix described above, whose rows are the characteristic vectors of the independent sets in $\mathcal{I}(G)$, defined over $V(G)$. The vector x for which we seek to maximize the objective function $c^T x$ has as its entries the values of a fractional clique at each vertex. The vector y for which we seek to minimize the objective function $y^T b$ has as its entries the values of a fractional coloring on each independent set.

In order to apply the Strong Duality Theorem, we need only establish that both LPs are feasible. Fortunately, this is easy: the zero vector is in the feasible region for the primary LP, and any proper coloring is in the feasible region for the dual. Thus, we may conclude that both objective functions have the same optimal value; i.e., that for a graph G, we have $\omega_F(G) = \chi_F(G)$.

This equality gives us a means of calculating these parameters. Suppose that, for a graph G, we find a fractional clique with weight equal to r. Since the fractional clique number is the supremum of weights of fractional cliques, we can say that $r \leq \omega_F(G)$. Now suppose we also find a fractional coloring of weight r. Then, since the fractional chromatic number is the infimum of weights of fractional colorings, we obtain $\chi_F(G) \leq r$. Combining these with the equality we obtained from duality, we get that $\omega_F(G) = r = \chi_F(G)$. This is the method we use to prove our result.

Chapter 3

Fractional Colorings and Mycielski's Graphs

We have noted that the fractional clique number of a graph G is bounded from below by the integer clique number, and that it is equal to the fractional chromatic number, which is bounded from above by the integer chromatic number. In other words,

$$\omega(G) \leq \omega_F(G) = \chi_F(G) \leq \chi(G).$$

Given these relations, one natural question to ask is whether the differences $\omega_F(G) - \omega(G)$ and $\chi(G) - \chi_F(G)$ can be made arbitrarily large. We shall answer this question in the affirmative, by displaying a sequence of graphs for which both differences increase without bound.

3.1 The Mycielski transformation

The sequence of graphs we will consider is obtained by starting with a single edge K_2, and repeatedly applying a graph transformation, which we now define. Suppose we have a graph G, with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The **Mycielski transformation** of G, denoted $\mu(G)$, has for its vertex set the set $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z\}$ - for a total of $2n + 1$ vertices. As for adjacency, we put

$x_i \sim x_j$ in $\mu(G)$ if and only if $v_i \sim v_j$ in G,

$x_i \sim y_j$ in $\mu(G)$ if and only if $v_i \sim v_j$ in G, and $y_i \sim z$ in $\mu(G)$ for all $i \in \{1, 2, \ldots, n\}$. See Figure 4 below.

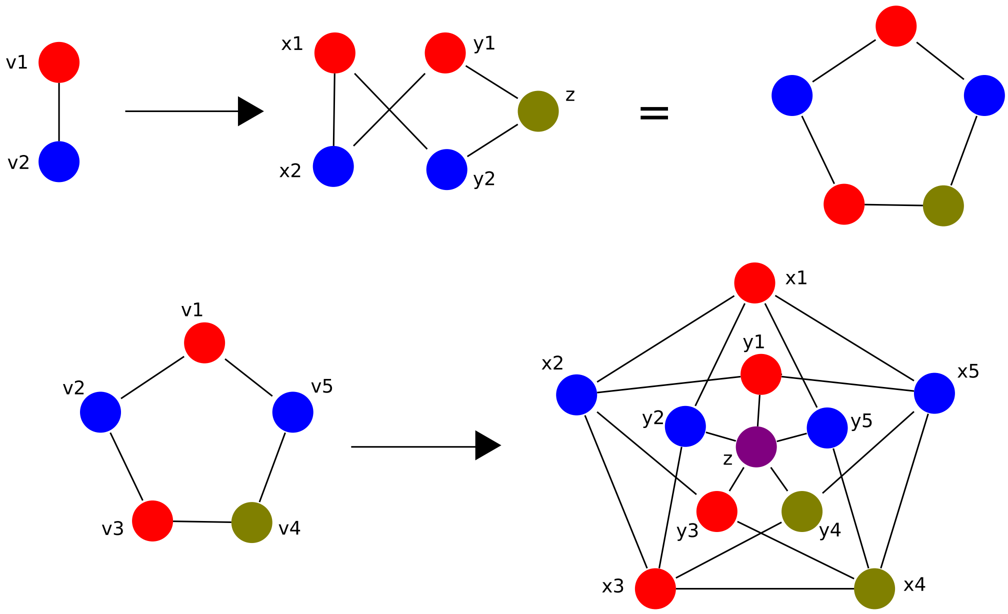

Figure 4: The Mycielski transformation

The theorem that we shall prove states that this transformation, applied to a graph G with at least one edge, results in a graph $\mu(G)$ with

(a) $\omega(\mu(G)) = \omega(G)$,

(b) $\chi(\mu(G)) = \chi(G) + 1$, and

(c) $\chi_F(\mu(G)) = \chi_F(G) + \frac{1}{\chi_F(G)}$.

3.2 Main result

We prove each of the three statements above in order:

3.2.1 Proof of part (a)

First we note that the vertices $x_1, x_2, \ldots, x_n$ form a subgraph of $\mu(G)$ which is isomorphic to G. Thus, any clique in G also appears as a clique in $\mu(G)$, so we have that $\omega(\mu(G)) \geq \omega(G)$.

To obtain the opposite inequality, consider cliques in $\mu(G)$. First, any clique containing the vertex z can contain only one other vertex, since z is only adjacent to the y vertices, none of which are adjacent to each other. Now consider a clique $\{x_{i(1)}, \ldots, x_{i(r)}, y_{j(1)}, \ldots, y_{j(s)}\}$. From the definition of the Mycielski transformation, we can see that the sets $\{i(1), \ldots, i(r)\}$ and $\{j(1), \ldots, j(s)\}$ are disjoint, and that the set $\{v_{i(1)}, \ldots, v_{i(r)}, v_{j(1)}, \ldots, v_{j(s)}\}$ is a clique in G. Thus, having considered cliques with and without vertex z, we see that for every clique in $\mu(G)$, there is a clique of equal size in G, or in other words, $\omega(\mu(G)) \leq \omega(G)$.

Combining these inequalities, we have $\omega(\mu(G)) = \omega(G)$, as desired.

3.2.2 Proof of part (b)

Suppose we have that $\chi(G) = k$. We must show that $\chi(\mu(G)) = k + 1$. First, we shall construct a proper k+1-coloring of $\mu(G)$, which will show that $\chi(\mu(G)) \leq k + 1$. Suppose that f is a proper k-coloring of G, understood as a mapping $f : V(G) \to \{1, \ldots, k\}$. We define a proper $k + 1$-coloring, $h : V(\mu(G)) \to \{1, \ldots, k + 1\}$ as follows. We set $h(x_i) = h(y_i) = f(v_i)$, for each $i \in \{1, \ldots, n\}$. Now set $h(z) = k + 1$. From the way $\mu(G)$ was constructed, we can see that this is a proper coloring, so we have $\chi(\mu(G)) \leq k + 1$.

For the inequality in the opposite direction, we will show that, given any coloring of $\mu(G)$, we can obtain a coloring of G with one color fewer. Since the chromatic number of G is k, this will show that $\chi(\mu(G)) - 1 \geq k$, or equivalently, $\chi(\mu(G)) \geq k + 1$. So, consider a proper coloring h of $\mu(G)$. We define a function f on the vertices of G as follows: $f(v_i) = h(x_i)$ if $h(x_i) \neq h(z)$, and $f(v_i) = h(y_i)$ if $h(x_i) = h(z)$. From the construction of $\mu(G)$, it should be clear that this is a proper coloring. It does not use the color $h(z)$, so it uses one color fewer than h uses, and thus we have that $\chi(\mu(G)) \geq k + 1$.

Again, combining our two inequalities, we obtain $\chi(\mu(G)) = \chi(G) + 1$, as desired.

3.2.3 Proof of part (c)

Now we will show that $\chi_F(\mu(G)) = \chi_F(G) + \frac{1}{\chi_F(G)}$, or in other words, we want to show that if $\chi_F(G) = \frac{a}{b}$, then $\chi_F(\mu(G)) = \frac{a^2+b^2}{ab}$. Our strategy will be to construct a fractional coloring and a fractional clique on $\mu(G)$, each with the appropriate weight, and then invoke our strong duality result.

Suppose we have a proper a/b-coloring of G, understood in terms of sets of colors assigned to each vertex. Thus, each vertex in G is assigned some subset of b colors out of a set of size a. We suppose, somewhat whimsically, that each of the a colors has a "offspring", b of them "male" and $a - b$ of them "female". Taking all of these offspring as distinct, we have obtained a^2 offspring colors. To these, add a set of b^2 new colors, and we have a total of $a^2 + b^2$ colors with which to color the vertices of $\mu(G)$. We assign them as follows. To each of vertex x_i, we assign all the offspring of all the colors associated with vertex v_i, a offspring of each of b colors for a total of ab colors. To each vertex y_i, we assign all the female offspring of all the colors associated with v_i (there are $b(a - b)$), and all of the new colors (there are b^2). To vertex z, we assign all of the male offspring of all the original colors, which is ab distinct colors. We see that we have assigned ab colors to each vertex, and it is easy to check that the coloring is a proper $(a^2 + b^2)/ab$-coloring of $\mu(G)$. The existence of this coloring proves that $\chi_F(\mu(G)) \leq \frac{a^2+b^2}{ab}$.

Our final, and most complicated, step is to construct a fractional clique of weight $\omega_F(G) + \frac{1}{\omega_F(G)}$ on $\mu(G)$. Suppose we have a fractional clique on G that achieves the optimal weight $\omega_F(G)$. Recall that this fractional clique is understood as a mapping $f : V(G) \to [0, 1]$ which sums to at most 1 on each independent set, and whose values all together sum to $\omega_F(G)$. Now we define a mapping

$g : V(\mu(G)) \to [0,1]$ as follows:

$$\begin{aligned} g(x_i) &= \left(1 - \frac{1}{\omega_F(G)}\right) f(v_i) \\ g(y_i) &= \frac{1}{\omega_F(G)} f(v_i) \\ g(z) &= \frac{1}{\omega_F(G)} \end{aligned}$$

We must show that this is a fractional clique. In other words, we must establish that it maps its domain into $[0,1]$, and that its values sum to at most 1 on each independent set in $\mu(G)$. The codomain is easy to establish: the range of f lies between 0 and 1, and since $\omega_F(G) \geq \omega(G) \geq 2 > 1$, then $0 < \frac{1}{\omega_F(G)} < 1$. Thus each expression in the definition of g yields a number between 0 and 1. It remains to show that the values of g are sufficiently bounded on independent sets.

We introduce a notation: for $M \subseteq V(G)$, we let $x(M) = \{x_i | v_i \in M\}$ and $y(M) = \{y_i | v_i \in M\}$. Now we will consider two types of independent sets in $\mu(G)$: those containing z and those not containing z.

Any independent set $S \subseteq V(\mu(G))$ that contains z cannot contain any of the y_i vertices, so it must be of the form $S = \{z\} \cup x(M)$ for some independent set M in $V(G)$. Summing the values of g over all vertices in the independent set, we obtain:

$$\begin{aligned} \sum_{v \in S} g(v) &= g(z) + \sum_{v \in x(M)} g(v) \\ &= \frac{1}{\omega_F(G)} + \left(1 - \frac{1}{\omega_F(G)}\right) \sum_{v \in M} f(v) \\ &\leq \frac{1}{\omega_F(G)} + \left(1 - \frac{1}{\omega_F(G)}\right) = 1 \end{aligned}$$

Now consider an independent set $S \subseteq V(\mu(G))$ with $z \notin S$. We can therefore say $S = x(M) \cup y(N)$ for some subsets of $V(G)$, M and N, and we know that M is an independent set. Since S is independent, then no vertex in $y(N)$ is adjacent to any vertex in $x(M)$, so we can express N as the union of two sets A and B, with $A \subseteq M$ and with none of the vertices in B adjacent to any vertex in M. Now we can sum the values of g over the vertices in $S = x(M) \cup y(N) = x(M) \cup y(A) \cup y(B)$:

$$
\begin{aligned}
\sum_{v\in S} g(v) &= \left(1-\frac{1}{\omega_F(G)}\right)\sum_{v\in M} f(v) + \frac{1}{\omega_F(G)}\sum_{v\in N} f(v) \\
&= \left(1-\frac{1}{\omega_F(G)}\right)\sum_{v\in M} f(v) + \frac{1}{\omega_F(G)}\sum_{v\in A} f(v) + \frac{1}{\omega_F(G)}\sum_{v\in B} f(v) \\
&\leq \left(1-\frac{1}{\omega_F(G)}\right)\sum_{v\in M} f(v) + \frac{1}{\omega_F(G)}\sum_{v\in M} f(v) + \frac{1}{\omega_F(G)}\sum_{v\in B} f(v) \\
&= \sum_{v\in M} f(v) + \frac{1}{\omega_F(G)}\sum_{v\in B} f(v)
\end{aligned}
$$

The first two equalities above are simply partitions of the sum into sub-sums corresponding to subsets. The inequality holds because $A \subseteq M$, and the final equality is just a simplification. It will now suffice to show that the final expression obtained above is less than or equal to 1.

Let us consider H, the subgraph of G induced by B. The graph H has some fractional chromatic number, say r/s. Suppose we have a proper r/s-coloring of H. Recall that the color classes of a fractional coloring are independent sets, so we have r independent sets of vertices in $V(H) = B$; let us call them $C_1, \ldots, C_r$. Not only is each of the sets C_i independent in H, but it is also independent in G, and also $C_i \cup M$ is independent in G as well, because $C_i \subseteq B$.

For each i, we note that f is a fractional clique on G, and sum over the independent set $C_i \cup M$ to obtain:

$$\sum_{v\in M} f(v) + \sum_{v\in C_i} f(v) \leq 1$$

Summing these inequalities over each C_i, we get:

$$r\sum_{v\in M} f(v) + s\sum_{v\in B} f(v) \leq r$$

The second term on the left side of the inequality results because each vertex in B belongs to s different color classes in our proper r/s-coloring. Now we divide by r to obtain:

$$\sum_{v\in M} f(v) + \frac{s}{r}\sum_{v\in B} f(v) \leq 1$$

Since r/s is the fractional chromatic number of H, and H is a subgraph of G, we can say that $\frac{r}{s} \leq \chi_F(G) = \omega_F(G)$, or equivalently, $\frac{1}{\omega_F(G)} \leq \frac{s}{r}$. Thus:

$$\sum_{v\in M} f(v) + \frac{1}{\omega_F(G)}\sum_{v\in B} f(v) \leq \sum_{v\in M} f(v) + \frac{s}{r}\sum_{v\in B} f(v) \leq 1,$$

as required. We have shown that the mapping g that we defined is indeed a fractional clique on $\mu(G)$. We now check its weight.

$$\begin{aligned}
\sum_{v\in V(\mu(G))} g(v) &= \sum_{i=1}^{n} h(x_i) + \sum_{i=1}^{n} h(y_i) + h(z) \\
&= \left(1 - \frac{1}{\omega_F(G)}\right)\sum_{v\in V(G)} f(v) + \frac{1}{\omega_F(G)}\sum_{v\in V(G)} f(v) + \frac{1}{\omega_F(G)} \\
&= \sum_{v\in V(G)} f(v) + \frac{1}{\omega_F(G)} \\
&= \omega_F(G) + \frac{1}{\omega_F(G)} = \chi_F(G) + \frac{1}{\chi_F(G)}
\end{aligned}$$

This is the required weight, so we have constructed a fractional coloring and a fractional clique on $\mu(G)$, both with weight $\chi_F(G) + \frac{1}{\chi_F(G)}$. We can now write the inequality

$$\chi_F(\mu(G)) \leq \chi_F(G) + \frac{1}{\chi_F(G)} \leq \omega_F(G)$$

and invoke strong duality to declare the terms at either end equal to each other, and thus to the middle term. □

3.3 Discussion of main result

Now that we have a theorem telling us how the Mycielski transformation affects the three parameters of clique number, chromatic number, and fractional chromatic number, let us apply this result in a concrete case, and iterate the Mycielski transformation to obtain a sequence of graphs $\{G_n\}$, with $G_{n+1} = \mu(G_n)$ for $n > 2$. For our starting graph G_2 we take a single edge, K_2, for which $\omega(G) = \chi_F(G) = \chi(G) = 2$.

Applying our theorem, first to clique numbers, we see that $\omega(G_n) = 2$ for all n. Considering chromatic numbers, we have $\chi(G_2) = 2$ and $\chi(G_{n+1}) = \chi(G_n)+1$; thus $\chi(G_n) = n$ for all n. Finally, the fractional chromatic number of G_n is determined by a sequence $\{a_n\}_{n\in\{2,3,\ldots\}}$ given by the recurrence: $a_2 = 2$ and $a_{n+1} = a_n + \frac{1}{a_n}$.

This sequence has been studied (see [5] or [1]), and it is known that for all n:

$$\sqrt{2n} \leq a_n \leq \sqrt{2n + \frac{1}{2}\ln n}.$$

Clearly, a_n grows without bound, but less quickly than any sequence of the form n^r for $r > \frac{1}{2}$. Thus, the difference between the fractional clique number and the clique number grows without limit, as does the difference between the chromatic number and the fractional chromatic number.

Chapter 4

The Zykov graphs

We have been considering a sequence of graphs, and the resulting sequences of clique numbers, fractional chromatic numbers, and chromatic numbers. In the sequence of Mycielski graphs, we saw that the clique number is constant at 2, while the chromatic number increases by 1 at each step. We now consider another sequence of graphs in which the clique numbers and chromatic numbers are the same as they are for the Mycielski sequence. We will obtain some evidence that this new sequence's fractional chromatic numbers are also the same as for the Mycielski graphs.

Zykov (see [6, p.215] or [7]) described a sequence of graphs as follows: We begin with Z_1, a single vertex. In order to construct Z_n, we begin with disjoint copies of $Z_1, \ldots, Z_{n-1}$. We add to the union of these graphs a set of new vertices, equal in number to the product $|V(Z_1)| \cdot \ldots \cdot |V(Z_{n-1})|$. For each sequence of vertices $v_1, \ldots, v_{n-1}$ with v_i being a vertex in our copy of Z_i, we connect one of the new vertices to each vertex in the sequence. See Figure 5 for the first four graphs in this sequence.

We observe that the vertex sets of these graphs get very large quickly. The sequence $\{a_i = |V(Z_i)| \mid i = 1, 2, \ldots\}$ is described by the recurrence relation:

$$\begin{aligned} a_1 &= 1 \\ a_{n+1} &= \sum_{k=1}^{n} a_k + \prod_{k=1}^{n} a_k \end{aligned}$$

This sequence begins $1, 2, 5, 18, 206, 37312, 1383566504, \ldots$. We have investigated the first 5 graphs in the sequence in some detail.

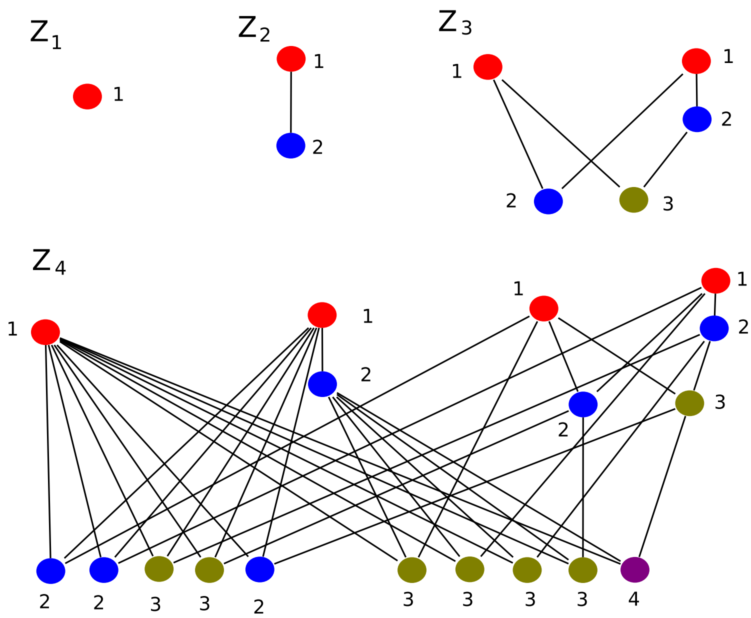

Figure 5: The first four Zykov graphs

It is easy to show that the graphs in this sequence are all triangle-free, so they all have clique number equal to 2. It also follows from the construction of these graphs that $\chi(Z_n) = n$ for all n. The first three graphs are already known to us: Z_1 is a single vertex, or K_1 ($\chi_F(Z_1) = 1$), Z_2 is the same as K_2 ($\chi_F(Z_2) = 2$), and Z_3 is the same as C_5 ($\chi_F(Z_3) = 5/2$).

The first interesting case is Z_4 (pictured above), which has 18 vertices and 2960 independent sets. The abundance of independent sets makes direct calculation of $\chi_F(Z_4)$ by means of linear programming prohibitive. However, we have two different techniques for reducing the size of the problem.

First, we can restrict ourselves to looking at *maximal* independent sets. That this restriction is justified is clear if we consider the dual linear program, which calculates fractional clique number. The constraints in this program specify that the "weights" assigned to the vertices in each independent set add up to no more than 1. If A and B are independent sets with $B \subseteq A$, and if the weights on the vertices in A add up to no more than 1, then total weight on B must also fall short of 1. Thus, the constraints corresponding to non-maximal independent sets are redundant, and the LP using only maximal sets as contstraints will be satisfied by the same optimal solution as the LP that takes all independent sets into account. The dual of the program that uses only maximal independent sets is a program that calculates a fractional coloring in which only maximal independent sets may serve as color classes. By the Strong Duality theorem, this LP will have the same optimal value as its dual, which has the same optimal value as our original LP.

Restricting ourselves to maximal independent sets, the study of Z_4 becomes much more accessible: of the 2960 independent sets, only 43 are maximal. Using Maple, we obtained a fractional clique and a fractional coloring for Z_4, both with weight equal to $\frac{29}{10}$, which is the same as the weight of the Mycielski transformation of the 5-cycle.

In pursuit of a pattern by which colorings and cliques for Zykov graphs may be generated, we found it desirable to calculate a fractional clique and fractional coloring for Z_5. This graph has 206 vertices and 98143 maximal independent sets. In order to analyze it, we employed another technique, this one relying on symmetries in the graph.

The Zykov graphs have some automorphisms which can be discovered by inspection of the first few graphs, particularly Z_4. Naturally, these automorphisms map vertices to vertices, and independent sets to independent sets. We set up equivalence relations on $V(Z_4)$ and $\mathcal{I}(Z_4)$, whereby vertices (and independent sets) are considered equivalent if a graph automorphism carries one to the other. Instead of considering individual vertices and independent sets, we can work with equivalence classes of vertices and independent sets, and ask whether we can still calculate fractional cliques and fractional colorings.

In order to "fold" the symmetries out of our LP, as it were, we simply redefine the vectors b and c and the matrix A that we used in Section 2.2.5. Vector b

in indexed by the equivalence classes of independent sets, and ith element is the number of independent sets in the ith equivalence class. Vector c, in the same way, counts the vertices in each equivalence class in our partition of $V(G)$. In matrix A we let entry (i, j) be the number of vertices of class j occuring in each independent set of class i, multiplied by the number of independent sets of in class i.

The matrix and vectors we have just described define an LP and its dual, which calculate fractional cliques and fractional colorings that are invariant under automorphisms. Such solutions must exist and be equal, due to the Strong Duality Theorem, just as before. Now we can handle the calculation of $\chi_F(Z_5)$. Although Z_5 has 98143 maximal independent sets, they fall into a mere 295 equivalence classes under graph automorphisms. Likewise, the 206 vertices fall into 11 classes, and we have a much more tractable problem. Running the resulting LP through Maple, we were able to determine that the fractional chromatic number of Z_5 is $\frac{941}{290}$, equal to the fractional chromatic number of the second Mycielski transformation of the 5-cycle.

Thus, we have Z_2, Z_3, Z_4 and Z_5 with the same parameter values as K_2, $\mu(K_2) = C_5$, $\mu(C_5)$ and $\mu(\mu(C_5))$, as far as clique number, chromatic number, and fractional chromatic number. We conjecture that this pattern continues. More formally:

4.0.1 Conjecture. *For all $n \in \{1, 2, \ldots\}$, the following hold:*

$$\begin{aligned} \omega(Z_n) &= 2 \\ \chi(Z_n) &= n \\ \chi_F(Z_n) &= a_n \end{aligned}$$

where a_n is determined by the recurrence $a_1 = 1$, $a_{n+1} = a_n + a_n^{-1}$.

The first two statements, regarding clique number and chromatic number, are easy to prove. We have verified the third for $n = 1, 2, 3, 4$ and 5.

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