A Sharp Upper Bounds for Largest Eigenvalue of the Laplacian Matrices of Tree

Tomohiro Kawasaki

July 28, 2011

Introduction

Motivation

Spectral Graph Theory :

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- In this project, we focus on an upper bound for the spectrum of the Laplacian matrix of a tree.

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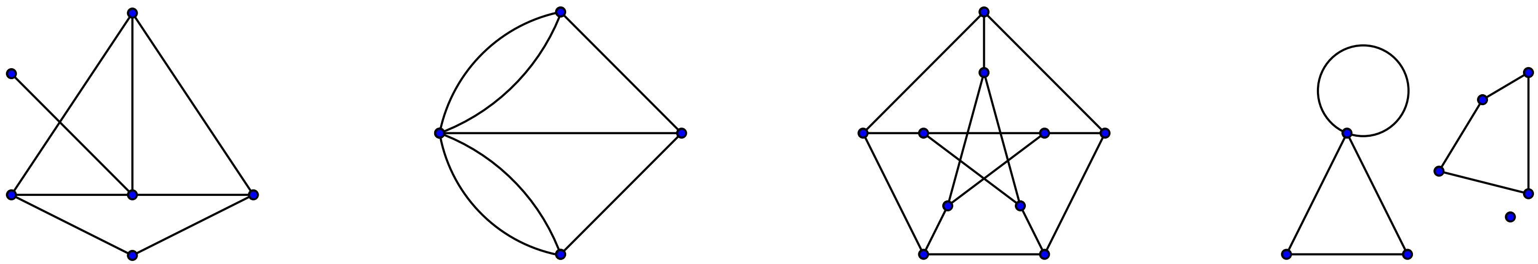
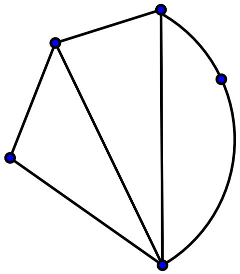
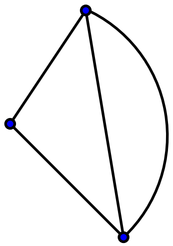
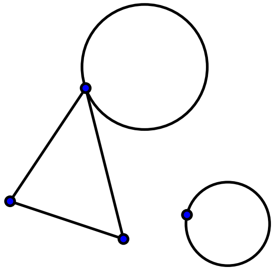
Examples:



Some particular type of graphs:







Preliminaries

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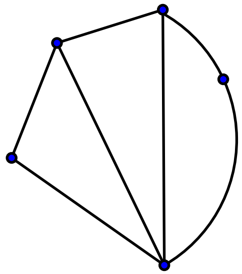
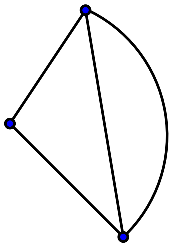
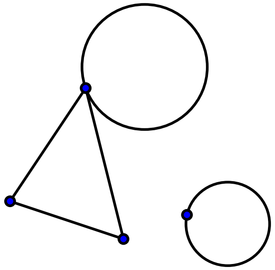
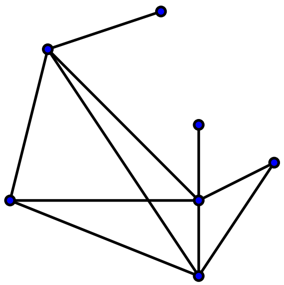
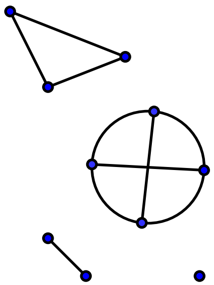

Simple graph


Multiple edges


Have a loop


Connected graph


Disconnected graph

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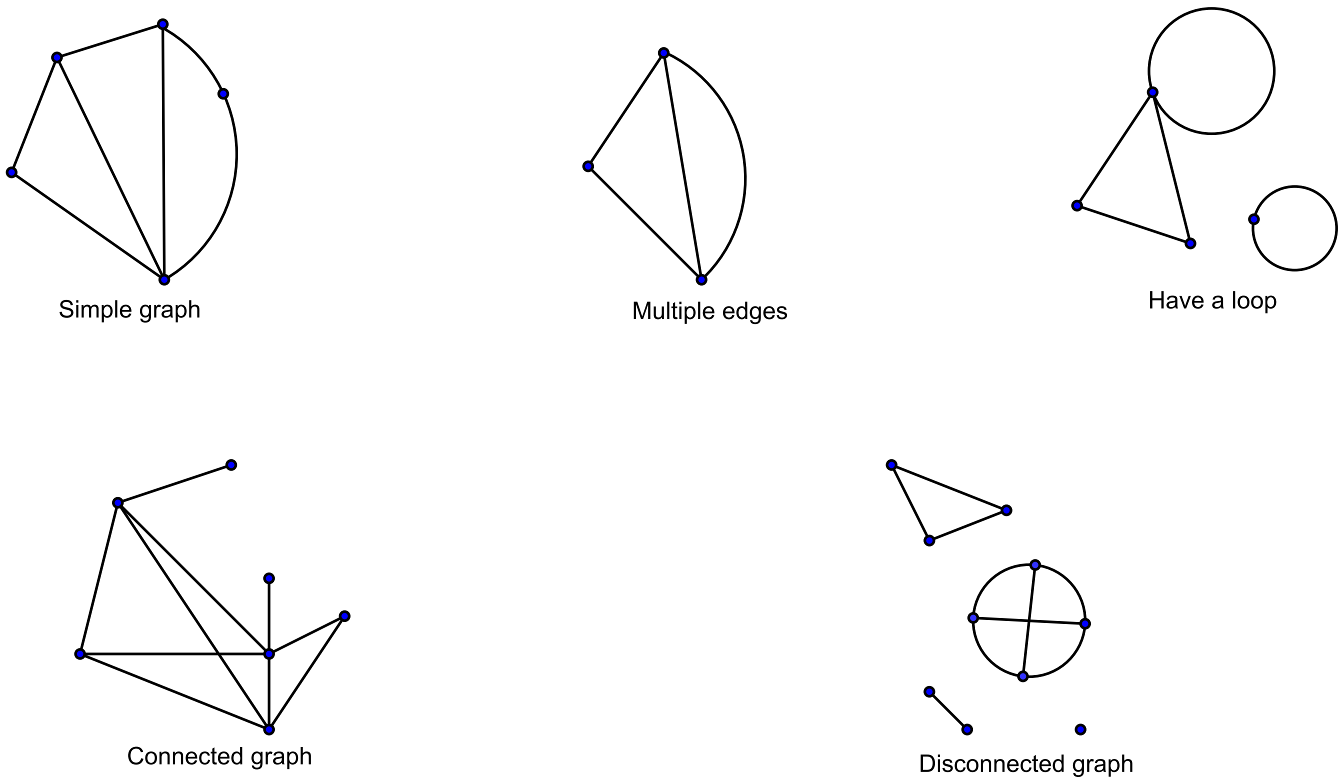

In this project, we assume that a graph G is simple connected.

Definition. *Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The* **adjacency matrix** $A = A(G)$ *is the $n \times n$ matrix (a_{ij}), where $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise.*

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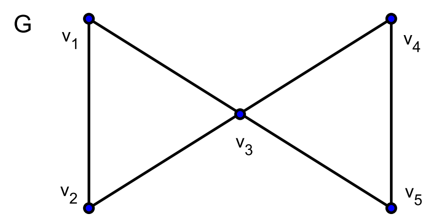

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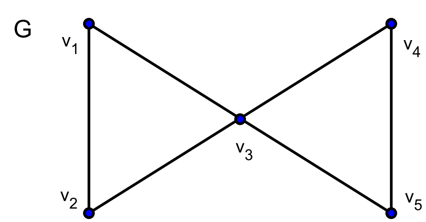

$$
\begin{array}{c c}
A(G) & \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\
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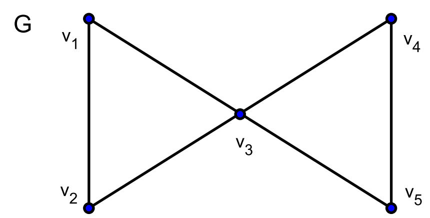

$A(G)$	v_1	v_2	v_3	v_4	v_5
v_1	0	1	1	0	0
v_2	1	0	1	0	0
v_3	1	1	0	1	1
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Note:

- □ The adjacency matrix $A(G)$ is symmetric.

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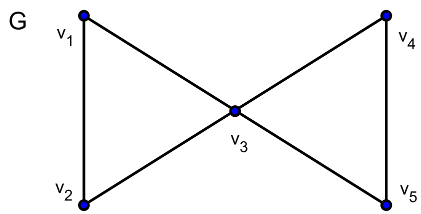

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Note:

- □ The adjacency matrix $A(G)$ is symmetric.
- □ The diagonal entries are always 0.

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Theorem (Perron-Frobenius Theorem)**.** *Suppose A is a real nonnegative $n \times n$ matrix whose underlying graph G is connected. Then, $\rho(A)$ is a simple eigenvalue of A. If x is an eigenvector for ρ, then no entries of x are zero, and all have the same sign.*

□ The Perron vector is a unique (up to scalar multiplication), positive, unit, and simple vector.

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Example: Take $A(G)$ previously obtained.

$$\begin{array}{c} A(G) \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

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$$\mathbf{x} = \{0.3941, 0.3941, 0.6154, 0.3941, 0.3941\}^T.$$

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- The **signless Laplacian matrix** Q to be the matrix $Q(G) = D(G) + A(G)$.
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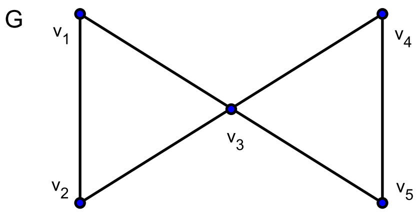

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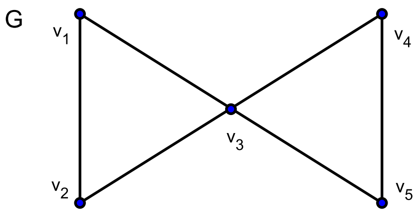

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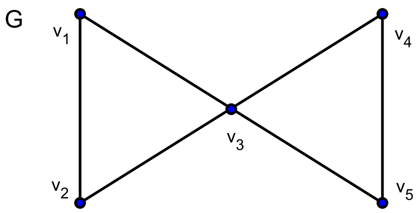

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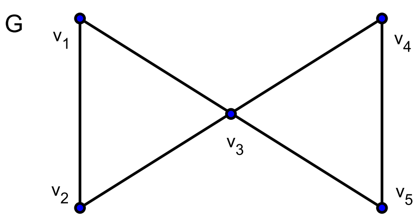

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$$L(G) \quad \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix} \end{array} \qquad Q(G) \quad \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 4 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

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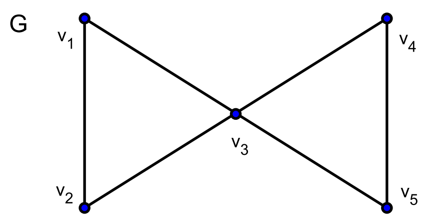

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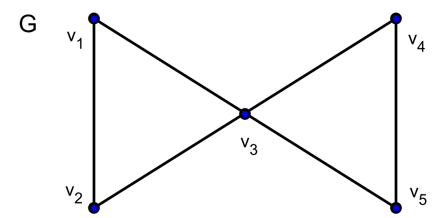

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v_3	−1	−1	4	−1	−1
v_4	0	0	−1	2	−1
v_5	0	0	−1	−1	2

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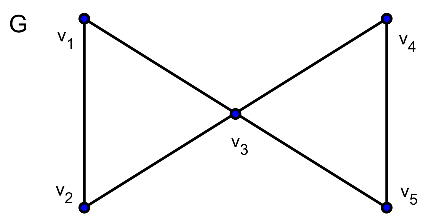

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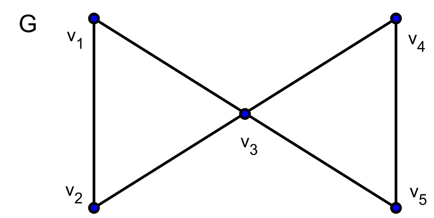

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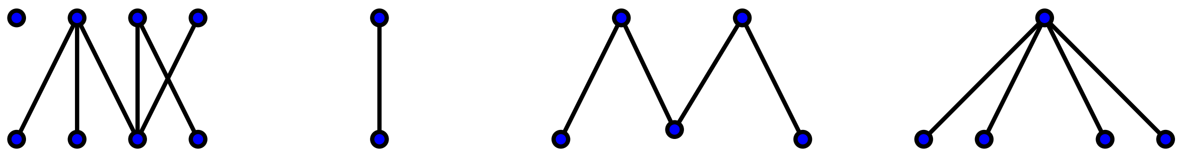

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- Let G be a simple graph and take $v \in V(G)$. Then, $N_G(v)$ denotes the set of vertices which are adjacent to the vertex v.

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- Let G be a simple graph and take $v \in V(G)$. Then, $N_G(v)$ denotes the set of vertices which are adjacent to the vertex v.
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- For a nonnegative integer n and k, $\mathscr{T}_{n,k}$ denotes the set of tree graphs with n vertices and k pendant vertices.
- For any fixed n and k, we define $T_{n,k} \in \mathscr{T}_{n,k}$ to be a tree graph obtained from a complete bipartite graph (we call this a *star* graph) $K_{1,k}$ and k paths of almost equal length, by joining each pendant vertex of $K_{1,k}$ to an end vertex of one path.

Objective

Example: How can I construct $T_{18,5}$?

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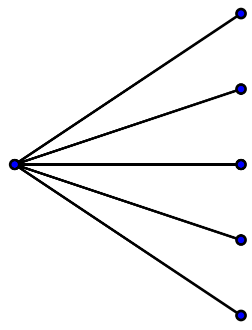

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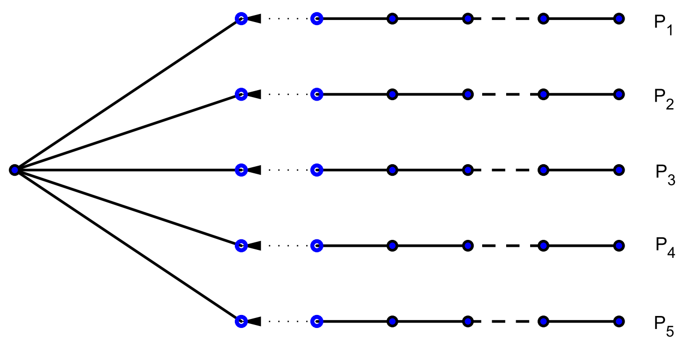

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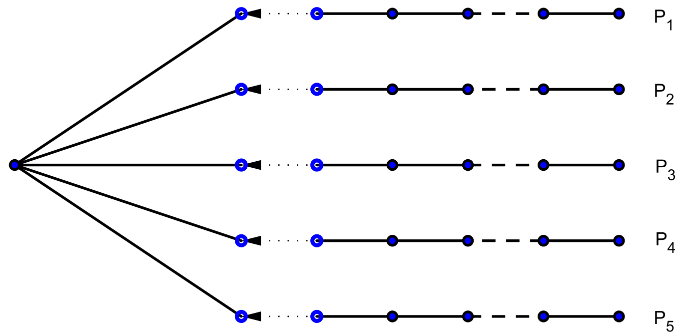

- ☐ Start with the star $K_{1,5}$
- ☐ We are attaching five paths to each pendant vertex in $K_{1,5}$
- ☐ What kind of paths do we need to add? The definition said "almost equal length"?!

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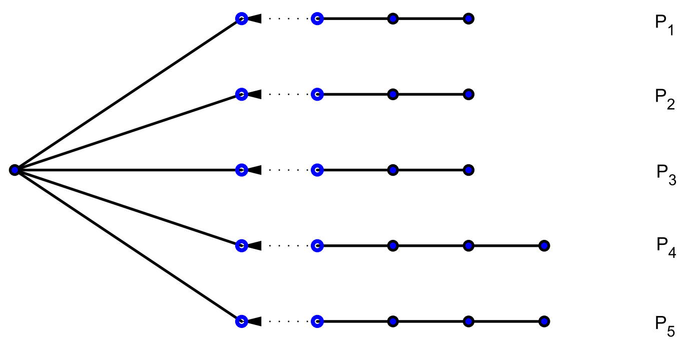

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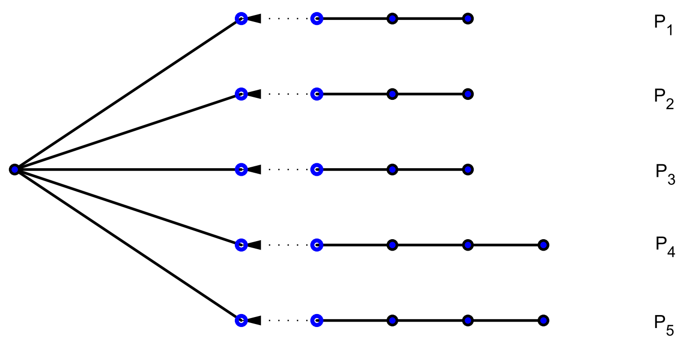

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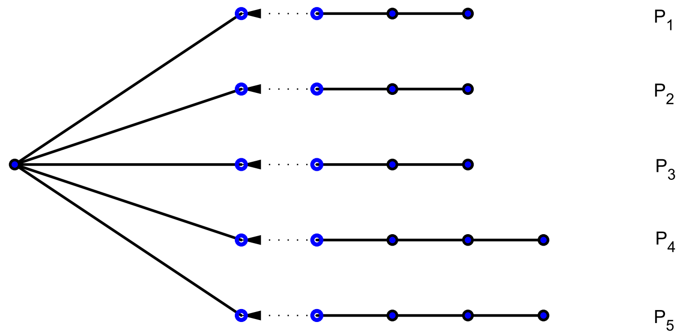

☐ We want five paths having the same number of vertices as much as possible, so consider the division algorithm
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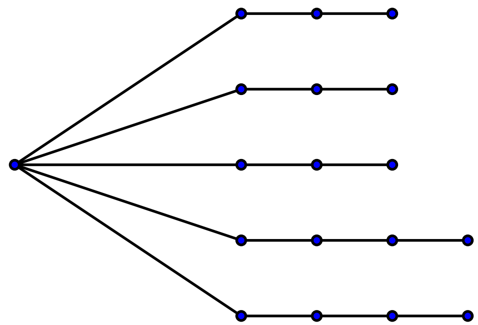

☐ We want five paths having the same number of vertices as much as possible, so consider the division algorithm $17 = 3 \cdot 5 + 2$.

☐ This quotient 3 represents the minimum number of vertices that each path has.

☐ But we still have two more vertices remaining...

☐ Thus, two paths have an extra vertex.

Objective

Introduction
Motivation
▷ Preliminaries
Objective
Summary
Some Lemmas
Main Results
The End

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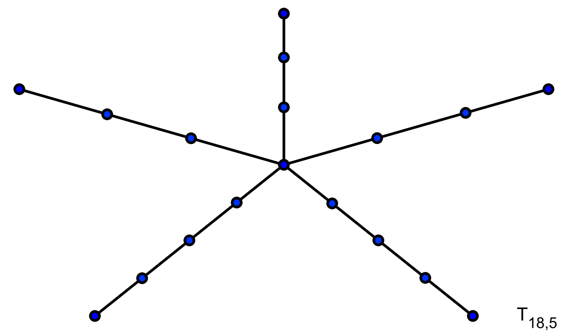

☐ Not every path has the same number of vertices, but each has an "almost equal" number of vertices!

Objective: For any $T \in \mathscr{T}_{n,k}$, its Laplacian spectral radius is bounded by the one for $T_{n,k}$. That is,

$$\mu(T) \leq \mu(T_{n,k})$$

Summary

Recall that $\mu(T) \leq \mu(T_{n,k})$ for all $T \in \mathscr{T}_{n,k}$. The following are notations used in this presentation.

- $A(G)$: adjacency matrix of G
- $\rho(G)$: spectral radius of G
- $L(G)$: Laplacian matrix of G
- $\mu(G)$: Laplacian spectral radius of G
- $Q(G)$: signless Laplacian matrix of G
- $\nu(G)$: signless Laplacian spectral radius of G
- $N_G(v)$: set of vertices adjacent to a vertex v in G
- $N_G(v)\backslash(N_G(u) \cup \{u\})$: set of neighbors of v, but do not include neighbors of u nor u itself
- $\mathscr{T}_{n,k}$: set of tree graphs with n vertices and k pendant vertices
- $T_{n,k}$: a tree graph by the construction just explained

Some Lemmas

Lemma 1

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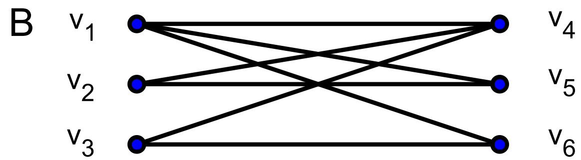

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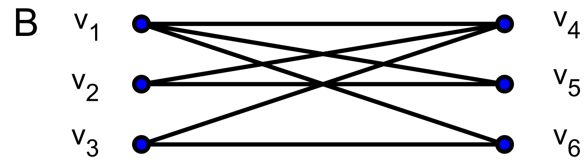

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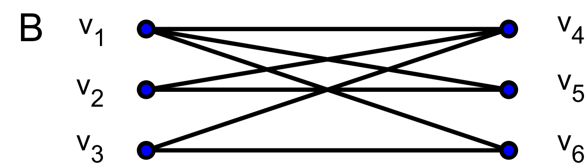

Their signless/Laplacian matrices and spectra are..

$L(B)$	v_1	v_2	v_3	v_4	v_5	v_6
v_1	3	0	0	−1	−1	−1
v_2	0	2	0	−1	−1	0
v_3	0	0	2	−1	0	−1
v_4	−1	−1	−1	3	0	0
v_5	−1	−1	0	0	2	0
v_6	−1	0	−1	0	0	2

$Q(B)$	v_1	v_2	v_3	v_4	v_5	v_6
v_1	3	0	0	1	1	1
v_2	0	2	0	1	1	0
v_3	0	0	2	1	0	1
v_4	1	1	1	3	0	0
v_5	1	1	0	0	2	0
v_6	1	0	1	0	0	2

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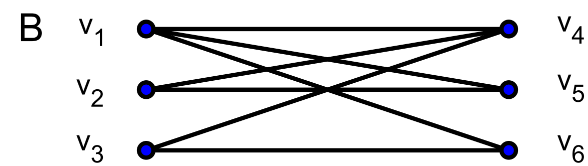

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v_4	−1	−1	−1	3	0	0
v_5	−1	−1	0	0	2	0
v_6	−1	0	−1	0	0	2

$Q(B)$	v_1	v_2	v_3	v_4	v_5	v_6
v_1	3	0	0	1	1	1
v_2	0	2	0	1	1	0
v_3	0	0	2	1	0	1
v_4	1	1	1	3	0	0
v_5	1	1	0	0	2	0
v_6	1	0	1	0	0	2

The spectrum of $L(B)$ is $\{-0.0000, 1.0000, 2.0000, 3.0000, 3.0000, 5.0000\}$, whereas the spectrum of $Q(B)$ is $\{-0.0000, 1.0000, 2.0000, 3.0000, 3.0000, 5.0000\}$, as desired.

Lemma 2. *Let u be a vertex of the connected graph G and for positive integers k and l, $G_{k,l}$ denote the graph obtained from G by adding pendant paths of length k and l at u. If $k \geq l \geq 1$, then*

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Let $k = 2$ and $l = 2$, and take following figures with u labeled.

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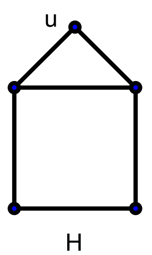

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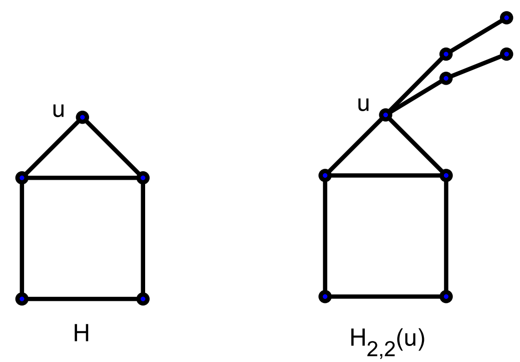

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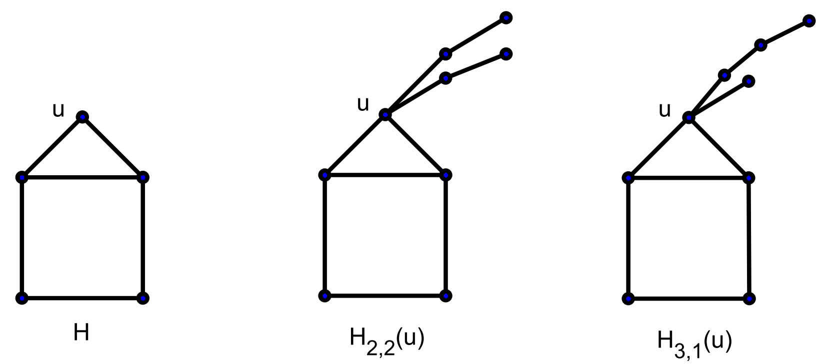

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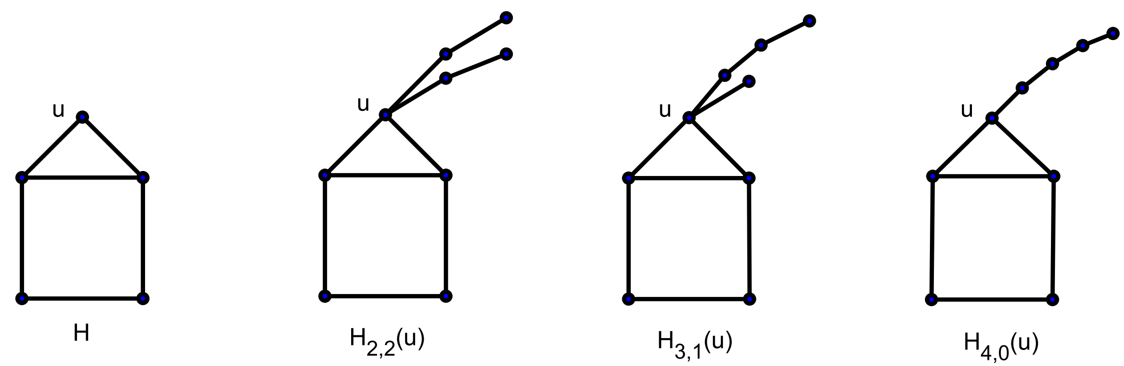

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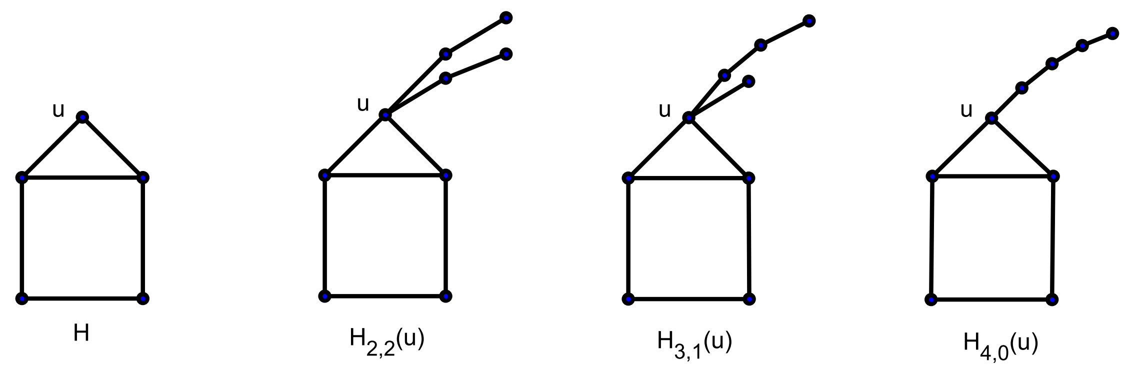

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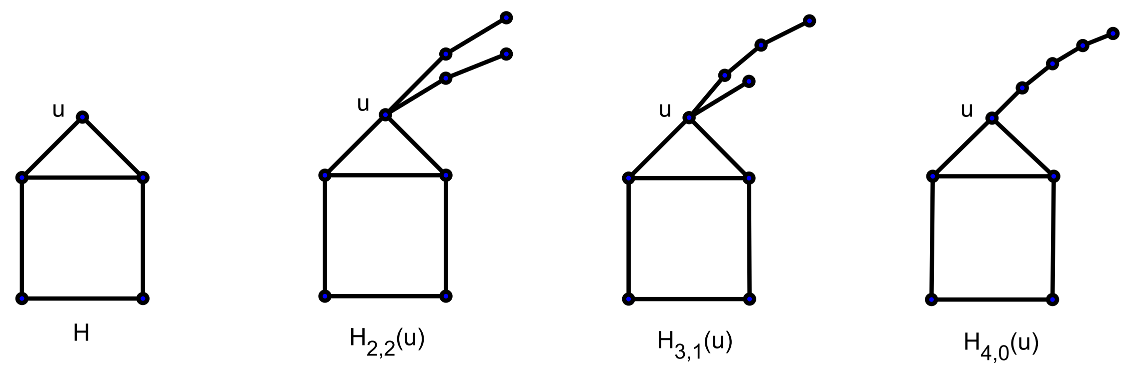

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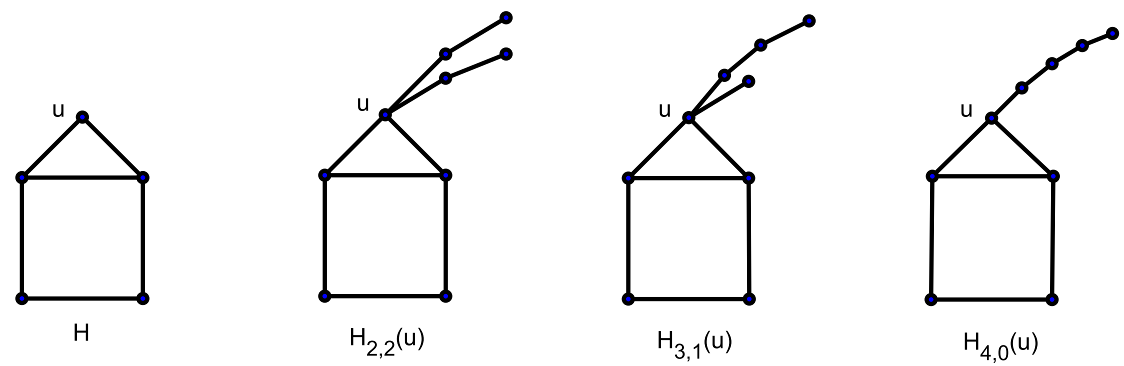

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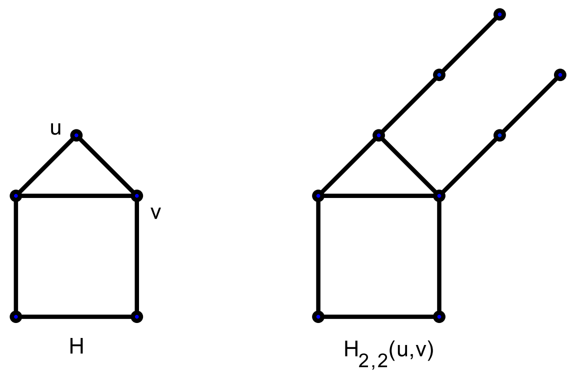

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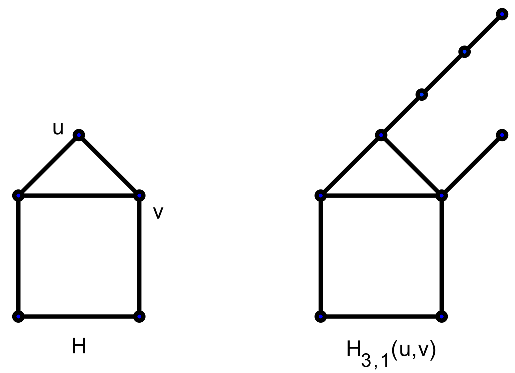

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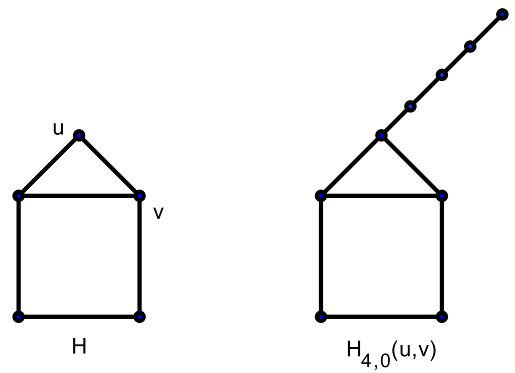

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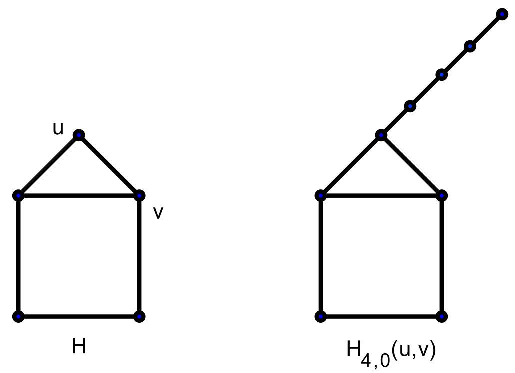

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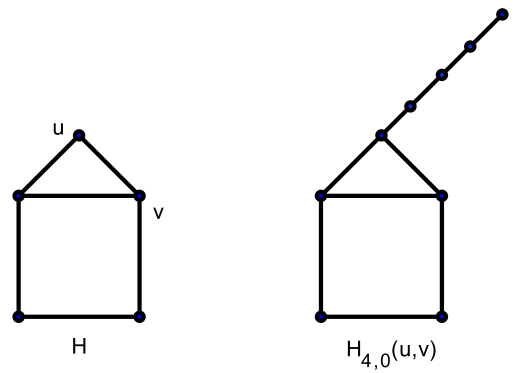

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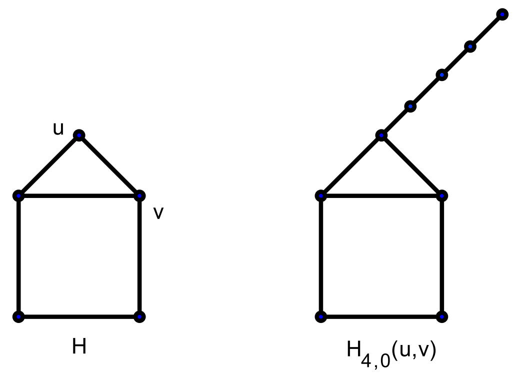

Spectral radii of these graphs are...

Graph	$H_{2,2}(u,v)$	$H_{3,1}(u,v)$	$H_{4,0}(u,v)$
S · R	2.6989	2.6839	2.5813

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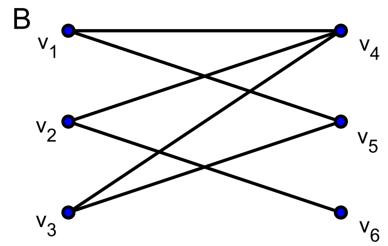

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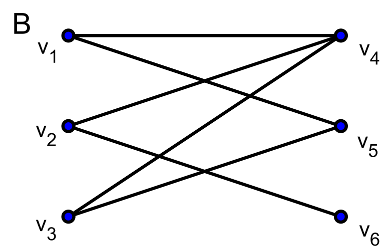

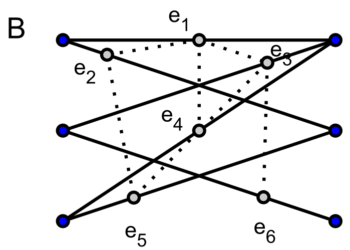

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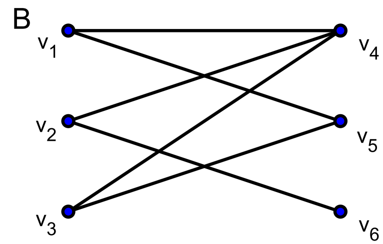

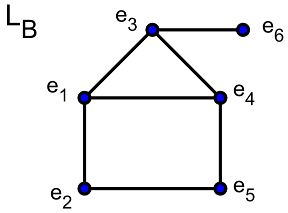

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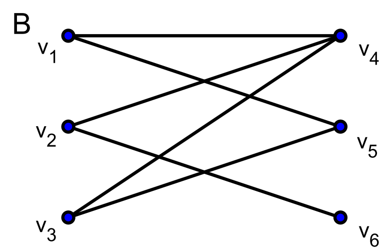

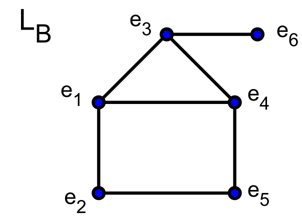

Example: Take a bipartite graph B as the following.

$L(B)$	v_1	v_2	v_3	v_4	v_5	v_6
v_1	2	0	0	−1	−1	0
v_2	0	2	0	−1	0	−1
v_3	0	0	2	−1	−1	0
v_4	−1	−1	−1	3	0	0
v_5	−1	0	−1	0	2	0
v_6	0	−1	0	0	0	1

$A(L_B)$	e_1	e_2	e_3	e_4	e_5	e_6
e_1	0	1	1	1	0	0
e_2	1	0	0	0	1	0
e_3	1	0	0	1	0	1
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e_5	0	1	0	1	0	0
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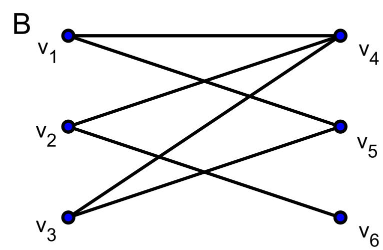

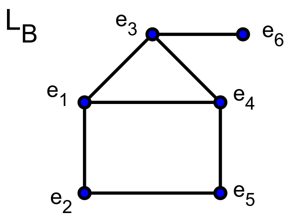

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spectrum of $L(B)$ is $\{-0.000, 0.438, 2.000, 2.000, 3.000, 4.561\}$

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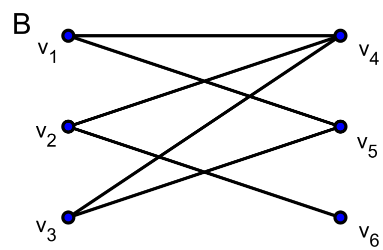

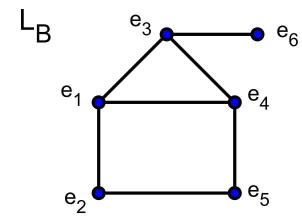

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Main Results

Idea

- Our goal is $\mu(T) \leq \mu(T_{n,k})$ for any $T \in \mathscr{T}_{n,k}$.

Idea

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- The idea is to reconstruct T to $T_{n,k}$ by deleting and adding edges one by one. Then, watching how the (signless) Laplacian spectral radius changes for each step.

Idea

For example, take $T \in \mathscr{T}_{8,6}$ as..

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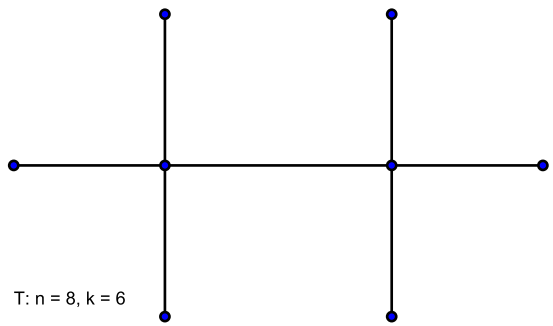

Idea

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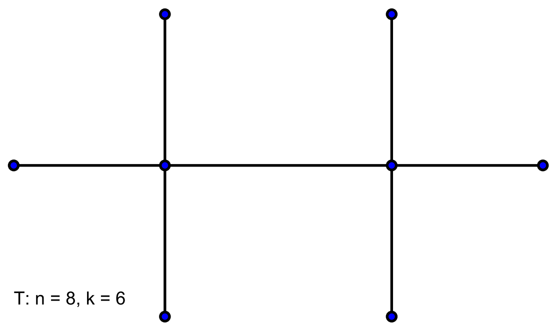
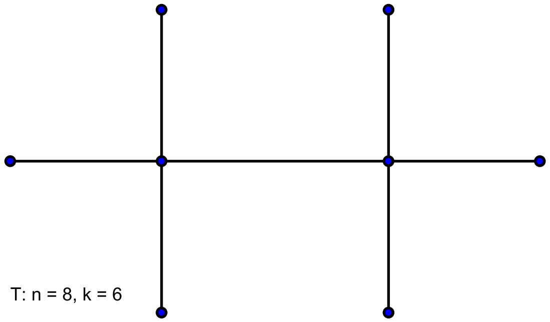


☐ First, construct $T_{8,6}$.

Idea

For example, take $T \in \mathscr{T}_{8,6}$ as..



- First, construct $T_{8,6}$.
- Having $n = 8$ and $k = 6$, We start with $K_{1,6}$.

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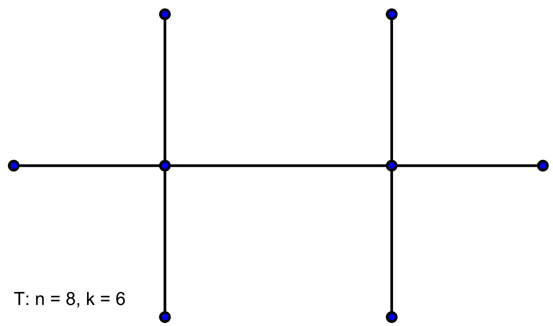

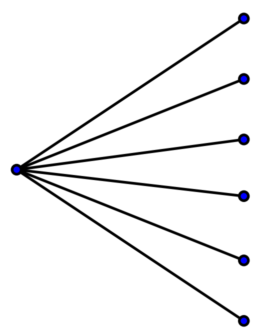

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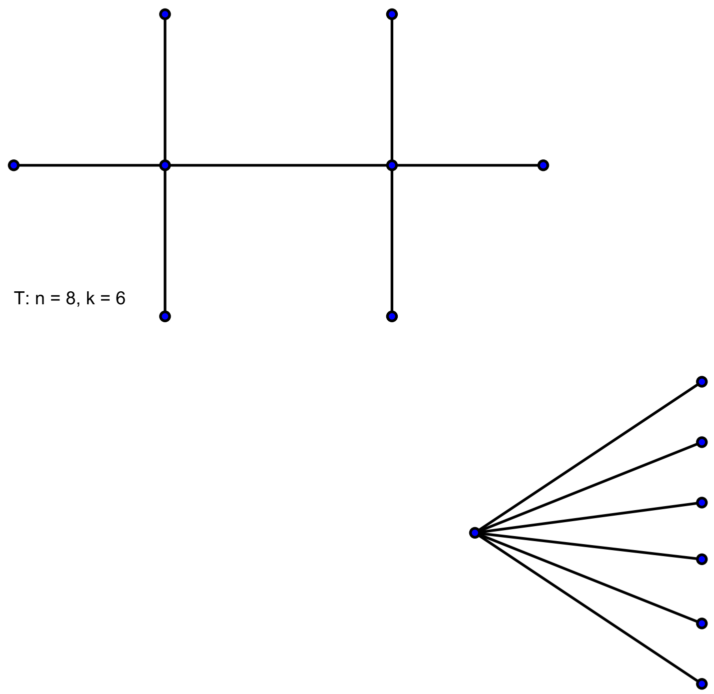

□ Since $7 = 1 \cdot 6 + 1$, we have...

Idea

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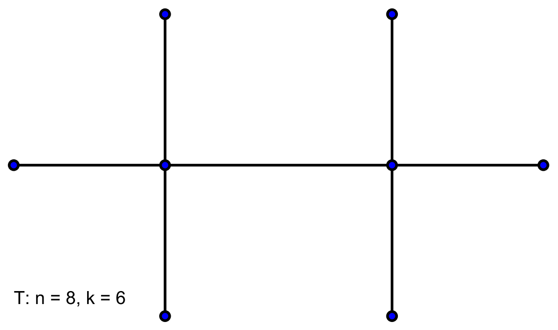

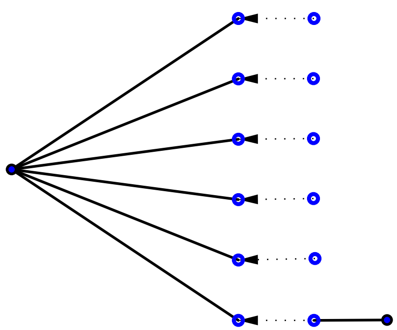

□ Since $7 = 1 \cdot 6 + 1$, we have... five paths of 1 vertex and one path of 2 vertices.

Idea

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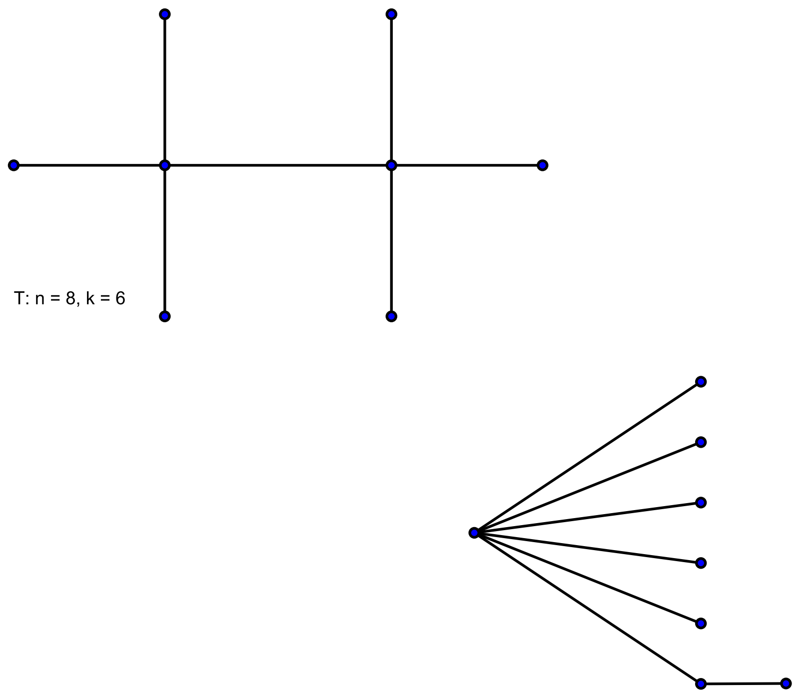
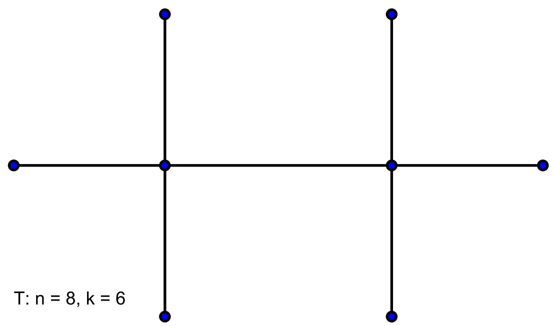
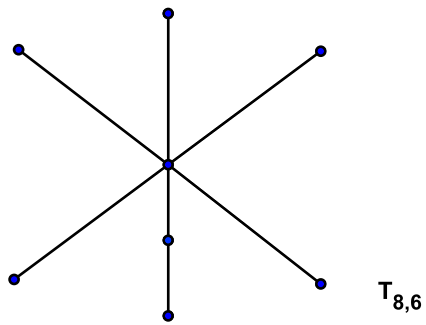


☐ Rearranging this graph, we obtain...

Idea

For example, take $T \in \mathscr{T}_{8,6}$ as..





☐ Rearranging this graph, we obtain... $T_{8,6}$

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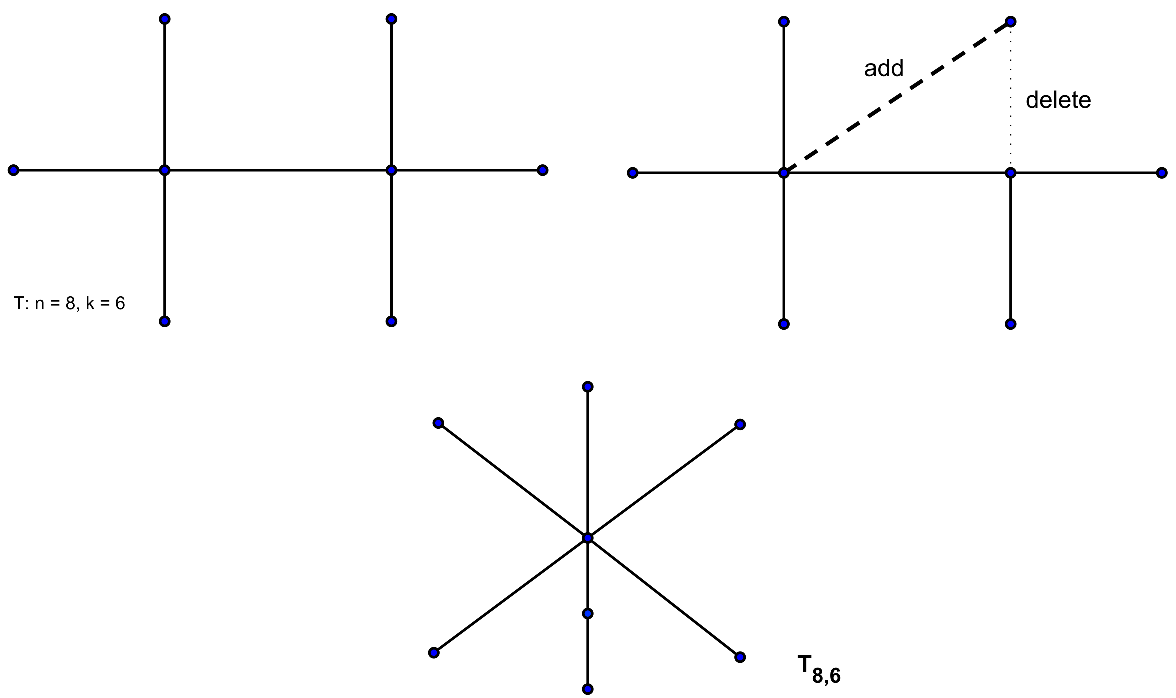

□ Back to T above,

Idea

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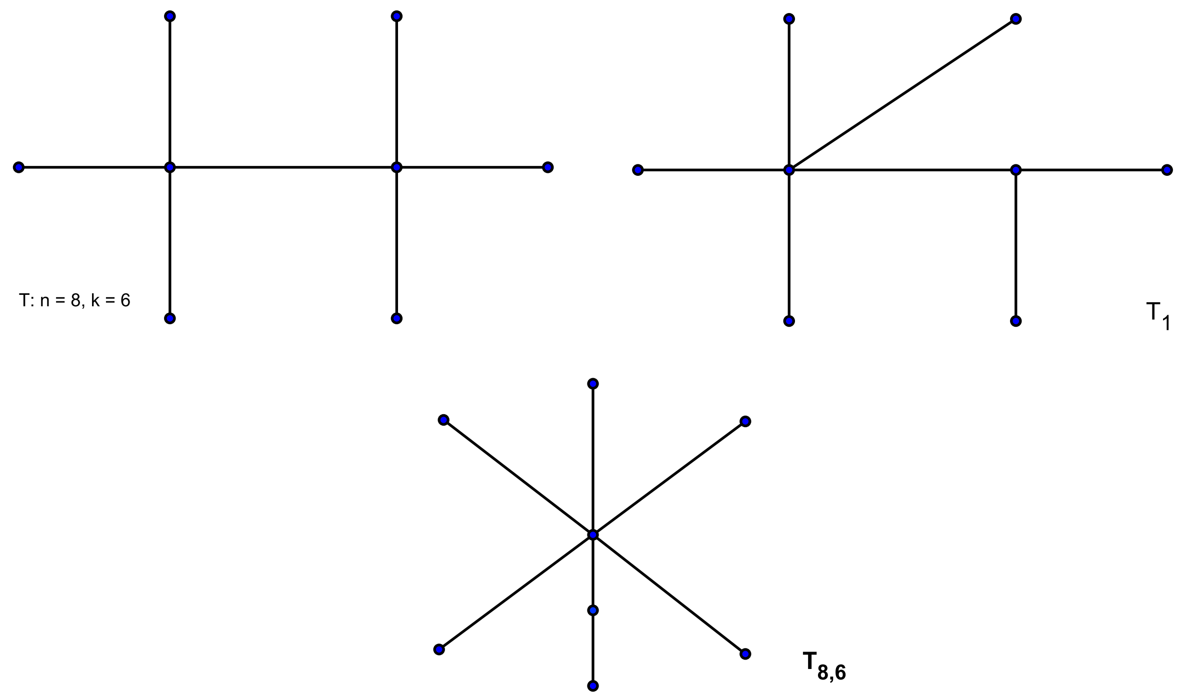

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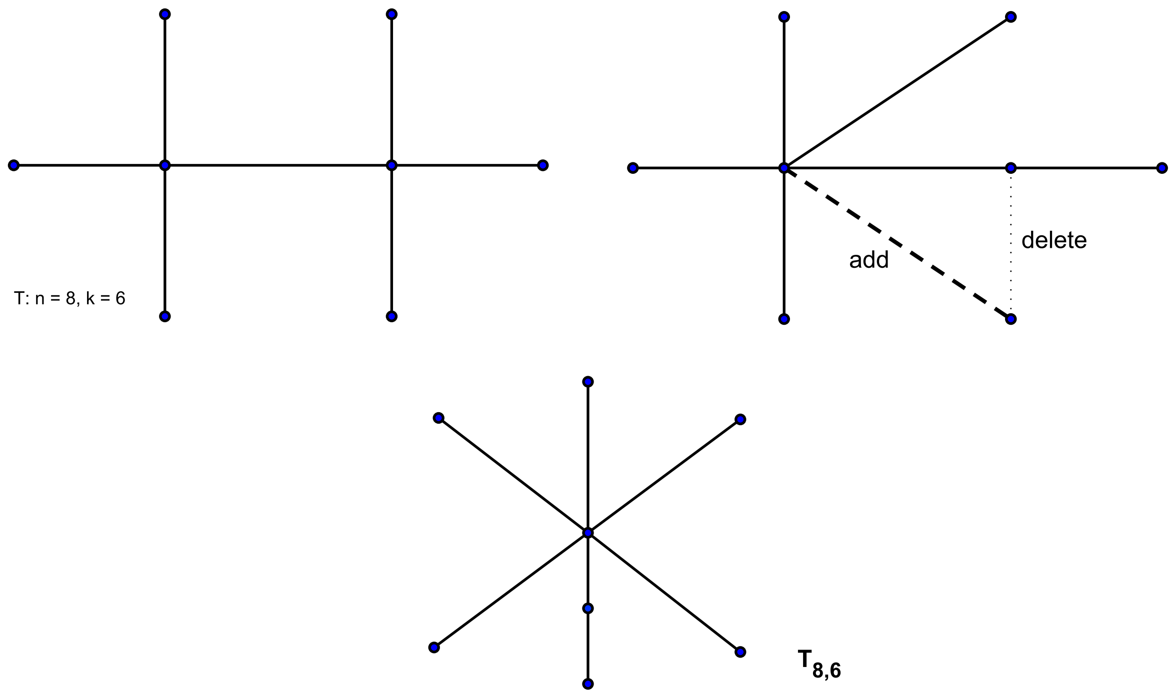

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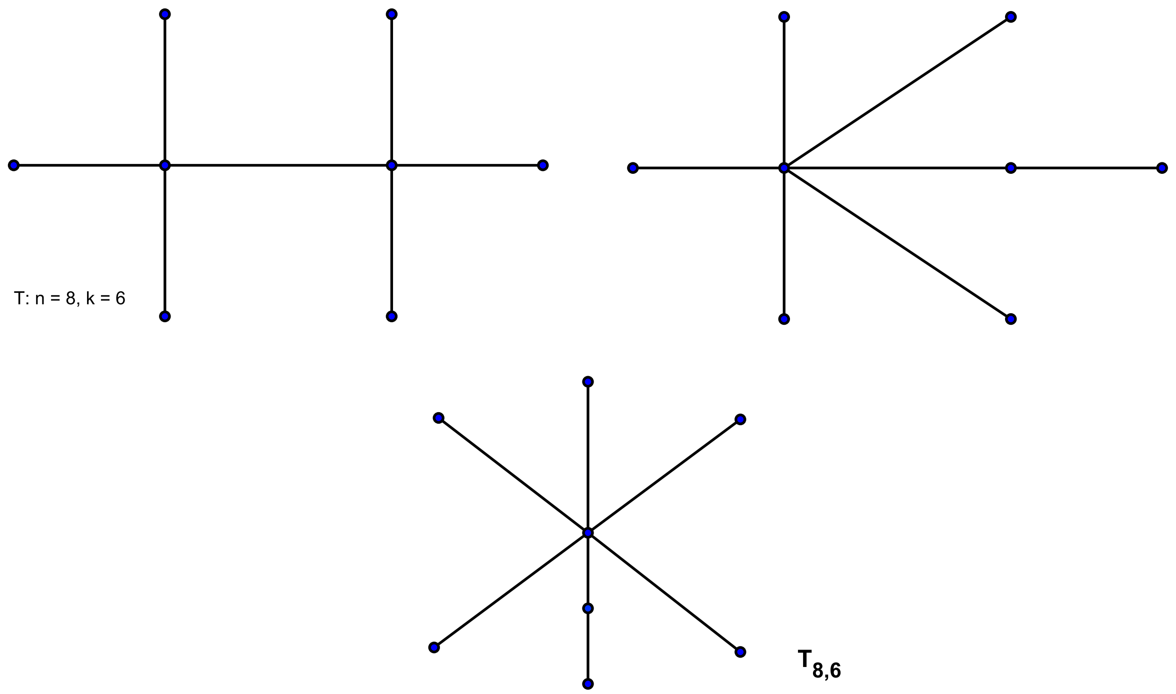

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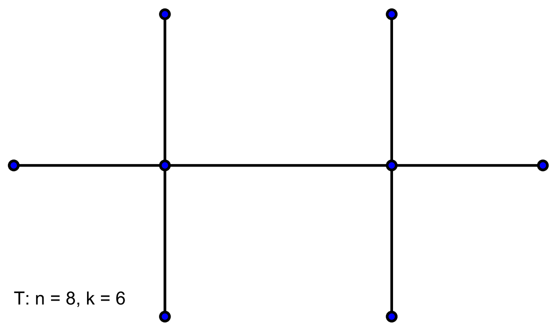
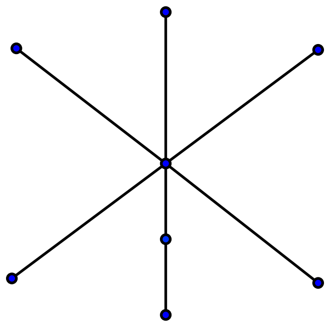
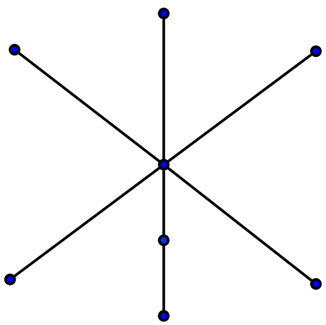
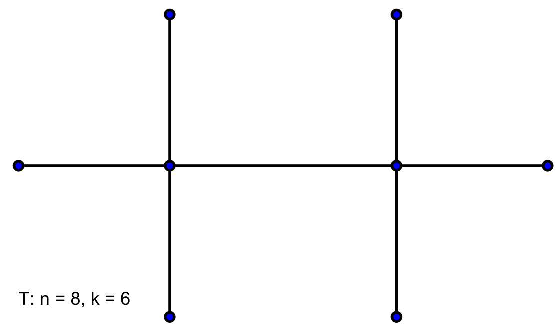


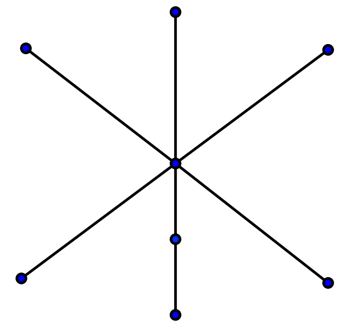


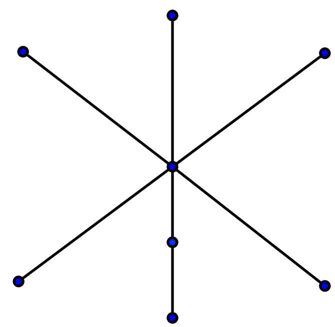


☐ Then, we just reconstructed T to $T_{n,k}$.

For example, take $T \in \mathscr{T}_{8,6}$ as..







$T_{8,6}$

☐ Now, their Laplacian spectral radii are...

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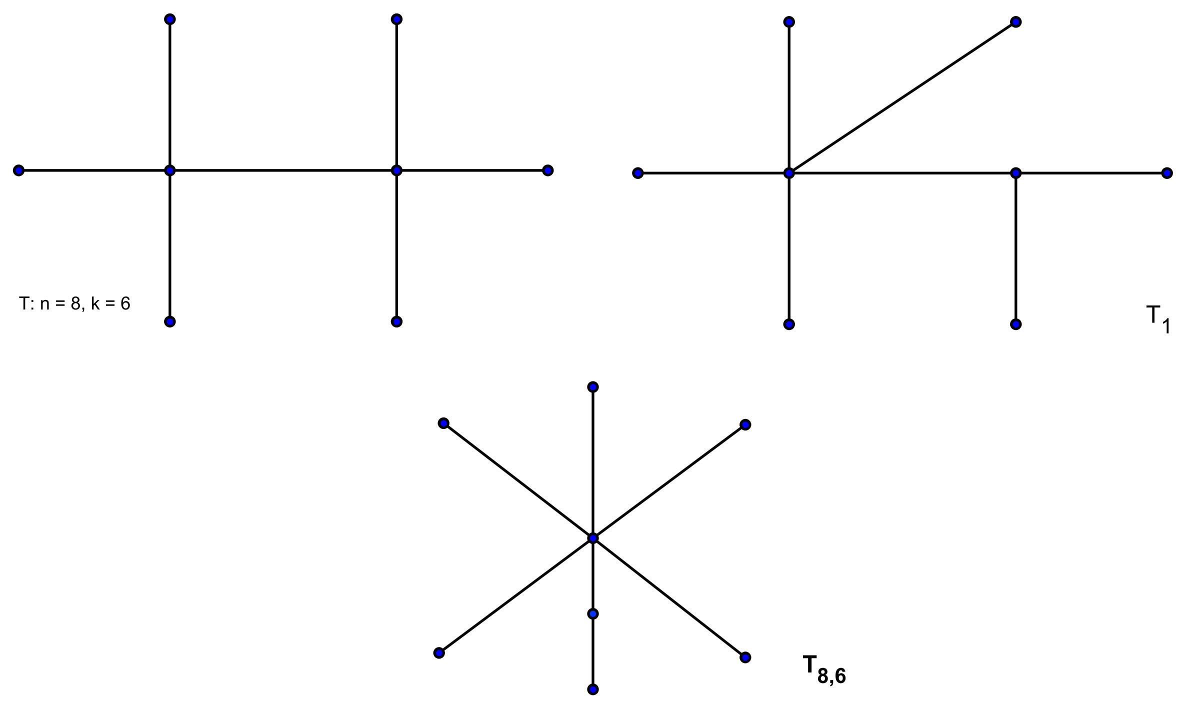

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☐ $\mu(T) = 5.6458$, $\mu(T_1) = 6.1413$, and $\mu(T_{8,6}) = 7.0340$, as desired.

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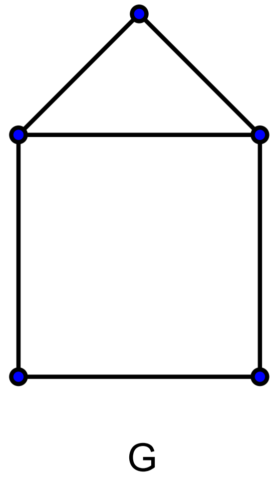

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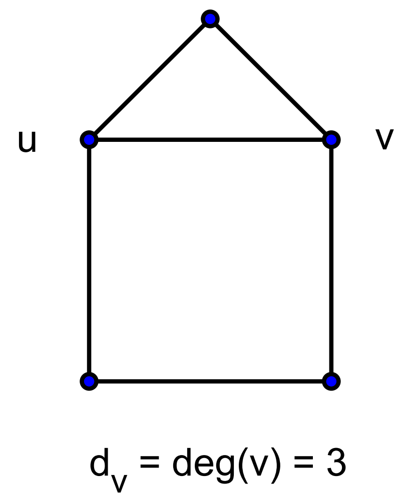

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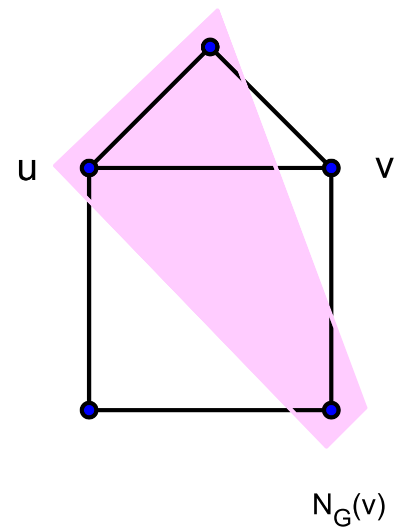

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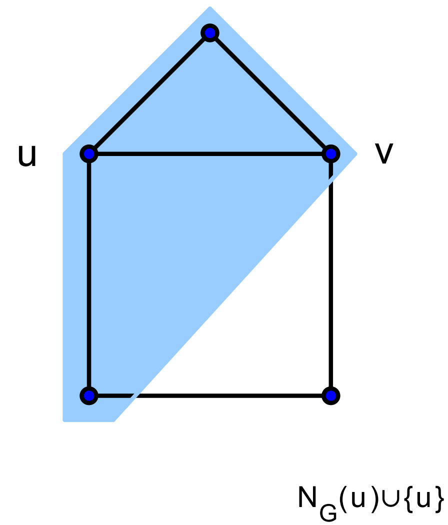

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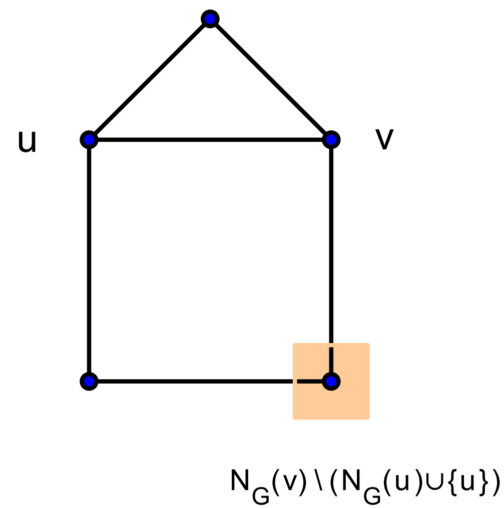

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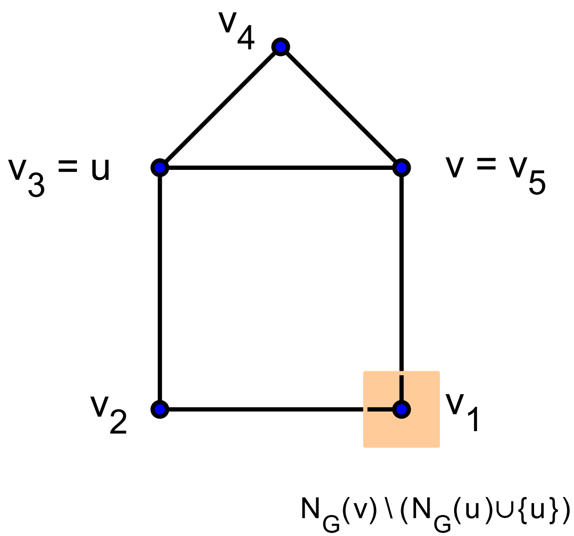

□ v_1 is only the vertex which belongs to $N_G(v)\backslash(N_G(u) \cup \{u\})$

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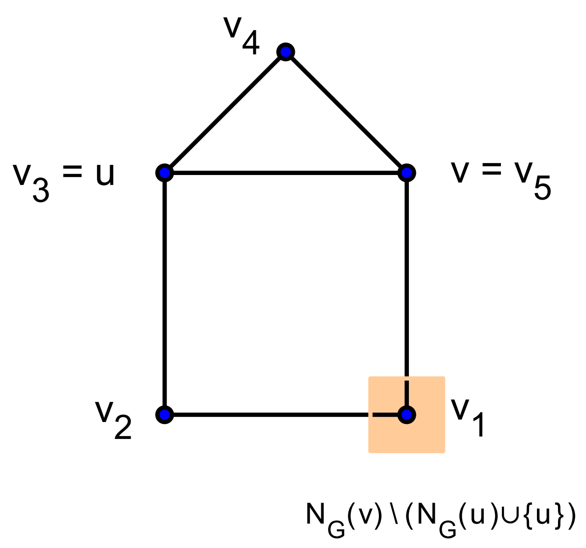

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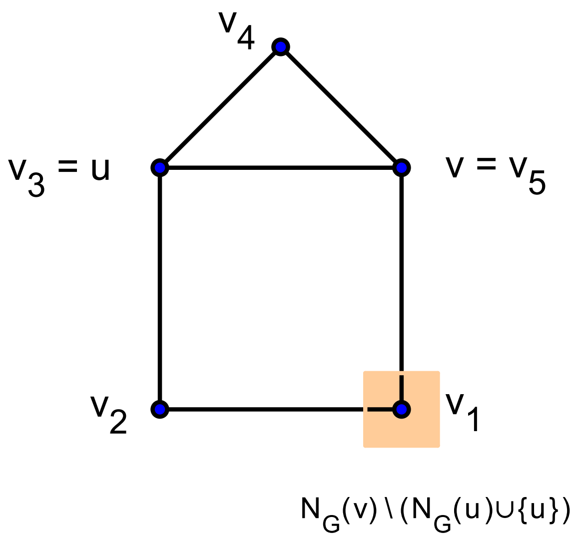
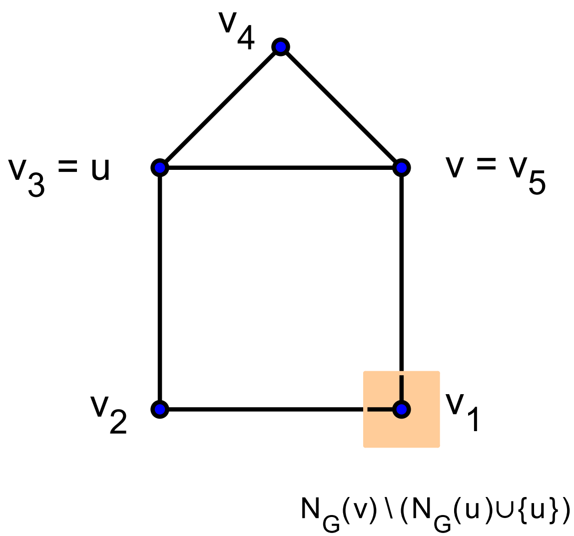


☐ The signless Laplacian matrix of G

$Q(G)$	v_1	v_2	v_3	v_4	v_5
v_1	2	1	0	0	1
v_2	1	2	1	0	0
v_3	0	1	3	1	1
v_4	0	0	1	2	1
v_5	1	0	1	1	3

Theorem. *Let u, v be two vertices of G and d_v be the degree of vertex v. Suppose $v_1, v_2, \ldots, v_s$ $(1 \leq s \leq d_v)$ are some vertices of $N_G(v)\backslash(N_G(u) \cup \{u\})$ and $x = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector of $D(G) + A(G)$, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges (v, v_i) and adding the edges (u, v_i) $(1 \leq i \leq s)$. If $x_u \geq x_v$, then $\nu(G) < \nu(G^*)$.*

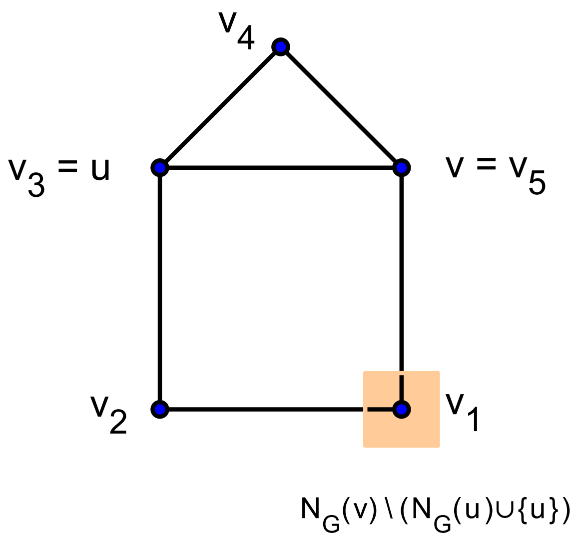


☐ Its spectrum is...

$$\{0.382, 1.139, 2.618, 2.745, \underline{5.114}\}$$

Theorem. *Let u, v be two vertices of G and d_v be the degree of vertex v. Suppose $v_1, v_2, \ldots, v_s$ $(1 \leq s \leq d_v)$ are some vertices of $N_G(v) \backslash (N_G(u) \cup \{u\})$ and $x = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector of $D(G) + A(G)$, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges (v, v_i) and adding the edges (u, v_i) $(1 \leq i \leq s)$. If $x_u \geq x_v$, then $\nu(G) < \nu(G^*)$.*

Theorem 1

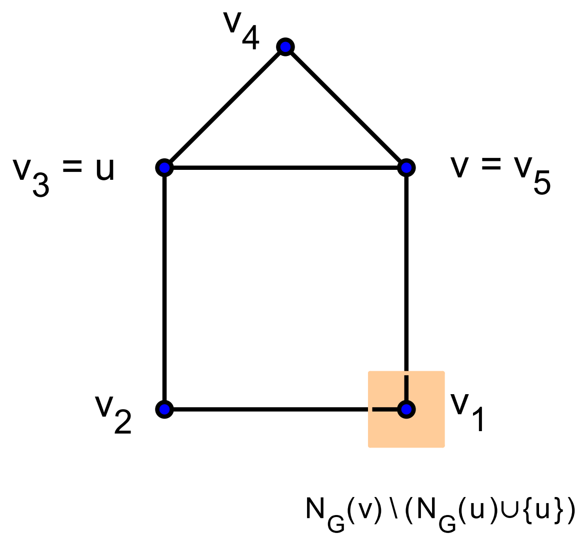


□ Its Perron vector is...

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ 0.2796 & 0.2796 & 0.5914 & 0.3797 & 0.5914 \end{bmatrix}^T$$

Theorem. *Let u, v be two vertices of G and d_v be the degree of vertex v. Suppose $v_1, v_2, \ldots, v_s$ $(1 \le s \le d_v)$ are some vertices of $N_G(v) \backslash (N_G(u) \cup \{u\})$ and $x = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector of $D(G) + A(G)$, where x_i corresponds to the vertex v_i $(1 \le i \le n)$. Let G^* be the graph obtained from G by deleting the edges (v, v_i) and adding the edges (u, v_i) $(1 \le i \le s)$. If $x_u \ge x_v$, then $\nu(G) < \nu(G^*)$.*

Theorem 1



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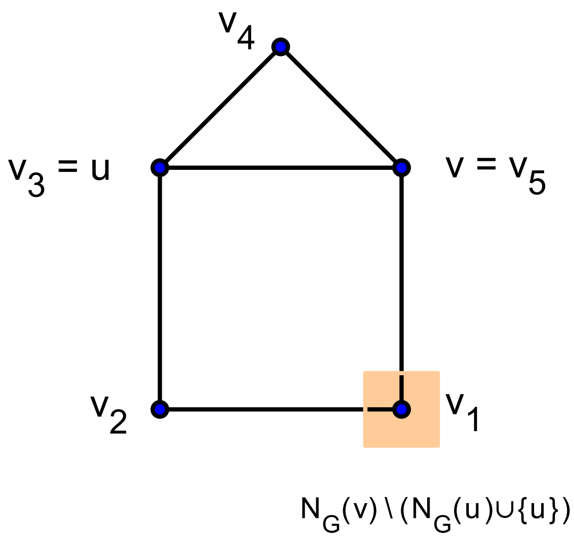
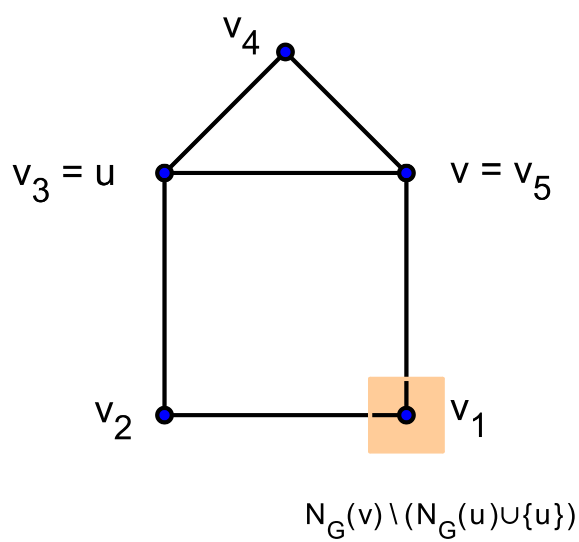
Theorem 1



☐ Its Perron vector is...

$$\mathbf{x} = \begin{bmatrix} \overset{v_1}{\updownarrow} & \overset{v_2}{\updownarrow} & \overset{v_3}{\updownarrow} & \overset{v_4}{\updownarrow} & \overset{v_5}{\updownarrow} \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ 0.2796 & 0.2796 & 0.5914 & 0.3797 & 0.5914 \end{bmatrix}^T$$

Theorem. *Let* u, v *be two vertices of* G *and* d_v *be the degree of vertex* v. *Suppose* $v_1, v_2, \ldots, v_s$ $(1 \leq s \leq d_v)$ *are some vertices of* $N_G(v)\backslash(N_G(u) \cup \{u\})$ *and* $x = (x_1, x_2, \ldots, x_n)^T$ *is the Perron vector of* $D(G) + A(G)$, *where* x_i *corresponds to the vertex* v_i $(1 \leq i \leq n)$. *Let* G^* *be the graph obtained from* G *by deleting the edges* (v, v_i) *and adding the edges* (u, v_i) $(1 \leq i \leq s)$. *If* $x_u \geq x_v$, *then* $\nu(G) < \nu(G^*)$.



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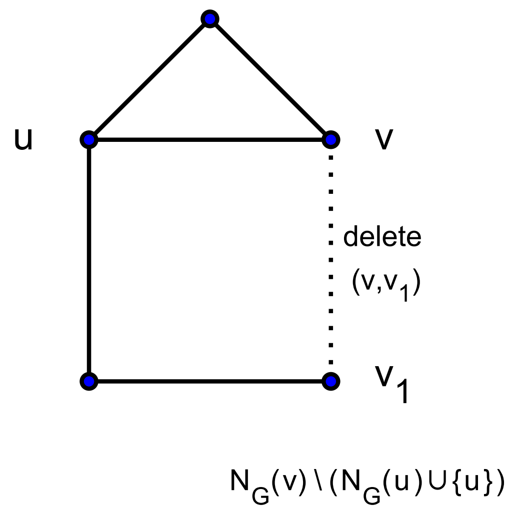

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$$\mathbf{x} = \begin{bmatrix} \overset{v_1}{\updownarrow} & \overset{v_2}{\updownarrow} & \overset{v_3}{\updownarrow} & \overset{v_4}{\updownarrow} & \overset{v_5}{\updownarrow} \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ 0.2796 & 0.2796 & 0.5914 & 0.3797 & 0.5914 \end{bmatrix}^T$$

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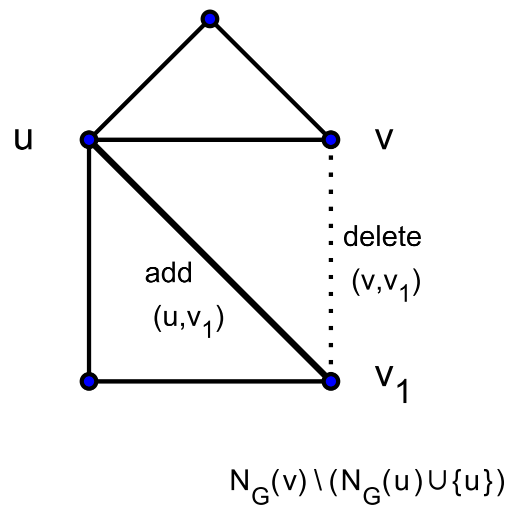

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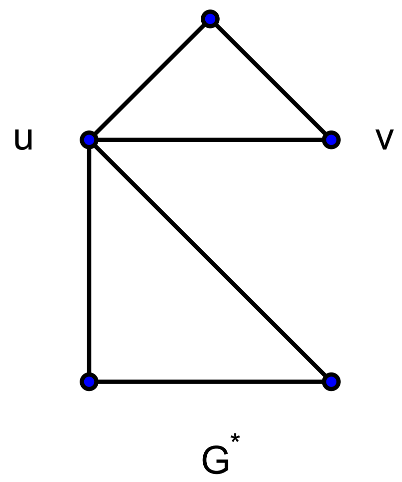

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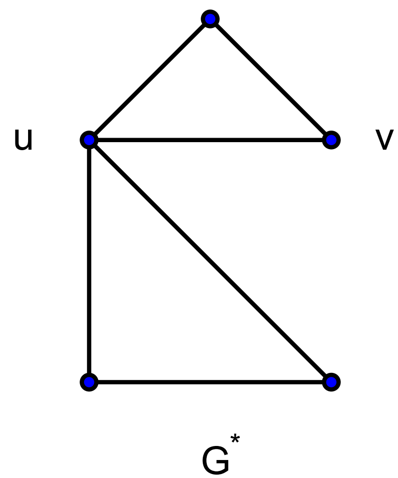

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$$\mathbf{x} = \begin{bmatrix} \overset{v_1}{\updownarrow} & \overset{v_2}{\updownarrow} & \overset{v_3}{\updownarrow} & \overset{v_4}{\updownarrow} & \overset{v_5}{\updownarrow} \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ 0.2796 & 0.2796 & 0.5914 & 0.3797 & 0.5914 \end{bmatrix}^T$$

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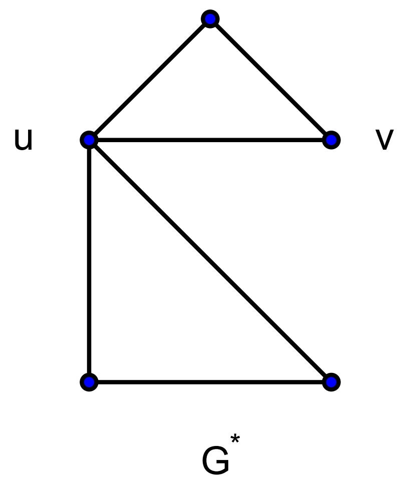
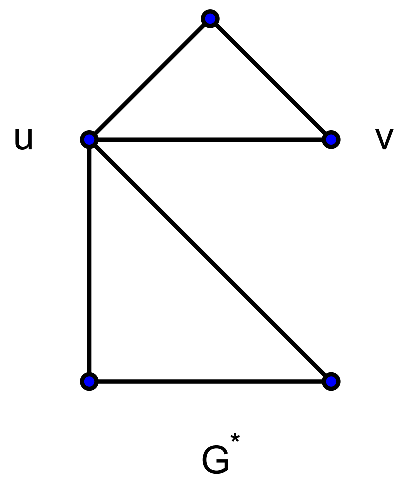


☐ We see that

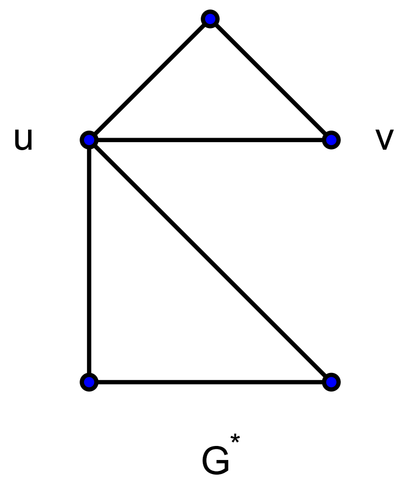
Theorem. *Let u, v be two vertices of G and d_v be the degree of vertex v. Suppose $v_1, v_2, \ldots, v_s$ $(1 \leq s \leq d_v)$ are some vertices of $N_G(v) \backslash (N_G(u) \cup \{u\})$ and $x = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector of $D(G) + A(G)$, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges (v, v_i) and adding the edges (u, v_i) $(1 \leq i \leq s)$. If $x_u \geq x_v$, then $\nu(G) < \nu(G^*)$.*

Theorem 1



□ We see that
$x_u = 0.5914 \geq 0.5914 = x_v$,

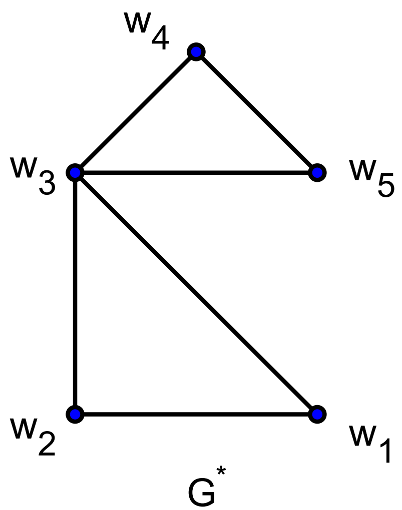
Theorem. *Let u, v be two vertices of G and d_v be the degree of vertex v. Suppose $v_1, v_2, \ldots, v_s$ $(1 \leq s \leq d_v)$ are some vertices of $N_G(v) \backslash (N_G(u) \cup \{u\})$ and $x = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector of $D(G) + A(G)$, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges (v, v_i) and adding the edges (u, v_i) $(1 \leq i \leq s)$. If $x_u \geq x_v$, then $\nu(G) < \nu(G^*)$.*



□ We see that $x_u = 0.5914 \geq 0.5914 = x_v$, satisfying the assumption.

Theorem. *Let u, v be two vertices of G and d_v be the degree of vertex v. Suppose $v_1, v_2, \ldots, v_s$ $(1 \leq s \leq d_v)$ are some vertices of $N_G(v) \backslash (N_G(u) \cup \{u\})$ and $x = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector of $D(G) + A(G)$, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges (v, v_i) and adding the edges (u, v_i) $(1 \leq i \leq s)$. If $x_u \geq x_v$, then $\nu(G) < \nu(G^*)$.*

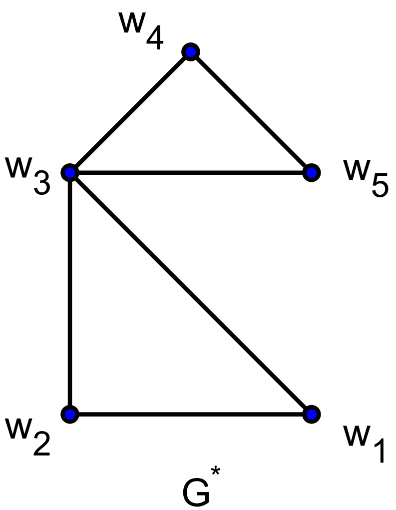
Theorem 1



☐ $Q(G^*)$ is...

$Q(G^*)$	w_1	w_2	w_3	w_4	w_5
w_1	2	1	1	0	0
w_2	1	2	1	0	0
w_3	1	1	4	1	1
w_4	0	0	1	2	1
w_5	0	0	1	1	2

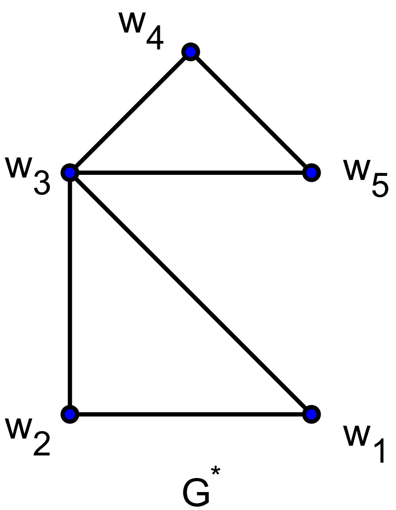
Theorem. *Let u, v be two vertices of G and d_v be the degree of vertex v. Suppose $v_1, v_2, \ldots, v_s$ $(1 \leq s \leq d_v)$ are some vertices of $N_G(v) \backslash (N_G(u) \cup \{u\})$ and $x = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector of $D(G) + A(G)$, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges (v, v_i) and adding the edges (u, v_i) $(1 \leq i \leq s)$. If $x_u \geq x_v$, then $\nu(G) < \nu(G^*)$.*



□ Its spectrum is...

$$\{1.000, 1.000, 1.438, 3.000, \underline{5.561}\}$$

Theorem. *Let u, v be two vertices of G and d_v be the degree of vertex v. Suppose $v_1, v_2, \ldots, v_s$ $(1 \leq s \leq d_v)$ are some vertices of $N_G(v) \backslash (N_G(u) \cup \{u\})$ and $x = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector of $D(G) + A(G)$, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges (v, v_i) and adding the edges (u, v_i) $(1 \leq i \leq s)$. If $x_u \geq x_v$, then $\nu(G) < \nu(G^*)$.*



Therefore, we have

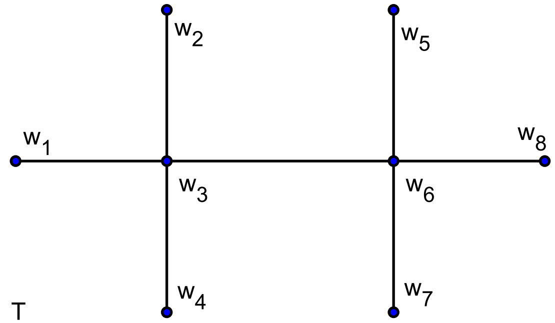
$$\nu(G) = 5.114 < 5.561 = \nu(G^*)$$

Theorem. *Let u, v be two vertices of G and d_v be the degree of vertex v. Suppose $v_1, v_2, \ldots, v_s$ $(1 \leq s \leq d_v)$ are some vertices of $N_G(v) \backslash (N_G(u) \cup \{u\})$ and $x = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector of $D(G) + A(G)$, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges (v, v_i) and adding the edges (u, v_i) $(1 \leq i \leq s)$. If $x_u \geq x_v$, then $\nu(G) < \nu(G^*)$.*

Example

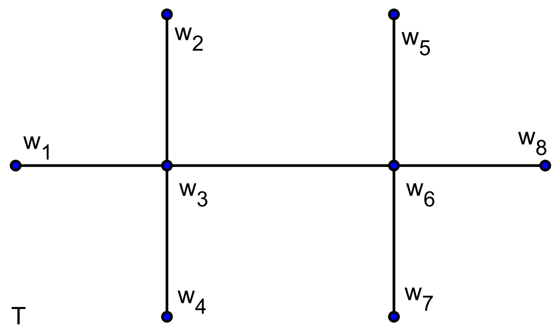
- ☐ Take the following figure with labels.

Example



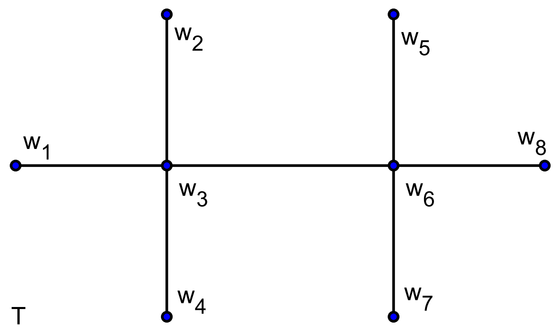
☐ Take the following figure with labels.

Example



- ☐ Take the following figure with labels.
- ☐ Its signless Laplacian matrix is...

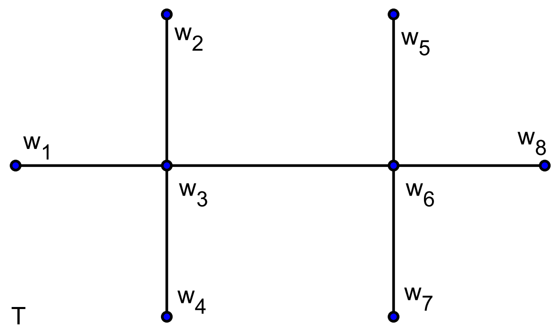
Example



$Q(T)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	4	1	0	1	0	0
w_4	0	0	1	1	0	0	0	0
w_5	0	0	0	0	1	1	0	0
w_6	0	0	1	0	1	4	1	1
w_7	0	0	0	0	0	1	1	0
w_8	0	0	0	0	0	1	0	1

- ☐ Take the following figure with labels.
- ☐ Its signless Laplacian matrix is...

Example

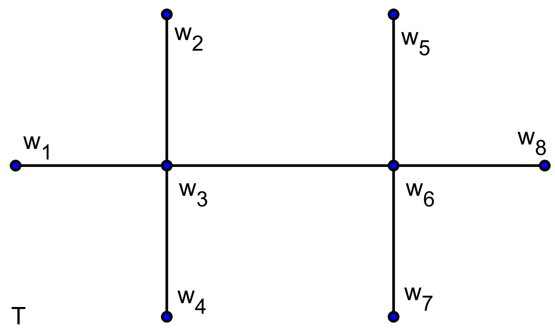


$Q(T)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	4	1	0	1	0	0
w_4	0	0	1	1	0	0	0	0
w_5	0	0	0	0	1	1	0	0
w_6	0	0	1	0	1	4	1	1
w_7	0	0	0	0	0	1	1	0
w_8	0	0	0	0	0	1	0	1

□ Then, its spectrum is...

Example

Introduction
Some Lemmas
Main Results
Idea
Theorem 1
▷ Example
Theorem 2
Case 1
Case 2
Case 3
Example
The End

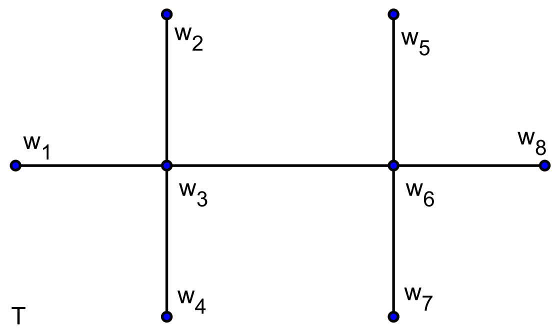


$Q(T)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	4	1	0	1	0	0
w_4	0	0	1	1	0	0	0	0
w_5	0	0	0	0	1	1	0	0
w_6	0	0	1	0	1	4	1	1
w_7	0	0	0	0	0	1	1	0
w_8	0	0	0	0	0	1	0	1

- Then, its spectrum is...

$$\{0.000, 0.354, 1.000, 1.000, 1.000, 1.000, 4.000, \underline{5.646}\}$$

Example



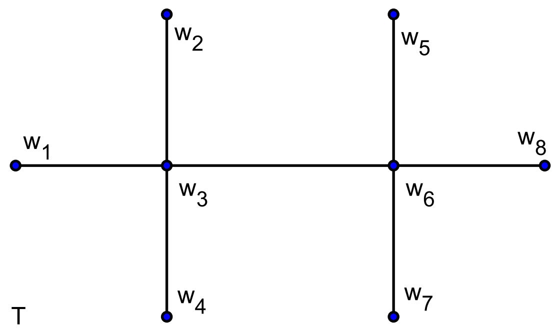
$Q(T)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	4	1	0	1	0	0
w_4	0	0	1	1	0	0	0	0
w_5	0	0	0	0	1	1	0	0
w_6	0	0	1	0	1	4	1	1
w_7	0	0	0	0	0	1	1	0
w_8	0	0	0	0	0	1	0	1

☐ Then, its spectrum is...

$$\{0.000, 0.354, 1.000, 1.000, 1.000, 1.000, 4.000, \underline{5.646}\}$$

so, $\nu(T) = 5.646.$

Example



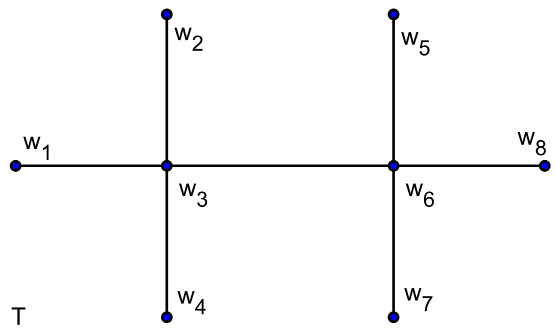
$Q(T)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	4	1	0	1	0	0
w_4	0	0	1	1	0	0	0	0
w_5	0	0	0	0	1	1	0	0
w_6	0	0	1	0	1	4	1	1
w_7	0	0	0	0	0	1	1	0
w_8	0	0	0	0	0	1	0	1

☐ Then, its spectrum is...

$$\{0.000, 0.354, 1.000, 1.000, 1.000, 1.000, 4.000, \underline{5.646}\}$$

so, $\nu(T) = 5.646$. Its associated Perron vector is...

Example



$Q(T)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	4	1	0	1	0	0
w_4	0	0	1	1	0	0	0	0
w_5	0	0	0	0	1	1	0	0
w_6	0	0	1	0	1	4	1	1
w_7	0	0	0	0	0	1	1	0
w_8	0	0	0	0	0	1	0	1

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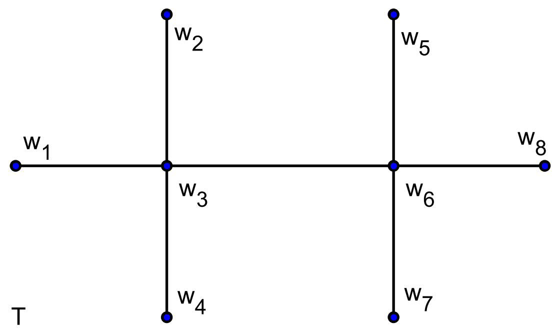
$$\{0.000, 0.354, 1.000, 1.000, 1.000, 1.000, 4.000, \underline{5.646}\}$$

so, $\nu(T) = 5.646$. Its associated Perron vector is...

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ 0.142 & 0.142 & 0.662 & 0.142 & 0.142 & 0.662 & 0.142 & 0.142 \end{bmatrix}^T$$

Example



$Q(T)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	4	1	0	1	0	0
w_4	0	0	1	1	0	0	0	0
w_5	0	0	0	0	1	1	0	0
w_6	0	0	1	0	1	4	1	1
w_7	0	0	0	0	0	1	1	0
w_8	0	0	0	0	0	1	0	1

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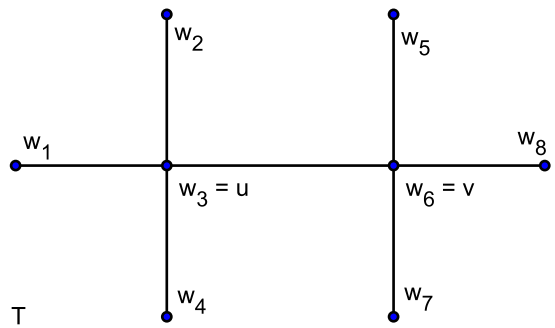
$$\{0.000, 0.354, 1.000, 1.000, 1.000, 1.000, 4.000, \underline{5.646}\}$$

so, $\nu(T) = 5.646$. Its associated Perron vector is...

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ 0.142 & 0.142 & 0.662 & 0.142 & 0.142 & 0.662 & 0.142 & 0.142 \end{bmatrix}^T$$

☐ For this example, let us choose $w_3 = u$ and $w_6 = v$, so that $x_u \geq x_v$ is preserved.



$Q(T)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	4	1	0	1	0	0
w_4	0	0	1	1	0	0	0	0
w_5	0	0	0	0	1	1	0	0
w_6	0	0	1	0	1	4	1	1
w_7	0	0	0	0	0	1	1	0
w_8	0	0	0	0	0	1	0	1

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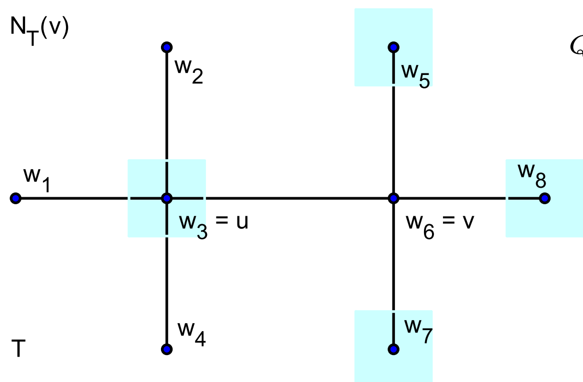
Example



$Q(T)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	4	1	0	1	0	0
w_4	0	0	1	1	0	0	0	0
w_5	0	0	0	0	1	1	0	0
w_6	0	0	1	0	1	4	1	1
w_7	0	0	0	0	0	1	1	0
w_8	0	0	0	0	0	1	0	1

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$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ 0.142 & 0.142 & 0.662 & 0.142 & 0.142 & 0.662 & 0.142 & 0.142 \end{bmatrix}^T$$

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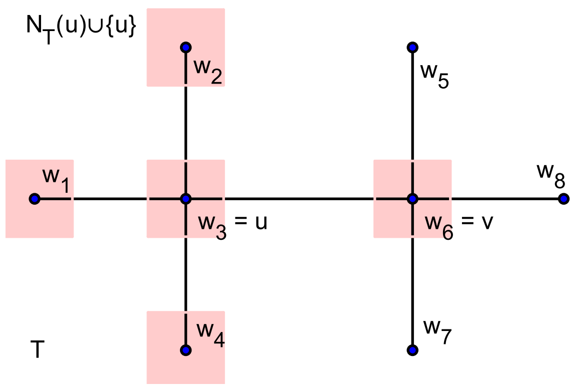
Example



$Q(T)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	4	1	0	1	0	0
w_4	0	0	1	1	0	0	0	0
w_5	0	0	0	0	1	1	0	0
w_6	0	0	1	0	1	4	1	1
w_7	0	0	0	0	0	1	1	0
w_8	0	0	0	0	0	1	0	1

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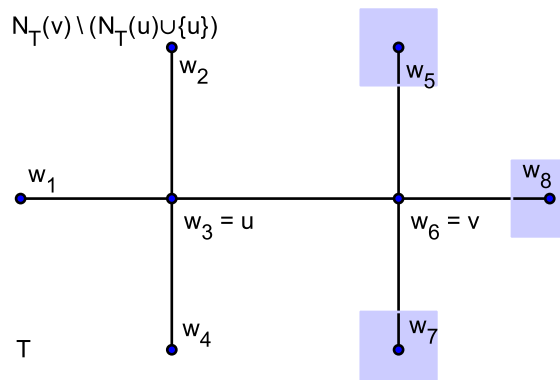

$Q(T)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	4	1	0	1	0	0
w_4	0	0	1	1	0	0	0	0
w_5	0	0	0	0	1	1	0	0
w_6	0	0	1	0	1	4	1	1
w_7	0	0	0	0	0	1	1	0
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$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ 0.142 & 0.142 & 0.662 & 0.142 & 0.142 & 0.662 & 0.142 & 0.142 \end{bmatrix}^T$$

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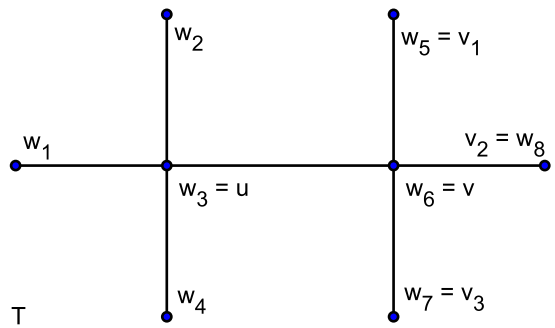
Example



$Q(T)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	4	1	0	1	0	0
w_4	0	0	1	1	0	0	0	0
w_5	0	0	0	0	1	1	0	0
w_6	0	0	1	0	1	4	1	1
w_7	0	0	0	0	0	1	1	0
w_8	0	0	0	0	0	1	0	1

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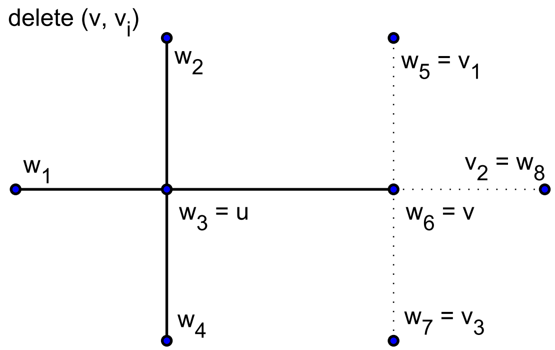
Example



$Q(T)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	4	1	0	1	0	0
w_4	0	0	1	1	0	0	0	0
w_5	0	0	0	0	1	1	0	0
w_6	0	0	1	0	1	4	1	1
w_7	0	0	0	0	0	1	1	0
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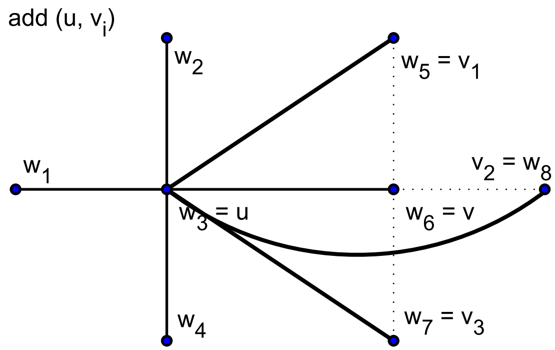

$Q(T)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	4	1	0	1	0	0
w_4	0	0	1	1	0	0	0	0
w_5	0	0	0	0	1	1	0	0
w_6	0	0	1	0	1	4	1	1
w_7	0	0	0	0	0	1	1	0
w_8	0	0	0	0	0	1	0	1

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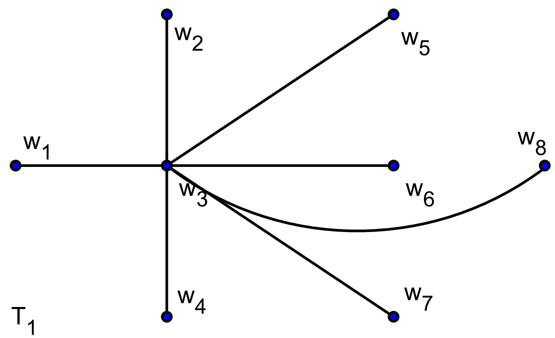
Example



$Q(T)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	4	1	0	1	0	0
w_4	0	0	1	1	0	0	0	0
w_5	0	0	0	0	1	1	0	0
w_6	0	0	1	0	1	4	1	1
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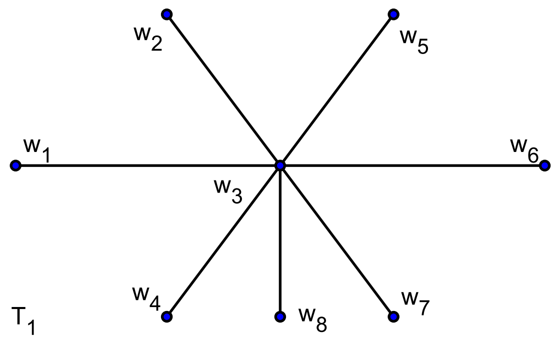
Example



$Q(T)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	4	1	0	1	0	0
w_4	0	0	1	1	0	0	0	0
w_5	0	0	0	0	1	1	0	0
w_6	0	0	1	0	1	4	1	1
w_7	0	0	0	0	0	1	1	0
w_8	0	0	0	0	0	1	0	1

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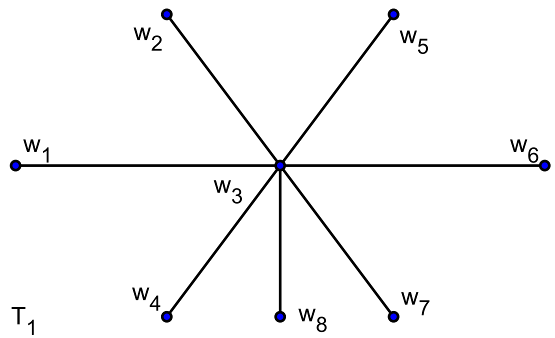
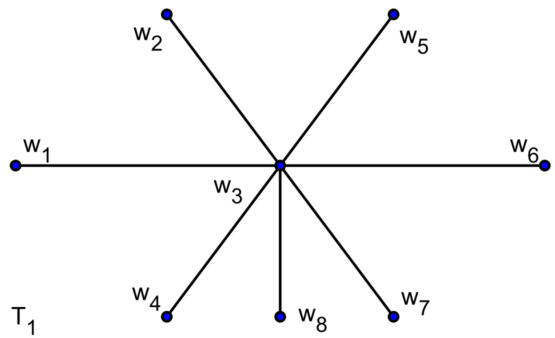
Example



$Q(T)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	4	1	0	1	0	0
w_4	0	0	1	1	0	0	0	0
w_5	0	0	0	0	1	1	0	0
w_6	0	0	1	0	1	4	1	1
w_7	0	0	0	0	0	1	1	0
w_8	0	0	0	0	0	1	0	1

☐ Now, the signless Laplacian Matrix for T_1 is...

Example

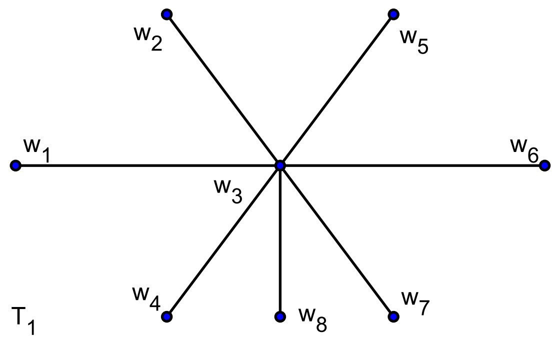


$Q(T_1)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	7	1	1	1	1	1
w_4	0	0	1	1	0	0	0	0
w_5	0	0	1	0	1	0	0	0
w_6	0	0	1	0	0	1	0	0
w_7	0	0	1	0	0	0	1	0
w_8	0	0	1	0	0	0	0	1

☐ Now, the signless Laplacian Matrix for T_1 is...

$$\{0.000, 1.000, 1.000, 1.000, 1.000, 1.000, 1.000, \underline{8.000}\}$$

Example



$Q(T_1)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	7	1	1	1	1	1
w_4	0	0	1	1	0	0	0	0
w_5	0	0	1	0	1	0	0	0
w_6	0	0	1	0	0	1	0	0
w_7	0	0	1	0	0	0	1	0
w_8	0	0	1	0	0	0	0	1

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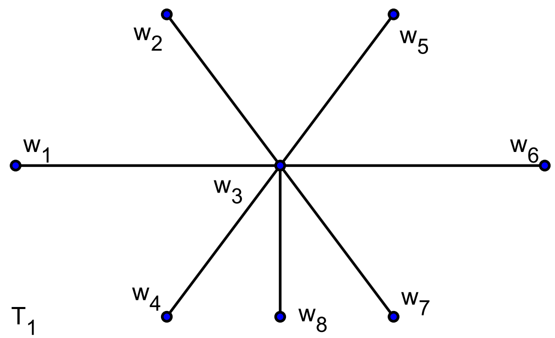
so, $\nu(T_1) = 8.000$, which we see that

Example



$Q(T_1)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	7	1	1	1	1	1
w_4	0	0	1	1	0	0	0	0
w_5	0	0	1	0	1	0	0	0
w_6	0	0	1	0	0	1	0	0
w_7	0	0	1	0	0	0	1	0
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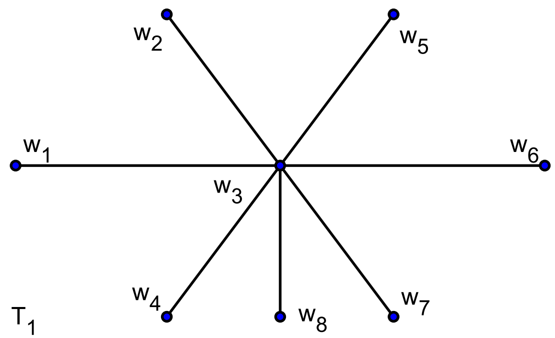
$$\nu(T) = 5.646 < 8.000 = \nu(T_1)$$

Example



$Q(T_1)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0	1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	0
w_3	1	1	7	1	1	1	1	1
w_4	0	0	1	1	0	0	0	0
w_5	0	0	1	0	1	0	0	0
w_6	0	0	1	0	0	1	0	0
w_7	0	0	1	0	0	0	1	0
w_8	0	0	1	0	0	0	0	1

☐ Now, the signless Laplacian Matrix for T_1 is...

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so, $\nu(T_1) = 8.000$, which we see that

$$\nu(T) = 5.646 < 8.000 = \nu(T_1)$$

implied by the theorem!

Theorem. *Let T be a tree with n vertices and k pendant vertices. Then*

$$\mu(T) \leq \mu(T_{n,k}),$$

where equality holds if and only if T is isomorphic to $T_{n,k}$.

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Combine all lemmas and the theorem, now how does this statement hold?

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Idea:

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Combine all lemmas and the theorem, now how does this statement hold?

Idea: Let t be the number of vertices whose degree is greater than 2.

Theorem. *Let T be a tree with n vertices and k pendant vertices. Then*

$$\mu(T) \leq \mu(T_{n,k}),$$

where equality holds if and only if T is isomorphic to $T_{n,k}$.

Combine all lemmas and the theorem, now how does this statement hold?

Idea: Let t be the number of vertices whose degree is greater than 2. We prove the statement for $t = 0$, $t = 1$, and $t > 1$.

Case 1

Theorem. *Let T be a tree with n vertices and k pendant vertices. Then $\mu(T) \le \mu(T_{n,k})$, where equality holds if and only if T is isomorphic to $T_{n,k}$.*

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Case 1

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□ t is the number of vertices whose degree is greater than 2.

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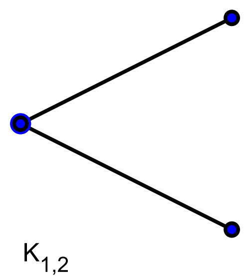

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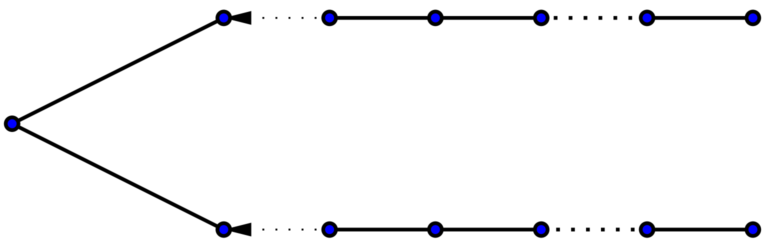

Attaching two paths to $K_{1,2}$

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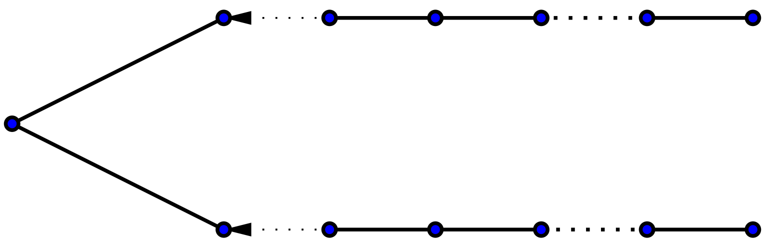

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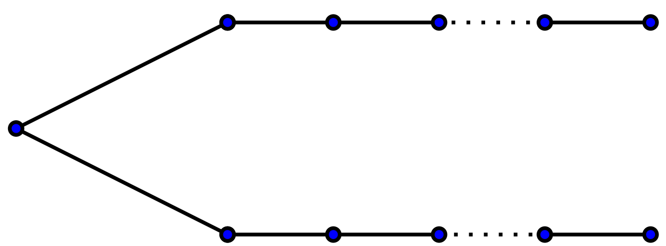

$T_{n,2} \cong P_n$

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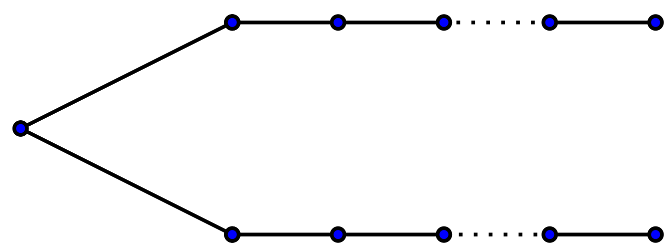

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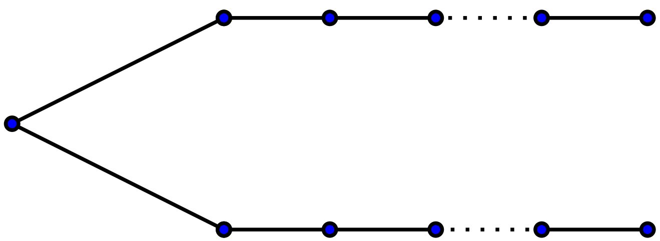

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CASE1 $\surd$

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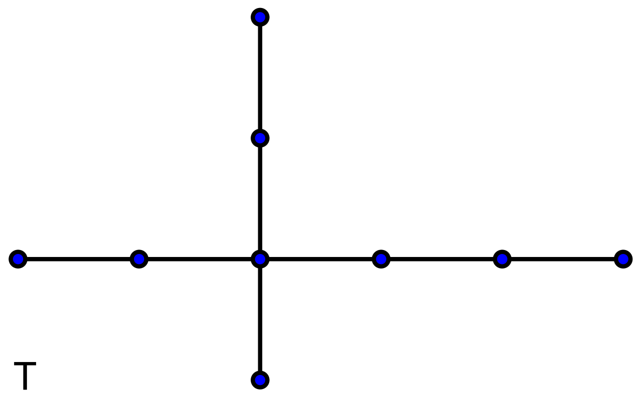

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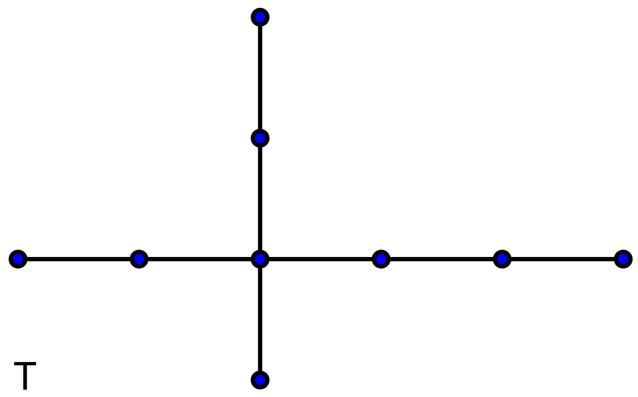

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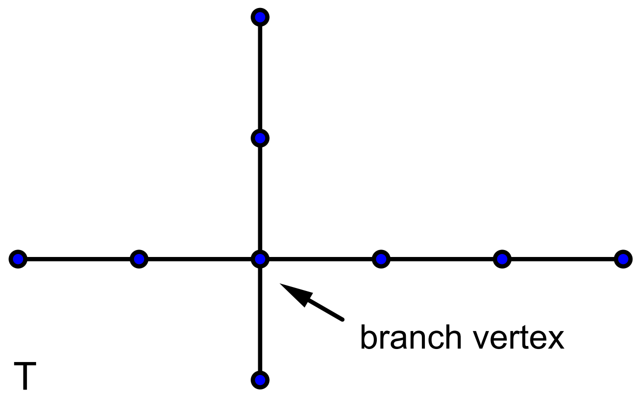

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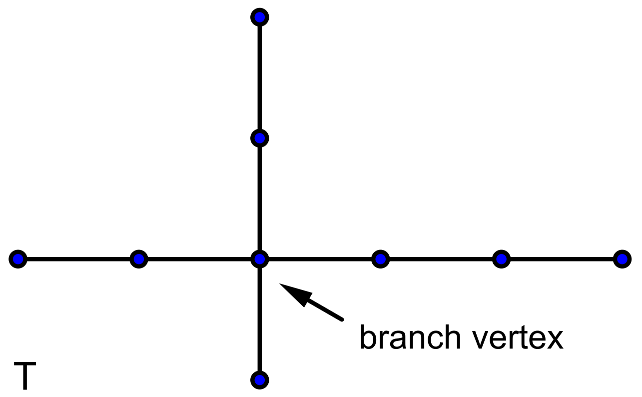

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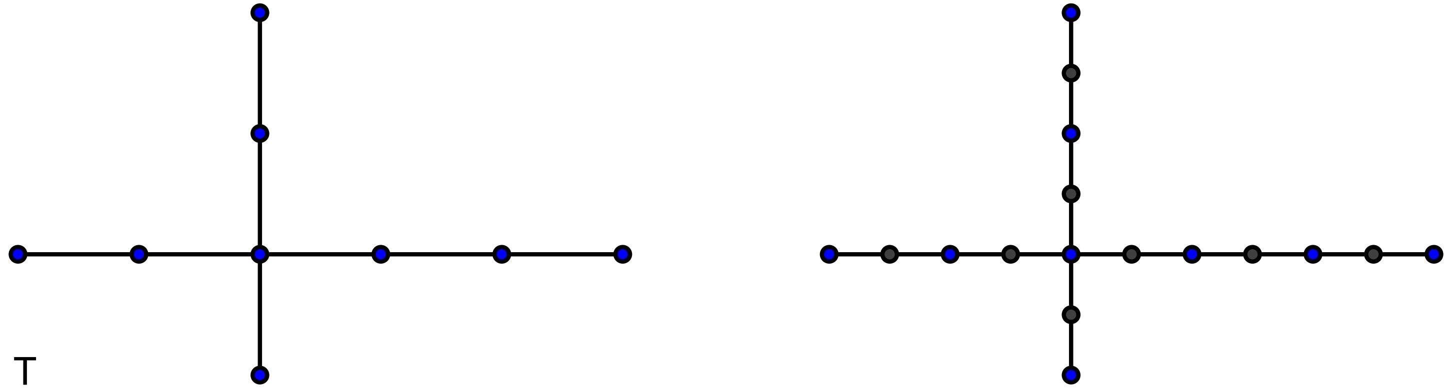

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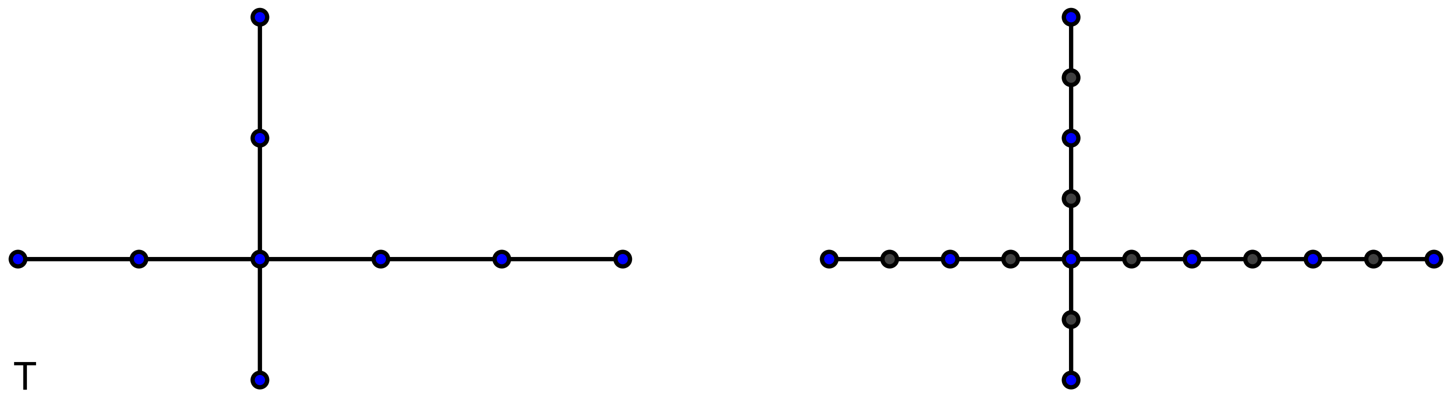

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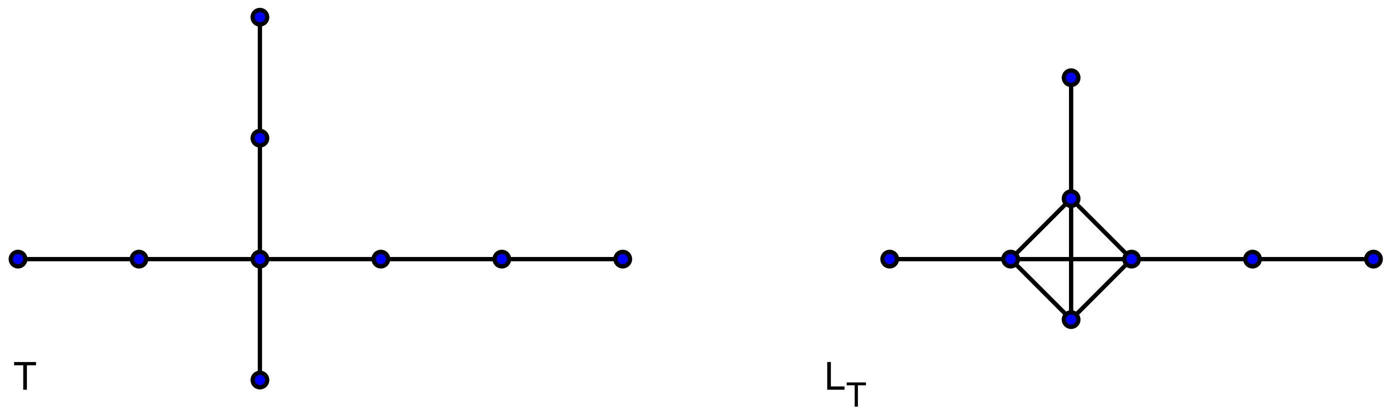

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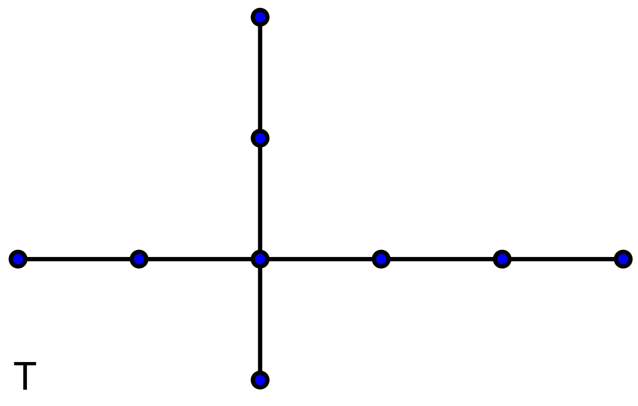

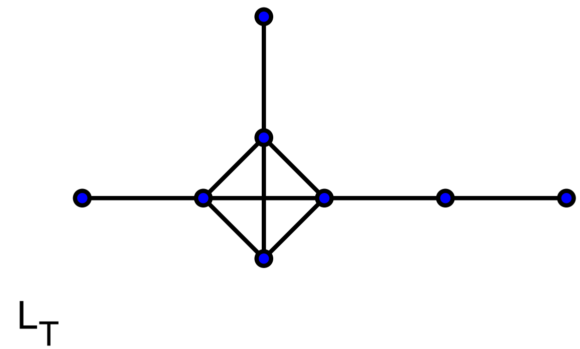

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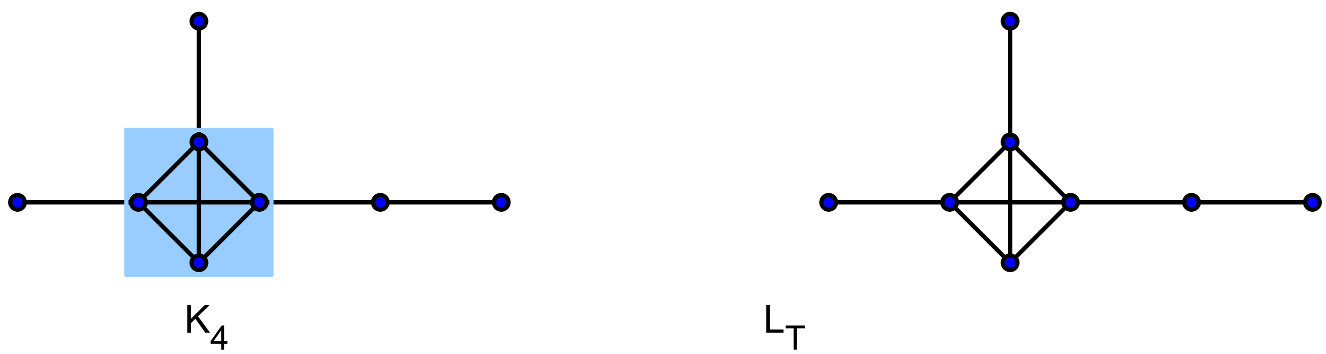

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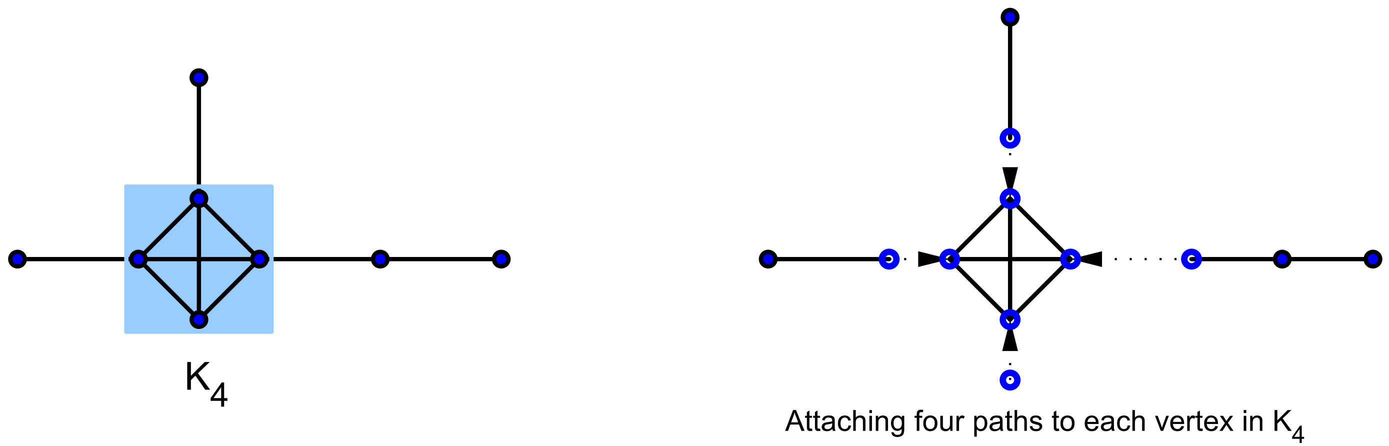

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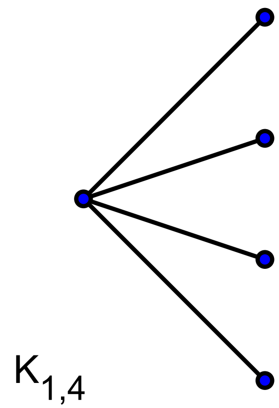

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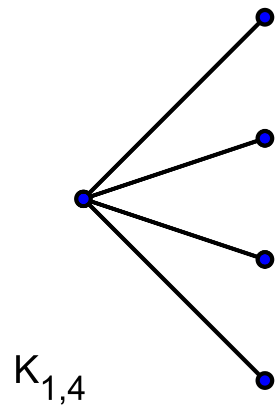

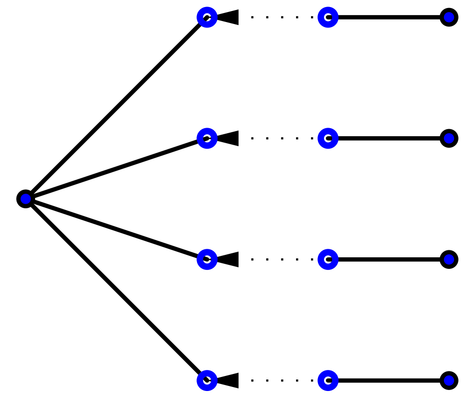

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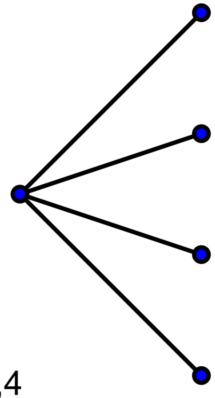
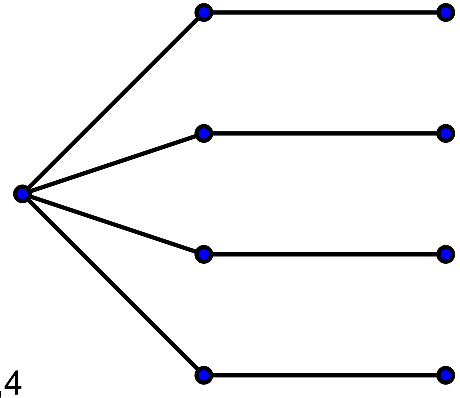


$K_{1,4}$



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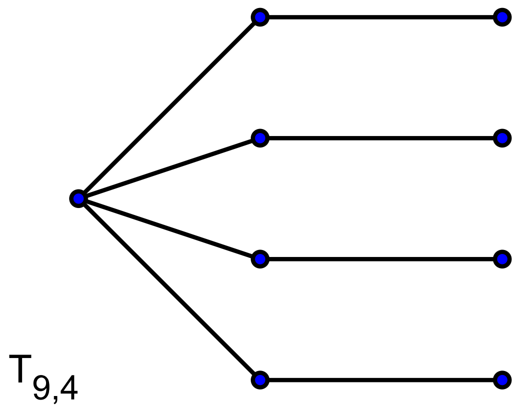

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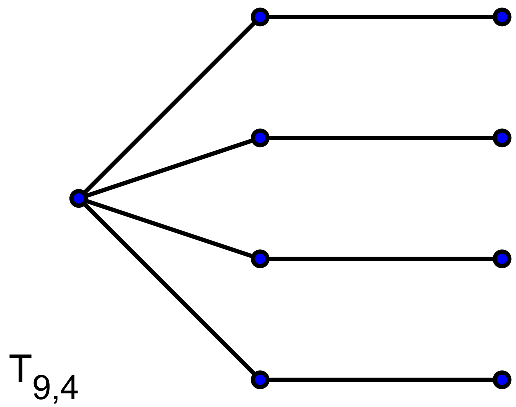

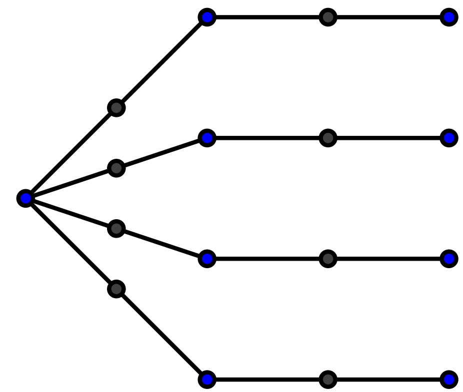

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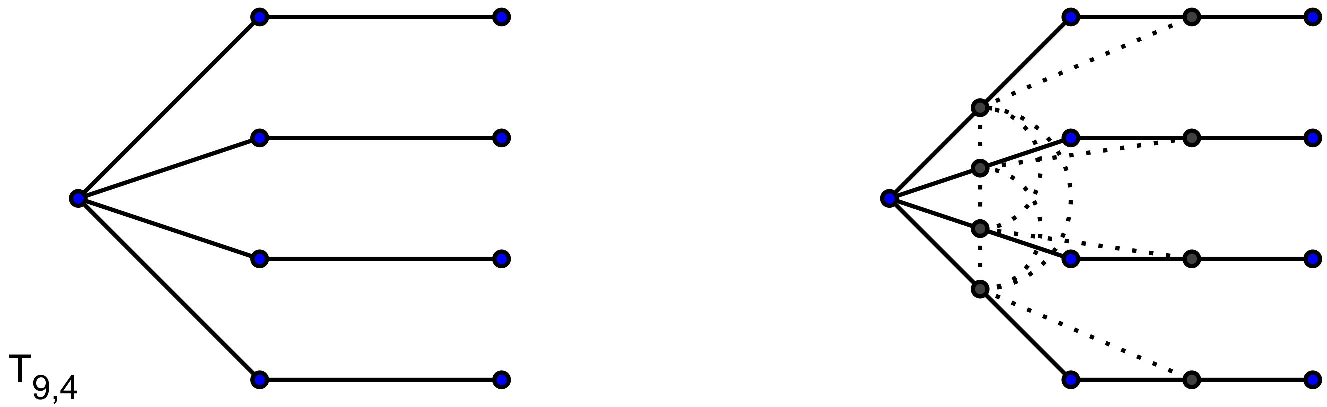

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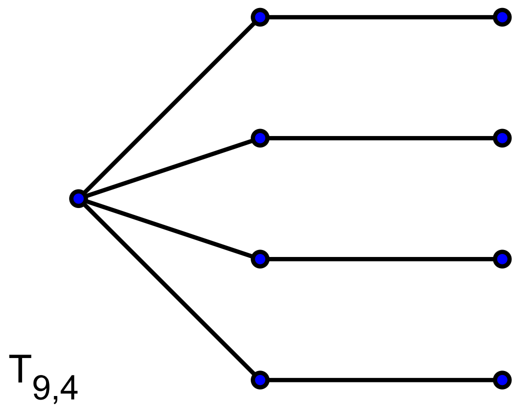

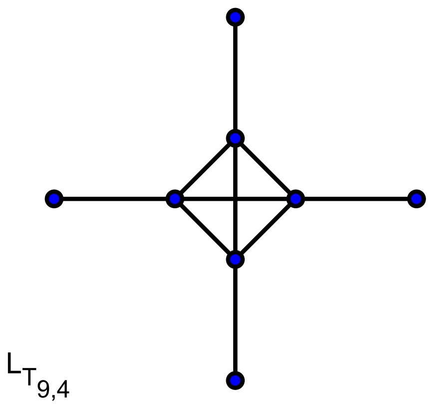

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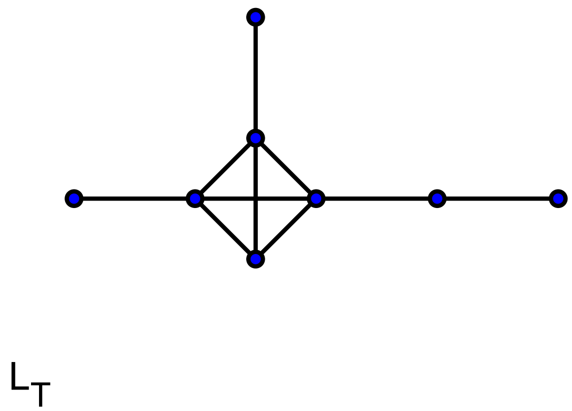

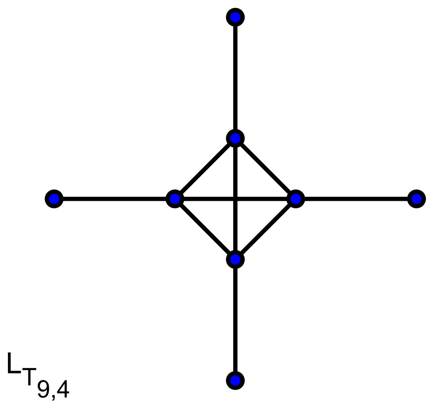

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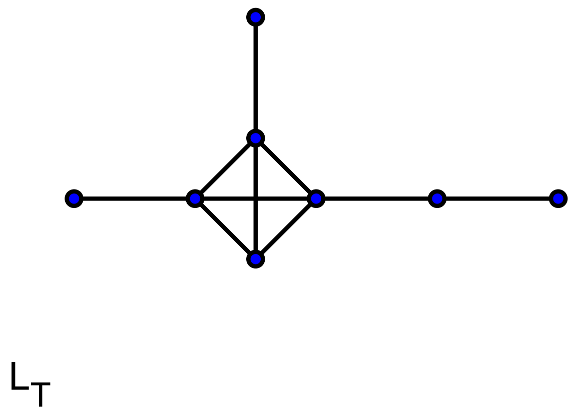

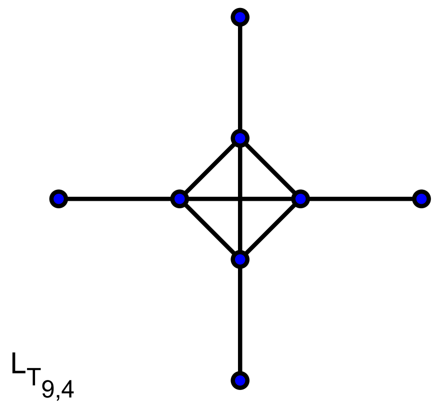

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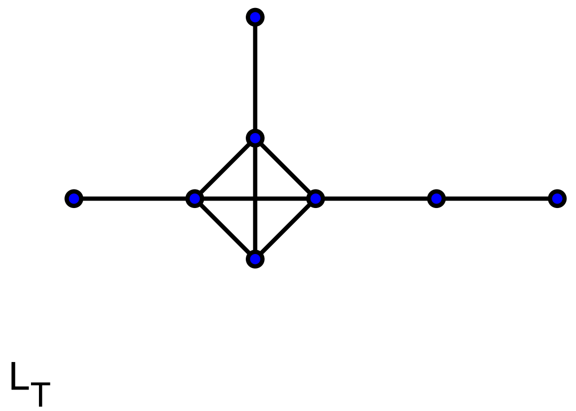

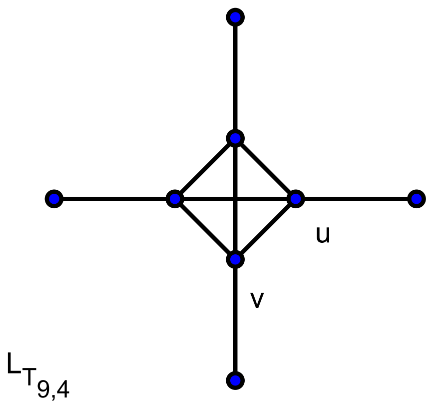

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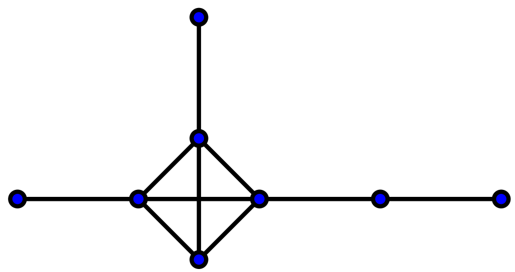
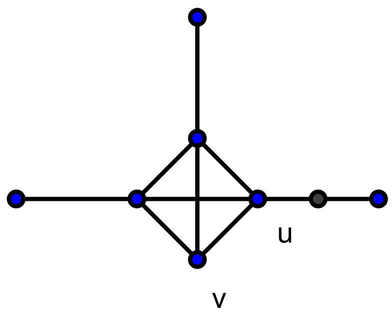


L_T



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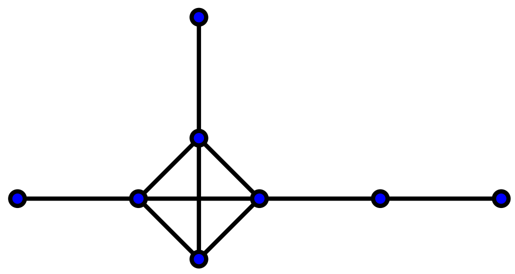
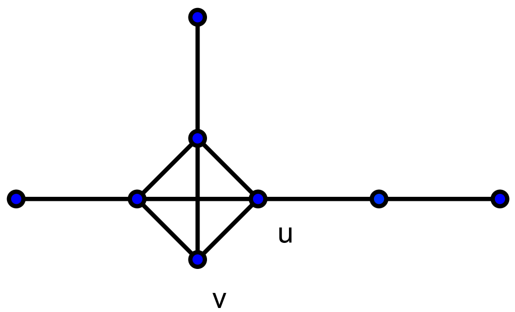


L_T



$L_{T_{9,4}} \cong L_T$

Case 2: $t = 1$.

- ☐ Also, consider $T_{n,k}$ and its line graph $L_{T_{n,k}}$.
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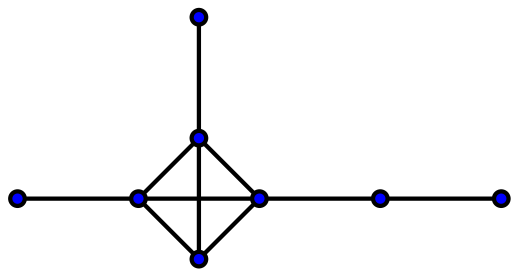

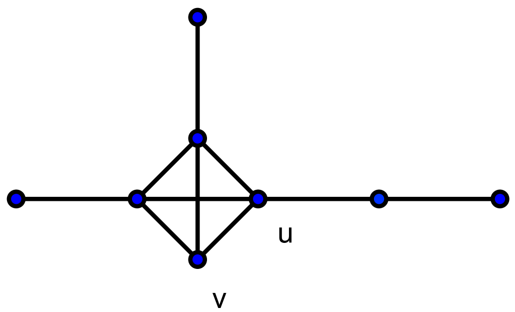

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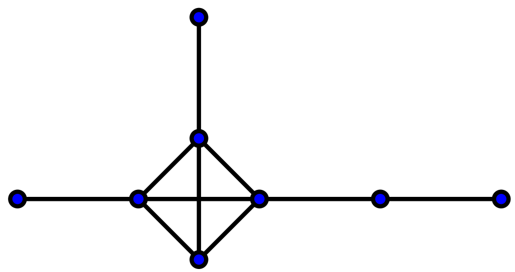

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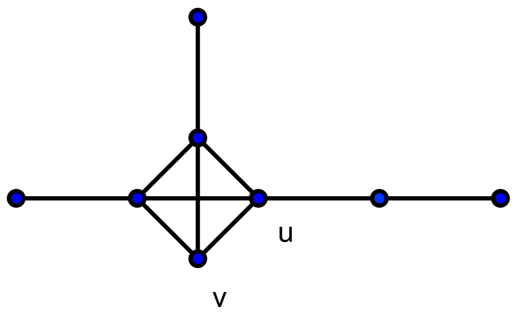

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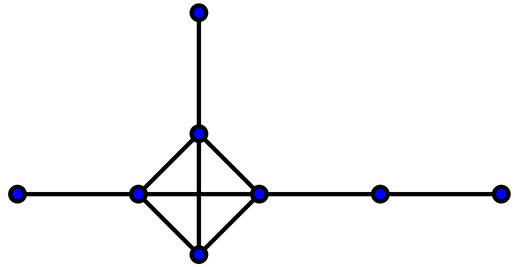

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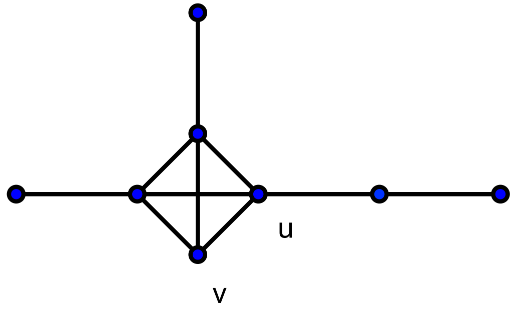

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Therefore, we get

$$\mu(T) = 2 + \rho(L_T) < 2 + \rho(L_{T_{n,k}}) = \mu(T_{n,k}).$$

Case 2

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CASE2 $\surd$

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Case 3

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Reconstruct T based on the method in theorem 1, so that the number of branch vertices can be reduced to 1, and then apply argument of case 2 (the proof for $t = 1$).

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Let us see little bit more detail with an example.

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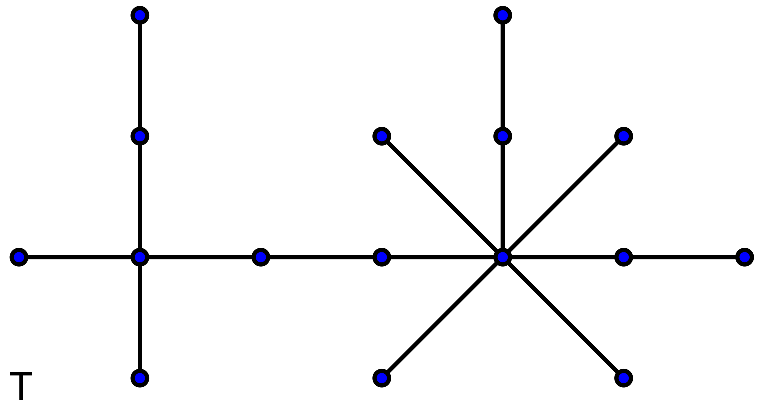

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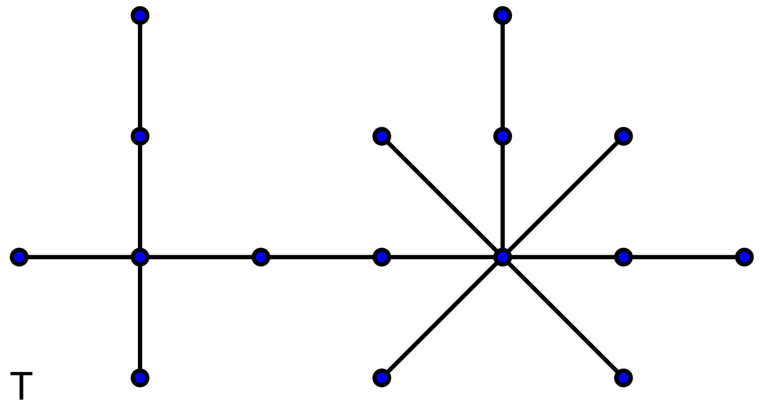

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□ To apply the method from theorem 1, label two branch vertices as u and v, and assume $x_u \geq x_v$.

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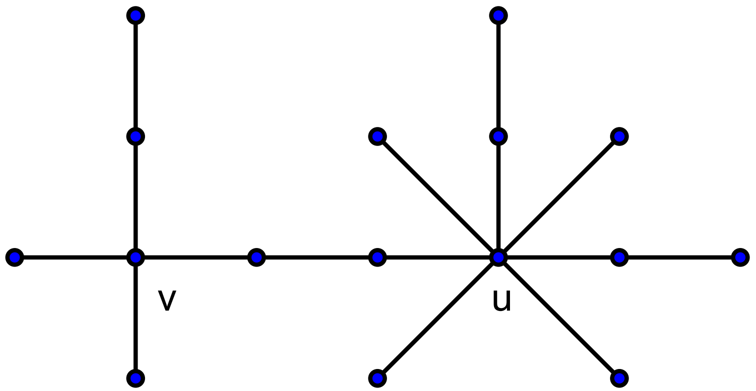

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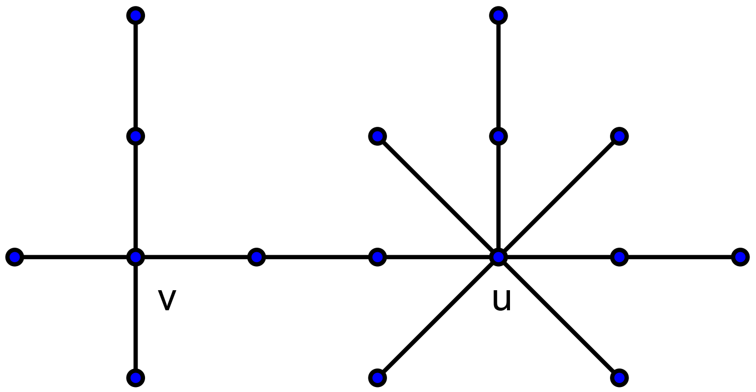

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☐ Recall: selecting two vertices in a tree graph determines a unique path.

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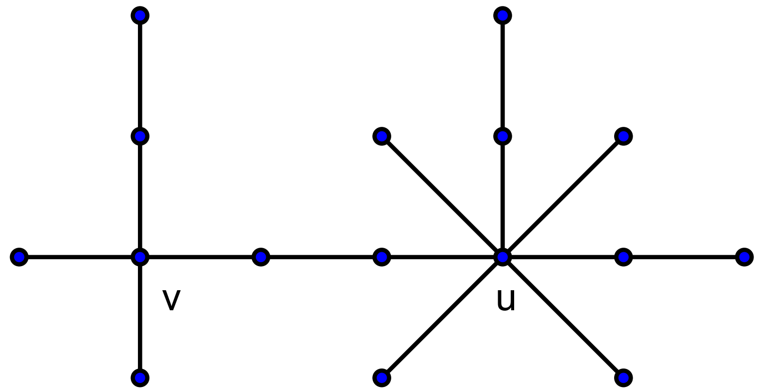

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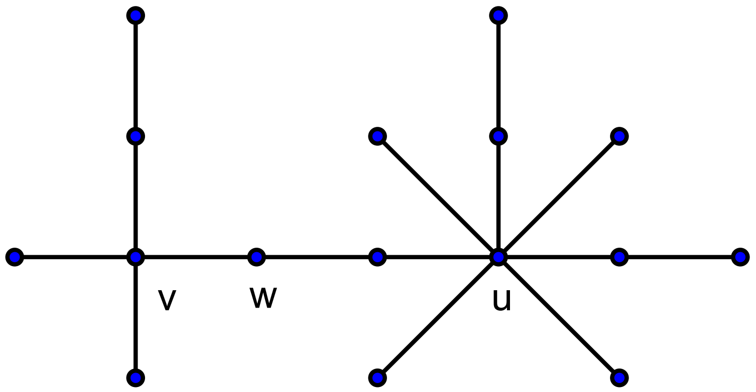

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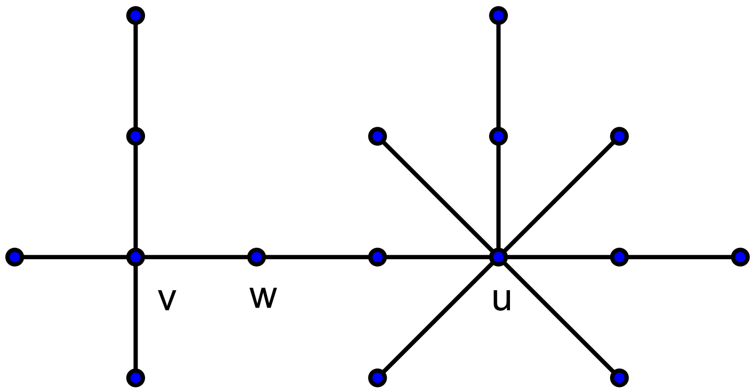

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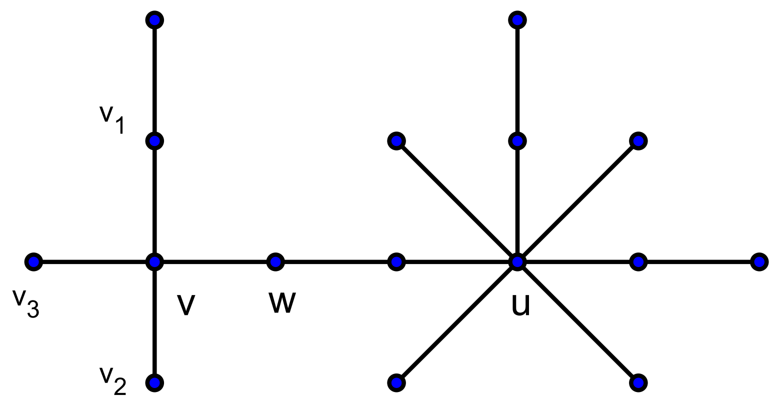

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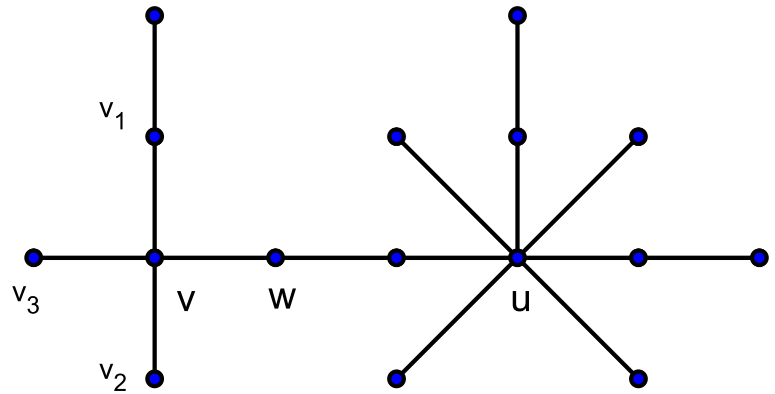

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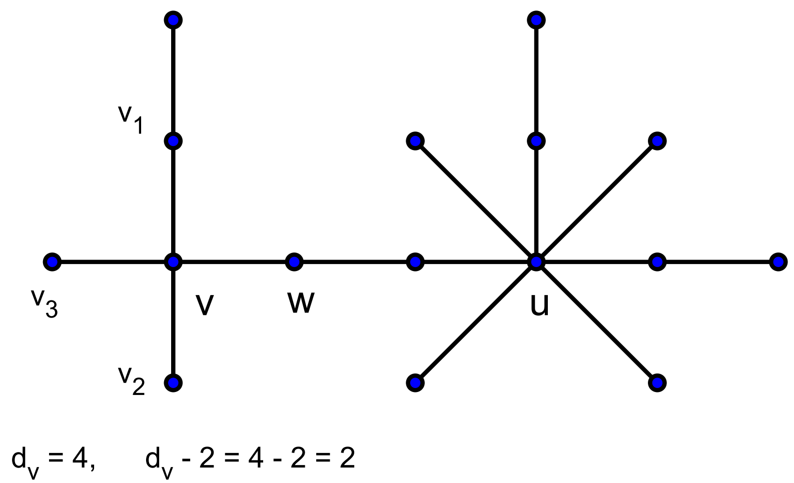

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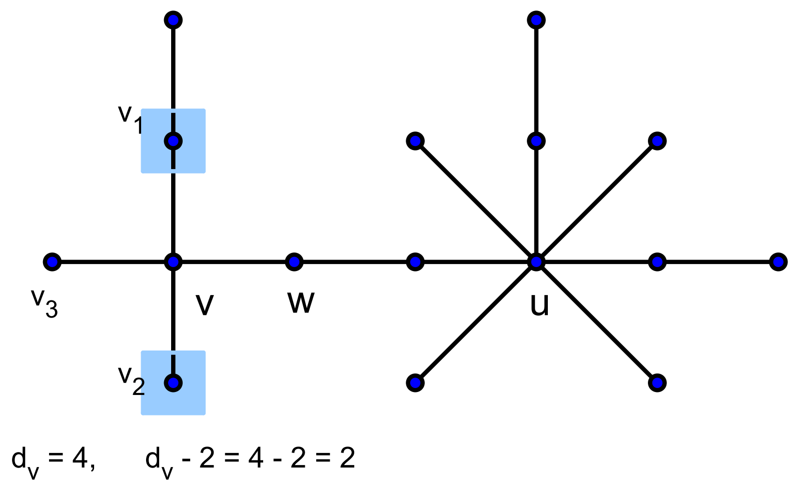

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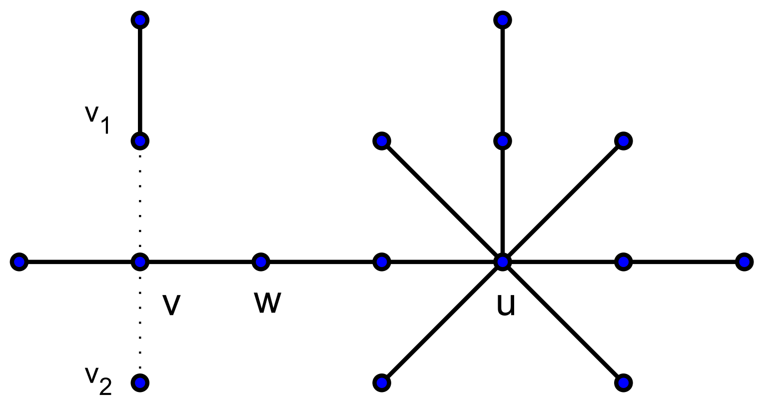

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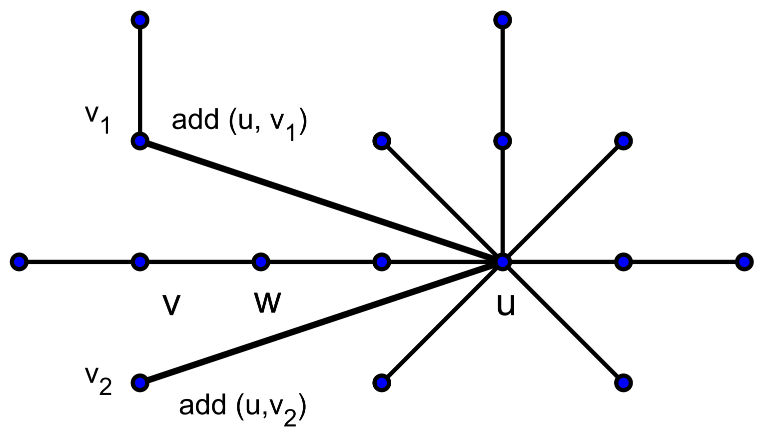

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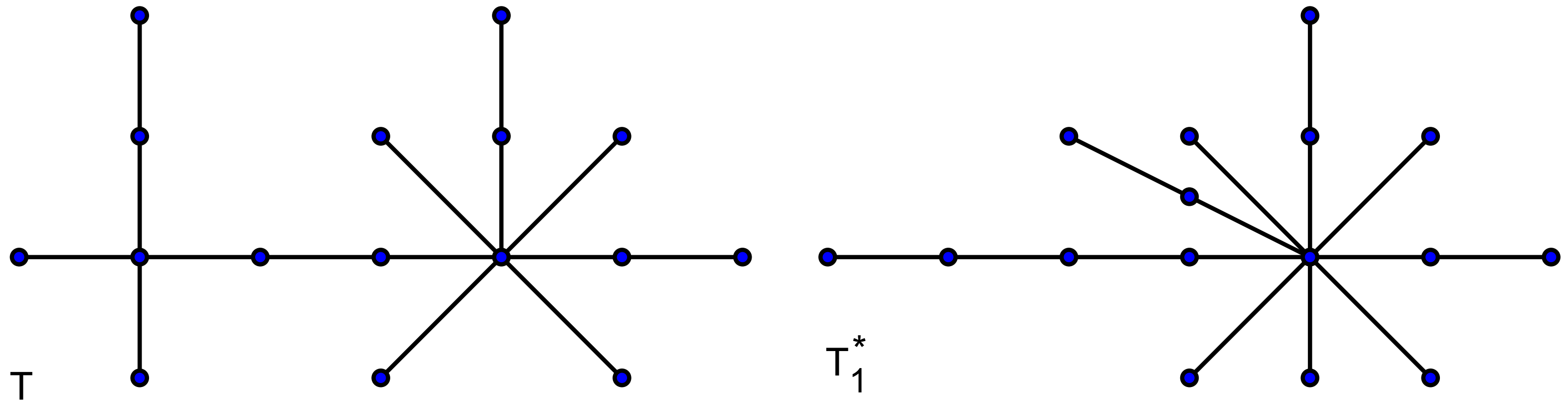

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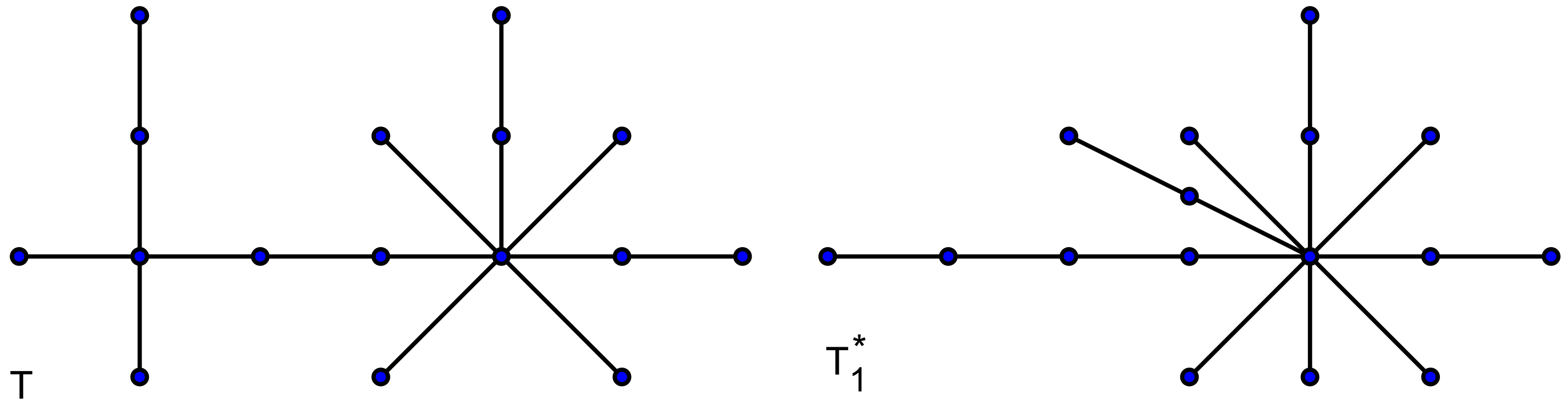

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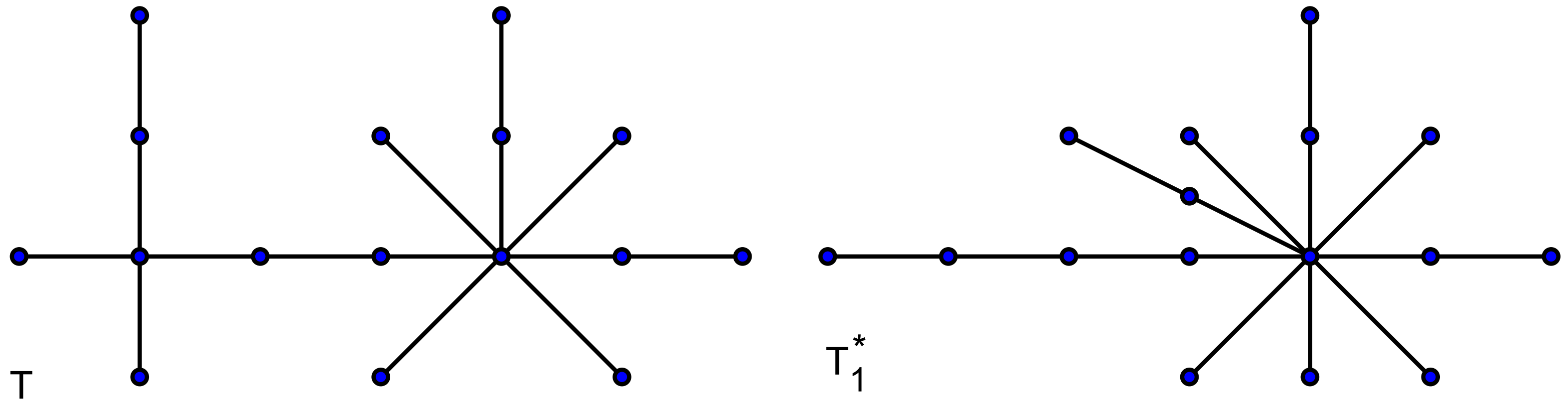

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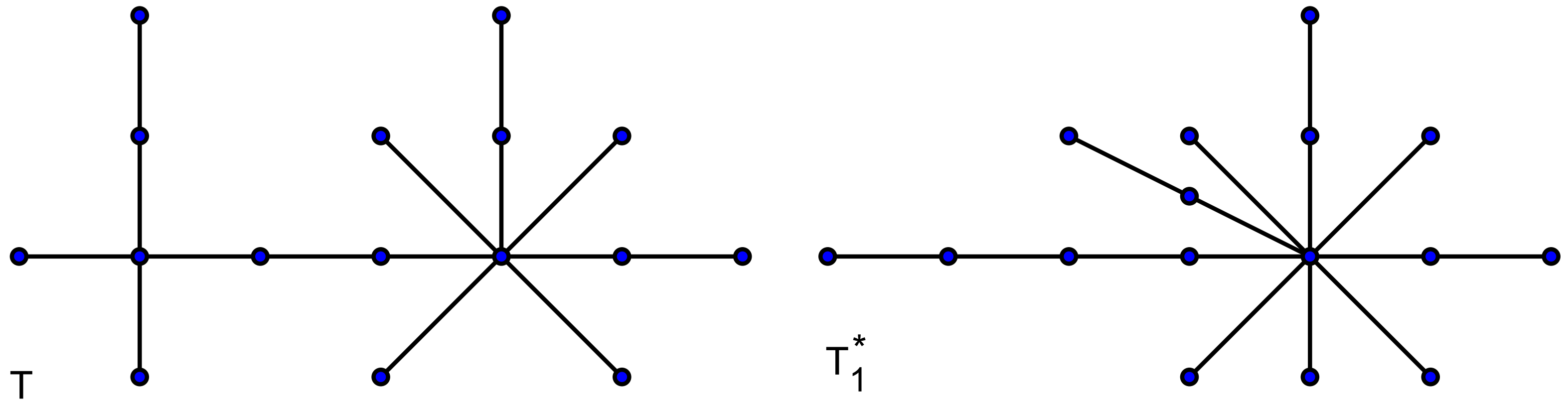

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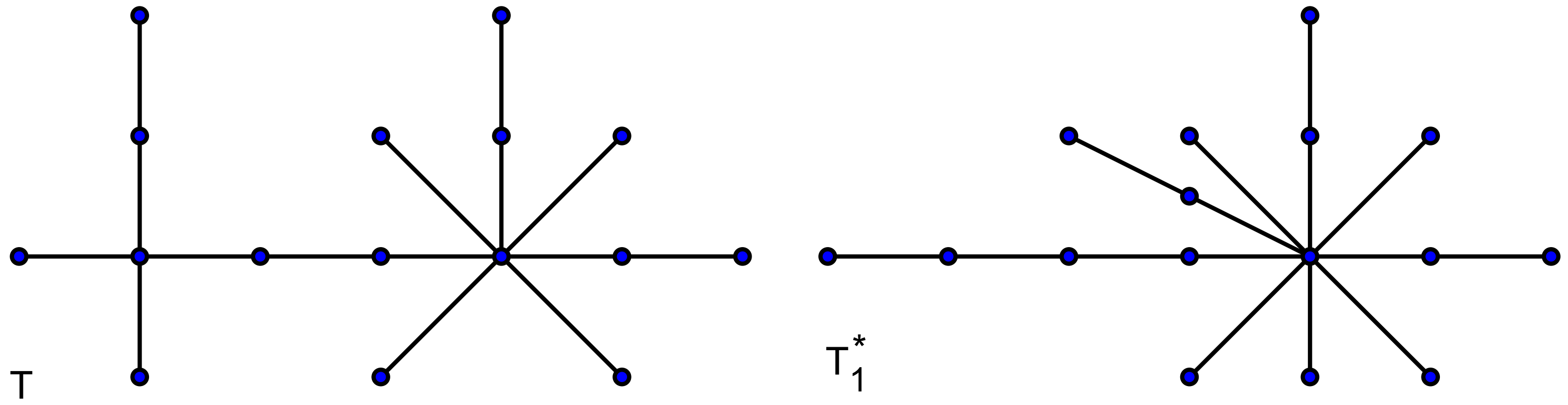

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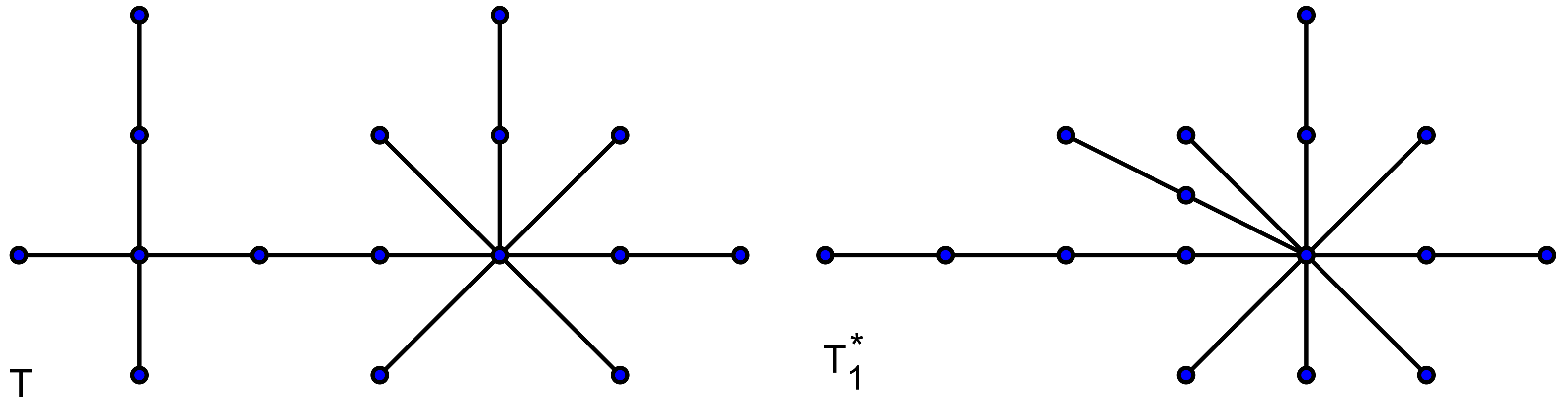

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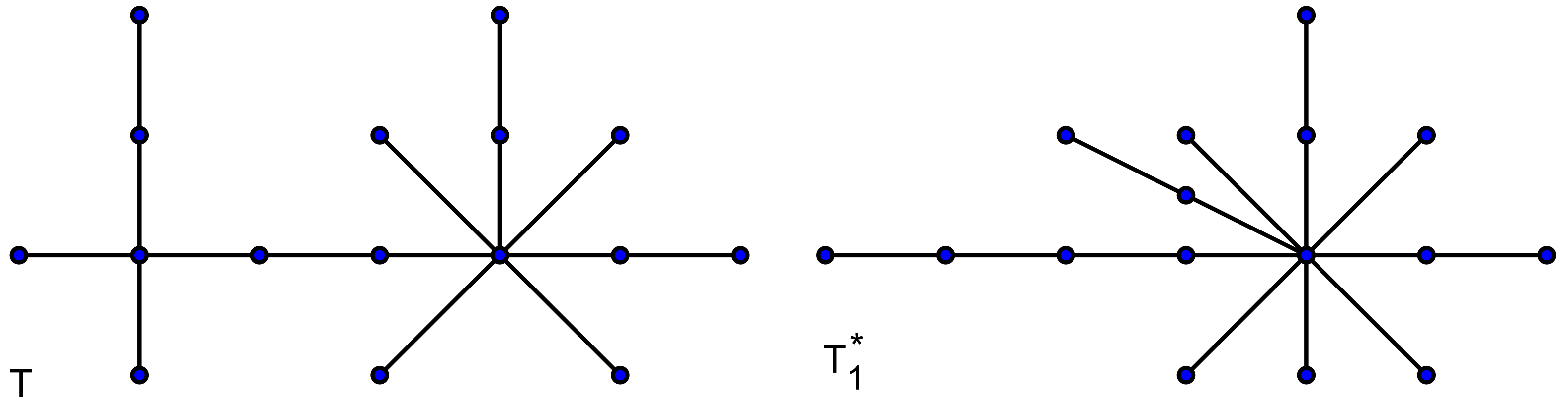

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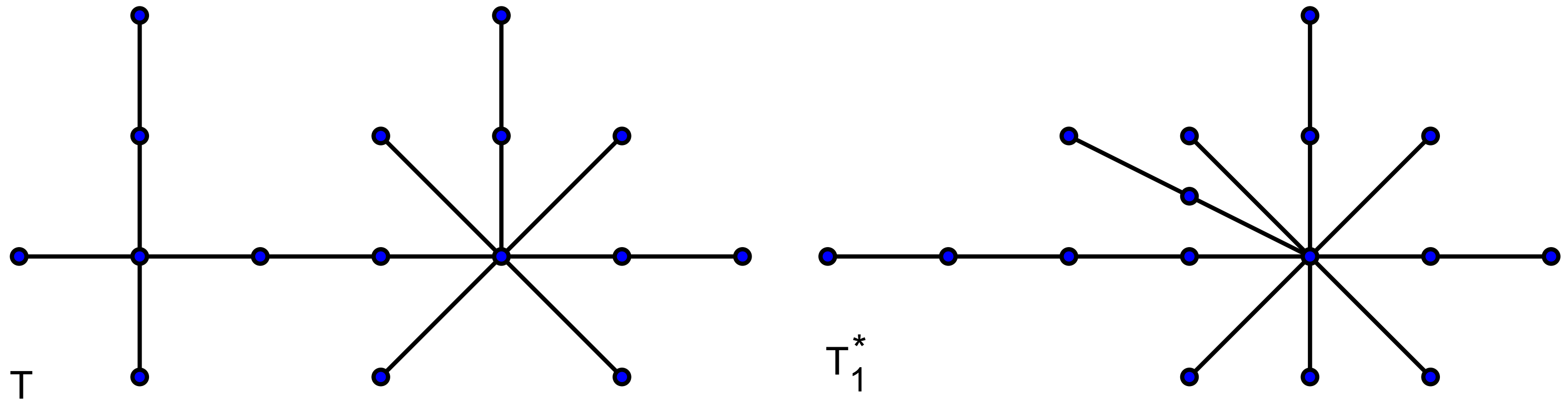

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Then the statement holds.

Case 3

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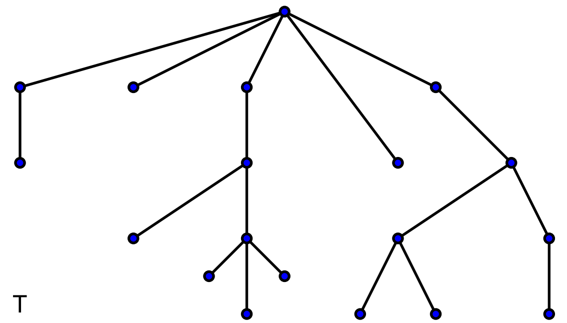
CASE3 $\surd$

Example (case 3)

□ Take $T \in \mathscr{T}_{19,10}$ as shown.

Example

Example (case 3)



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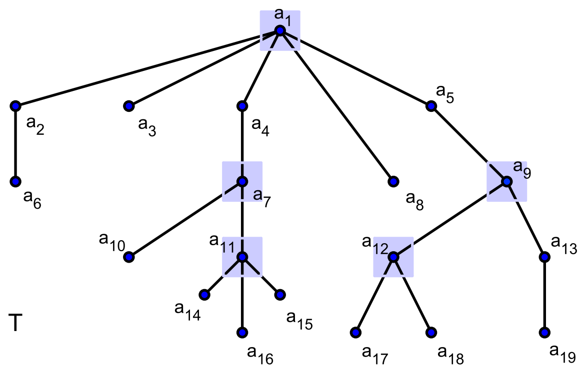

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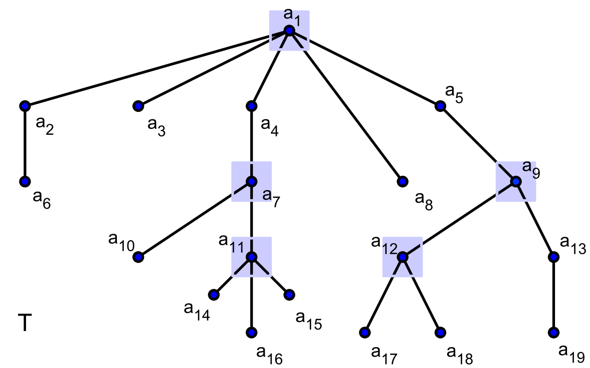
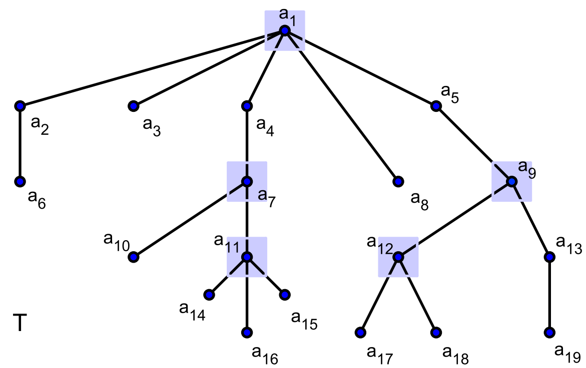


☐ Take $T \in \mathscr{T}_{19,10}$ as shown.
☐ First, find the signless Laplacian spectral radius and associated Perron vector.

Example

Example (case 3)



$\nu(T) = 6.1700$

Example

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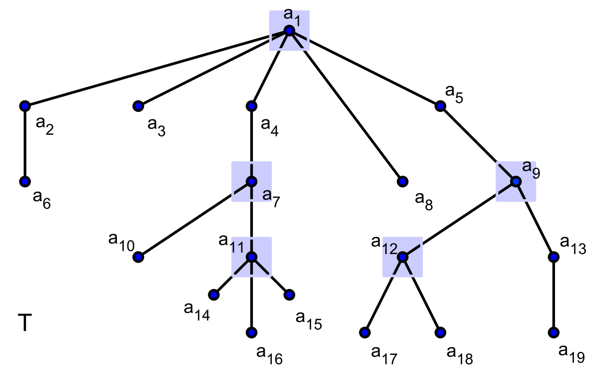


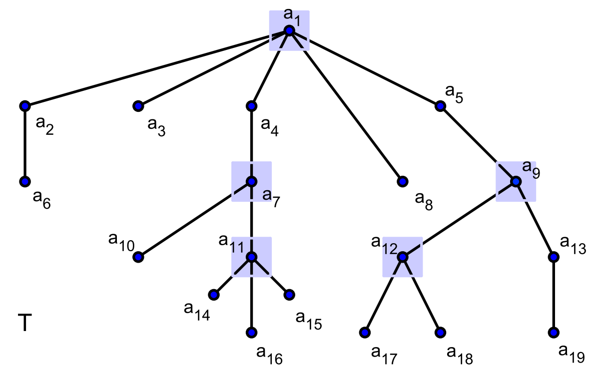
$$\nu(T) = 6.1700$$

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & \cdots \\ 0.8724 & 0.2194 & 0.1688 & 0.2330 & 0.2309 & 0.0424 & 0.0993 & 0.1688 & 0.0902 & \cdots \end{bmatrix}$$

$$\begin{matrix} x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} \\ 0.0192 & 0.0624 & 0.0324 & 0.0227 & 0.0121 & 0.0121 & 0.0121 & 0.0063 & 0.0063 & 0.0044 \end{matrix} \Bigg]^T$$

Example

Example (case 3)



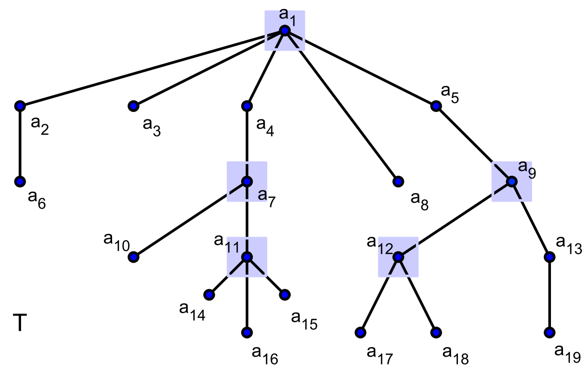
$\nu(T) = 6.1700$

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & \cdots \\ 0.8724 & 0.2194 & 0.1688 & 0.2330 & 0.2309 & 0.0424 & 0.0993 & 0.1688 & 0.0902 & \cdots \\ x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} \\ 0.0192 & 0.0624 & 0.0324 & 0.0227 & 0.0121 & 0.0121 & 0.0121 & 0.0063 & 0.0063 & 0.0044 \end{bmatrix}^T$$

☐ For each branch vertex, look at the corresponding entry in the Perron vector.

Example

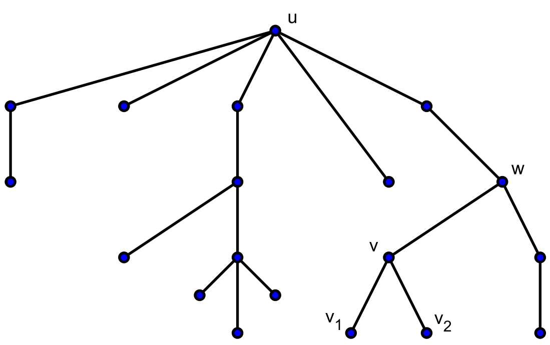
Example (case 3)



- ☐ Take $u = a_1$ and $v = a_{12}$ so that $x_u \geq x_v$ is preserved.
- ☐ The vertex w is uniquely determined.
- ☐ Label the rest of neighbors as v_1 and v_2
- ☐ $d_v - 2 = d_{a_{12}} - 2 = 3 - 2 = 1$, so we delete (v, v_1) and add (u, v_1).

Example

Example (case 3)



- ☐ Take $u = a_1$ and $v = a_{12}$ so that $x_u \geq x_v$ is preserved.
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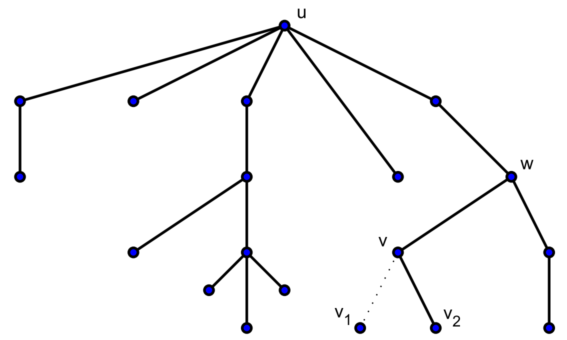
Example

Example (case 3)



- ☐ Take $u = a_1$ and $v = a_{12}$ so that $x_u \geq x_v$ is preserved.
- ☐ The vertex w is uniquely determined.
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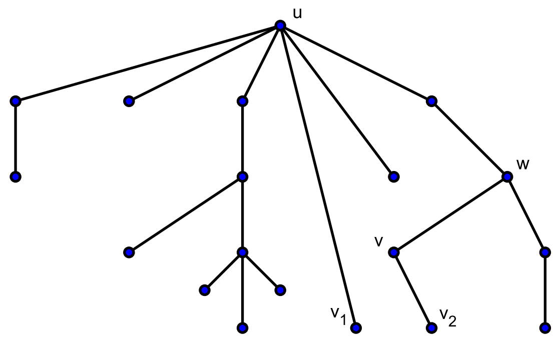
- ☐ $d_v - 2 = d_{a_{12}} - 2 = 3 - 2 = 1$, so we delete (v, v_1) and add (u, v_1).

Example

Example (case 3)



- ☐ Take $u = a_1$ and $v = a_{12}$ so that $x_u \geq x_v$ is preserved.
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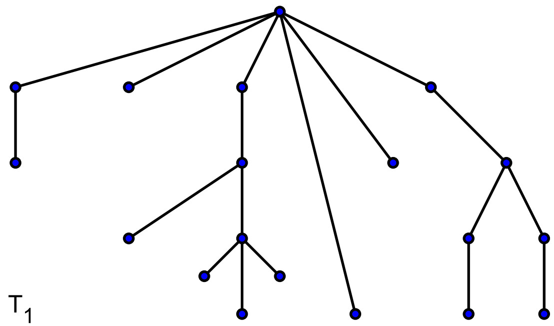
Example

Example (case 3)



- ☐ Take $u = a_1$ and $v = a_{12}$ so that $x_u \geq x_v$ is preserved.
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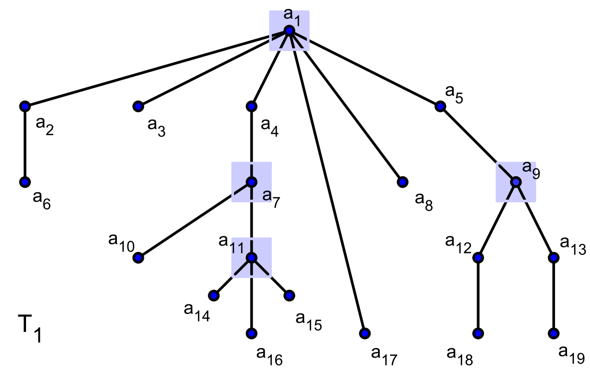
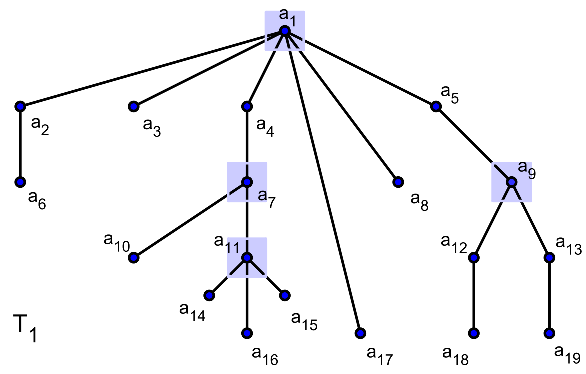
Example

Example (case 3)



- ☐ Take $u = a_1$ and $v = a_{12}$ so that $x_u \geq x_v$ is preserved.
- ☐ The vertex w is uniquely determined.
- ☐ Label the rest of neighbors as v_1 and v_2
- ☐ $d_v - 2 = d_{a_{12}} - 2 = 3 - 2 = 1$, so we delete (v, v_1) and add (u, v_1).

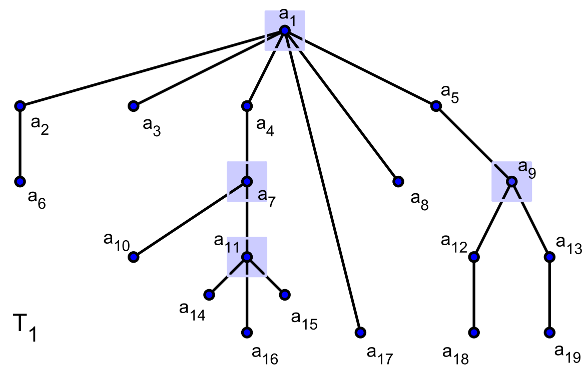
Example (case 3)



☐ For this new graph, T_1, find ν and associated Perron vector.

Example

Example (case 3)

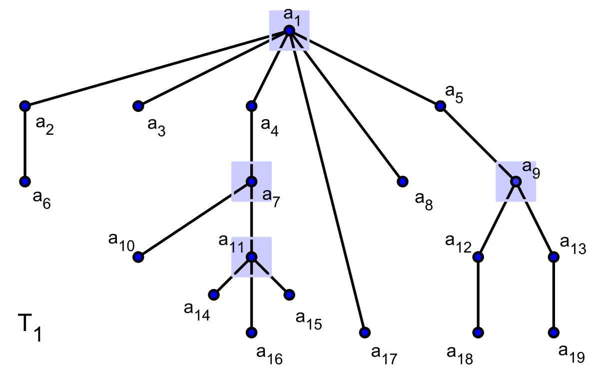


☐ For this new graph, T_1, find ν and associated Perron vector.

$\nu(T_1) = 7.1074$

Example

Example (case 3)

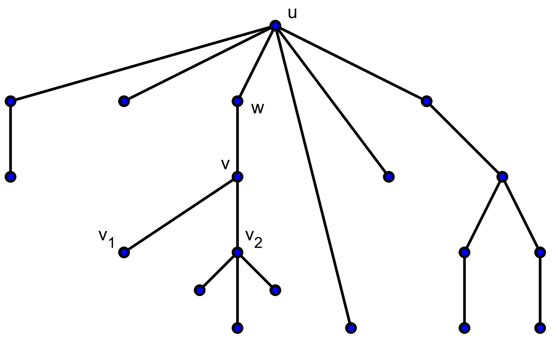


☐ For this new graph, T_1, find ν and associated Perron vector.

$$\nu(T_1) = 7.1074$$

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & \cdots \\ 0.8724 & 0.2194 & 0.1688 & 0.2330 & 0.2309 & 0.0424 & 0.0993 & 0.1688 & 0.0902 & \cdots \\ x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} \\ 0.0192 & 0.0624 & 0.0324 & 0.0227 & 0.0121 & 0.0121 & 0.0121 & 0.0063 & 0.0063 & 0.0044 \end{bmatrix}^T$$

Example (case 3)



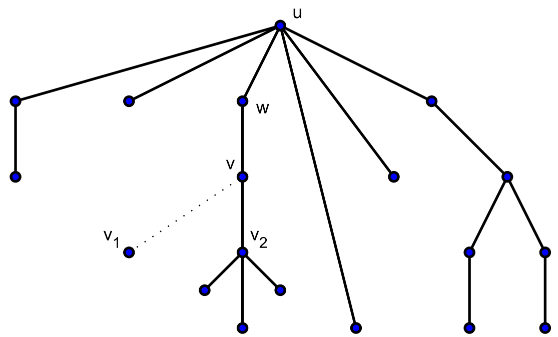
☐ For this new graph, T_1, find ν and associated Perron vector.

$\nu(T_1) = 7.1074$

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & \cdots \\ 0.8724 & 0.2194 & 0.1688 & 0.2330 & 0.2309 & 0.0424 & 0.0993 & 0.1688 & 0.0902 & \cdots \\ x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} \\ 0.0192 & 0.0624 & 0.0324 & 0.0227 & 0.0121 & 0.0121 & 0.0121 & 0.0063 & 0.0063 & 0.0044 \end{bmatrix}^T$$

Example

Example (case 3)



☐ For this new graph, T_1, find ν and associated Perron vector.

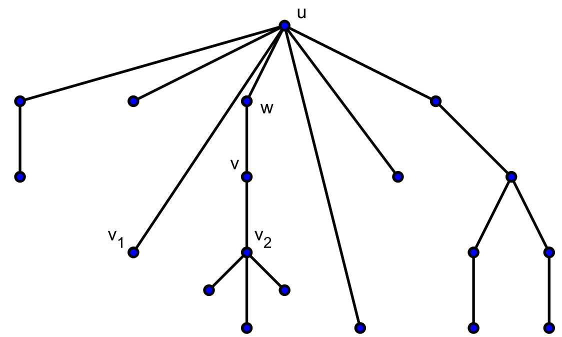
$\nu(T_1) = 7.1074$

$$\mathbf{x} = \left[\begin{array}{cccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & \cdots \\ 0.8724 & 0.2194 & 0.1688 & 0.2330 & 0.2309 & 0.0424 & 0.0993 & 0.1688 & 0.0902 & \cdots \end{array} \right.$$

$$\left. \begin{array}{cccccccccc} x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} \\ 0.0192 & 0.0624 & 0.0324 & 0.0227 & 0.0121 & 0.0121 & 0.0121 & 0.0063 & 0.0063 & 0.0044 \end{array} \right]^T$$

Example

Example (case 3)



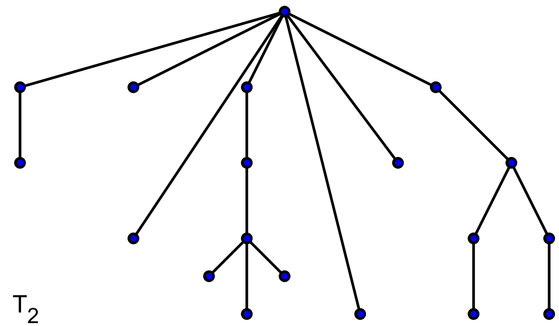
☐ For this new graph, T_1, find ν and associated Perron vector.

$\nu(T_1) = 7.1074$

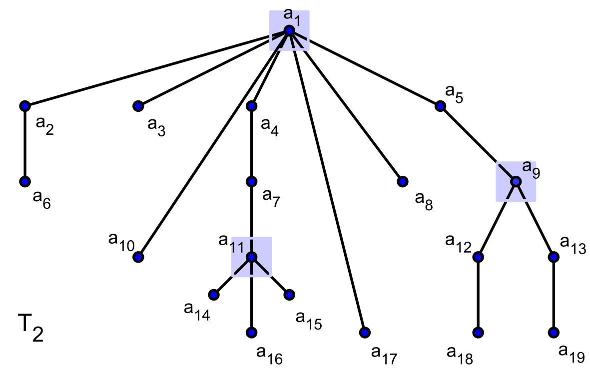
$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & \cdots \\ 0.8724 & 0.2194 & 0.1688 & 0.2330 & 0.2309 & 0.0424 & 0.0993 & 0.1688 & 0.0902 & \cdots \\ x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} \\ 0.0192 & 0.0624 & 0.0324 & 0.0227 & 0.0121 & 0.0121 & 0.0121 & 0.0063 & 0.0063 & 0.0044 \end{bmatrix}^T$$

Example

Example (case 3)



Example (case 3)



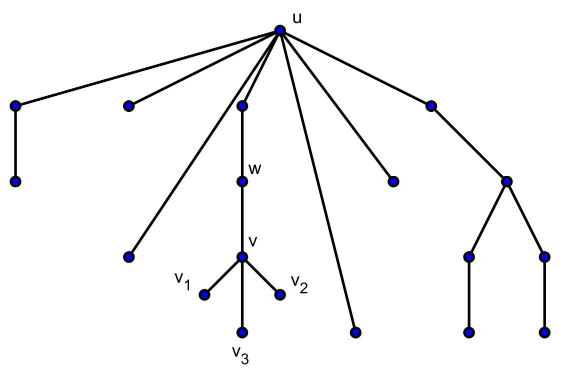
$$\nu(T_2) = 8.0740$$

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & \cdots \\ 0.8724 & 0.2194 & 0.1688 & 0.2330 & 0.2309 & 0.0424 & 0.0993 & 0.1688 & 0.0902 & \cdots \end{bmatrix}$$

$$\begin{bmatrix} x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} \\ 0.0192 & 0.0624 & 0.0324 & 0.0227 & 0.0121 & 0.0121 & 0.0121 & 0.0063 & 0.0063 & 0.0044 \end{bmatrix}^T$$

Example

Example (case 3)



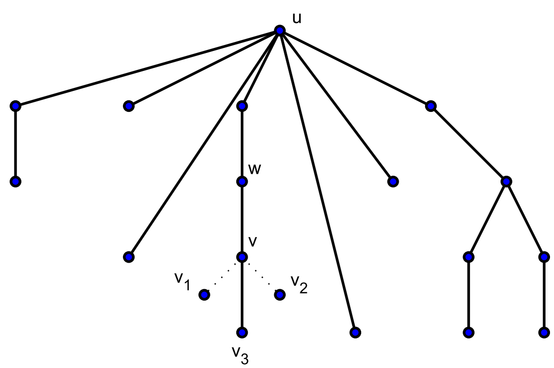
$$\nu(T_2) = 8.0740$$

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & \cdots \\ 0.8724 & 0.2194 & 0.1688 & 0.2330 & 0.2309 & 0.0424 & 0.0993 & 0.1688 & 0.0902 & \cdots \end{bmatrix}$$

$$\begin{bmatrix} x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} \\ 0.0192 & 0.0624 & 0.0324 & 0.0227 & 0.0121 & 0.0121 & 0.0121 & 0.0063 & 0.0063 & 0.0044 \end{bmatrix}^T$$

Example

Example (case 3)



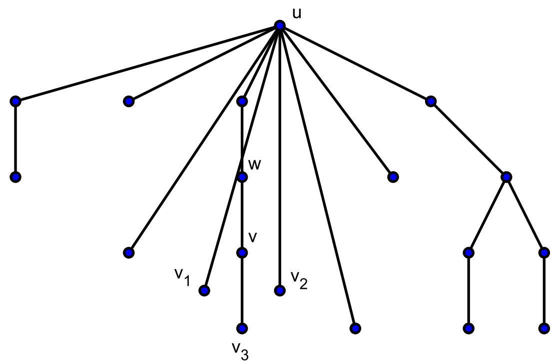
$\nu(T_2) = 8.0740$

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & \cdots \\ 0.8724 & 0.2194 & 0.1688 & 0.2330 & 0.2309 & 0.0424 & 0.0993 & 0.1688 & 0.0902 & \cdots \end{bmatrix}$$

$$\begin{matrix} x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} \\ 0.0192 & 0.0624 & 0.0324 & 0.0227 & 0.0121 & 0.0121 & 0.0121 & 0.0063 & 0.0063 & 0.0044 \end{matrix} \Bigg]^T$$

Example

Example (case 3)

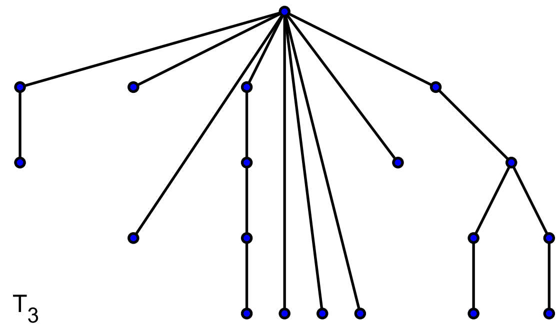


$$\nu(T_2) = 8.0740$$

$$\mathbf{x} = \left[\begin{array}{cccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & \cdots \\ 0.8724 & 0.2194 & 0.1688 & 0.2330 & 0.2309 & 0.0424 & 0.0993 & 0.1688 & 0.0902 & \cdots \end{array} \right.$$

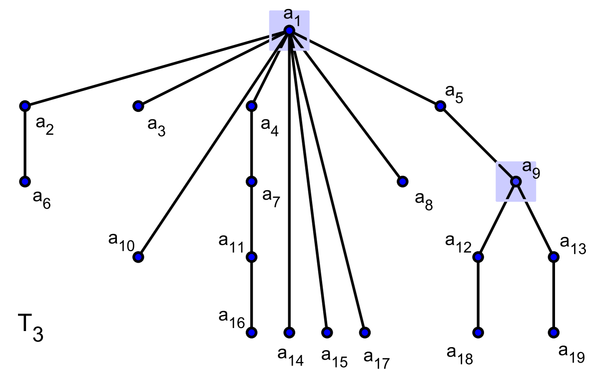
$$\left. \begin{array}{cccccccccc} x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} \\ 0.0192 & 0.0624 & 0.0324 & 0.0227 & 0.0121 & 0.0121 & 0.0121 & 0.0063 & 0.0063 & 0.0044 \end{array} \right]^T$$

Example (case 3)



Example

Example (case 3)



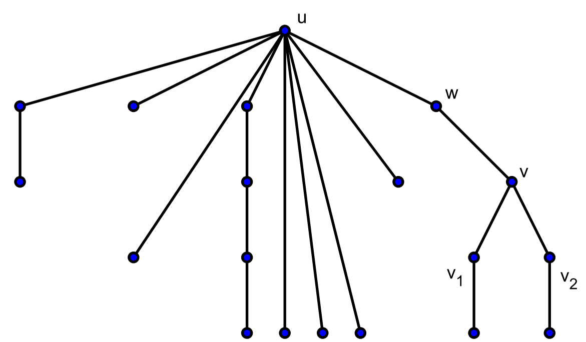
$$\nu(T_3) = 10.0426$$

$$\mathbf{x} = \left[\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & \cdots \\ 0.8724 & 0.2194 & 0.1688 & 0.2330 & 0.2309 & 0.0424 & 0.0993 & 0.1688 & 0.0902 & \cdots \end{matrix} \right.$$

$$\left. \begin{matrix} x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} \\ 0.0192 & 0.0624 & 0.0324 & 0.0227 & 0.0121 & 0.0121 & 0.0121 & 0.0063 & 0.0063 & 0.0044 \end{matrix} \right]^T$$

Example

Example (case 3)



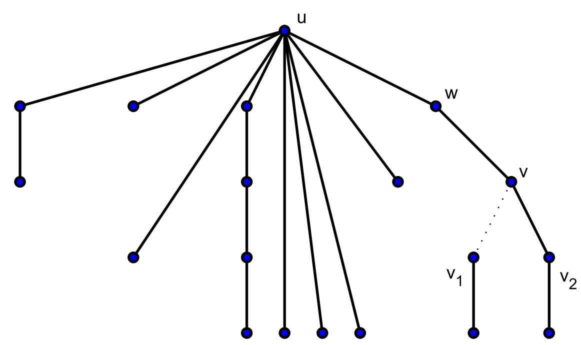
$$\nu(T_3) = 10.0426$$

$$\mathbf{x} = \left[\begin{array}{cccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & \cdots \\ 0.8724 & 0.2194 & 0.1688 & 0.2330 & 0.2309 & 0.0424 & 0.0993 & 0.1688 & 0.0902 & \cdots \end{array} \right.$$

$$\left. \begin{array}{cccccccccc} x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} \\ 0.0192 & 0.0624 & 0.0324 & 0.0227 & 0.0121 & 0.0121 & 0.0121 & 0.0063 & 0.0063 & 0.0044 \end{array} \right]^T$$

Example

Example (case 3)



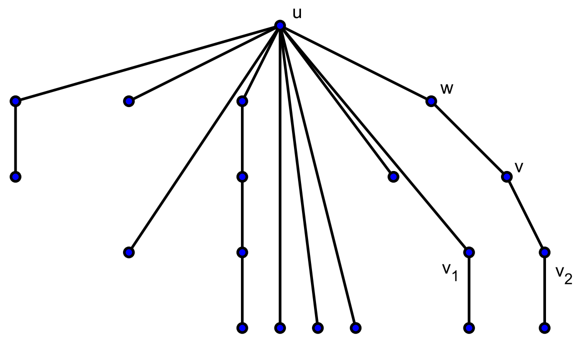
$$\nu(T_3) = 10.0426$$

$$\mathbf{x} = \left[\begin{array}{cccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & \cdots \\ 0.8724 & 0.2194 & 0.1688 & 0.2330 & 0.2309 & 0.0424 & 0.0993 & 0.1688 & 0.0902 & \cdots \end{array} \right.$$

$$\left. \begin{array}{cccccccccc} x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} \\ 0.0192 & 0.0624 & 0.0324 & 0.0227 & 0.0121 & 0.0121 & 0.0121 & 0.0063 & 0.0063 & 0.0044 \end{array} \right]^T$$

Example

Example (case 3)



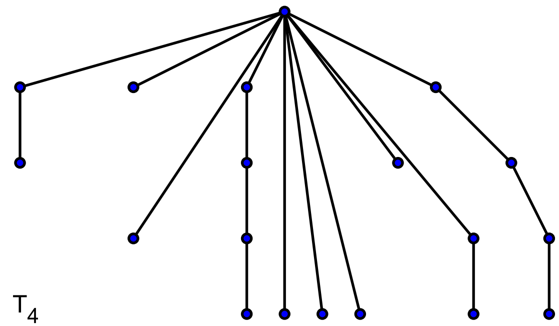
$$\nu(T_3) = 10.0426$$

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & \cdots \\ 0.8724 & 0.2194 & 0.1688 & 0.2330 & 0.2309 & 0.0424 & 0.0993 & 0.1688 & 0.0902 & \cdots \end{bmatrix.$$

$$\begin{matrix} x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} \\ 0.0192 & 0.0624 & 0.0324 & 0.0227 & 0.0121 & 0.0121 & 0.0121 & 0.0063 & 0.0063 & 0.0044 \end{matrix} \Big]^T$$

Example

Example (case 3)

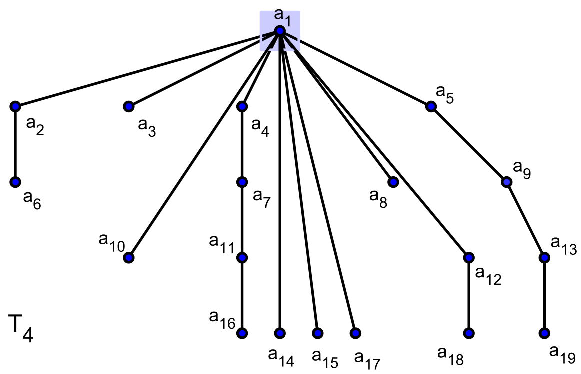


$\nu(T_3) = 10.0426$

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & \cdots \\ 0.8724 & 0.2194 & 0.1688 & 0.2330 & 0.2309 & 0.0424 & 0.0993 & 0.1688 & 0.0902 & \cdots \\ x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} \\ 0.0192 & 0.0624 & 0.0324 & 0.0227 & 0.0121 & 0.0121 & 0.0121 & 0.0063 & 0.0063 & 0.0044 \end{bmatrix}^T$$

Example

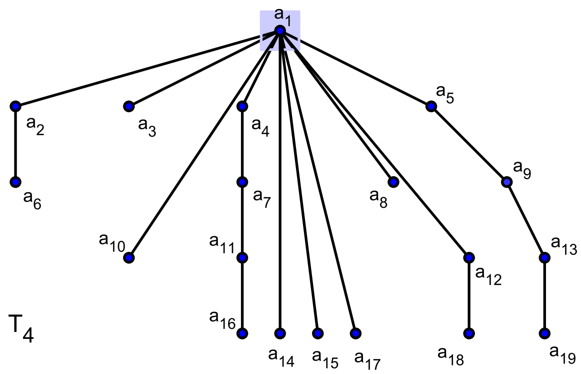
Example (case 3)



$$\nu(T) = 6.1700 \quad \nu(T_1) = 7.1074 \quad \nu(T_2) = 8.0740 \quad \nu(T_3) = 10.0426 \quad \nu(T_4) = 11.0448$$

Example

Example (case 3)



$\nu(T) = 6.1700 \;\; \nu(T_1) = 7.1074 \;\; \nu(T_2) = 8.0740 \;\; \nu(T_3) = 10.0426 \;\; \nu(T_4) = 11.0448$

Therefore,

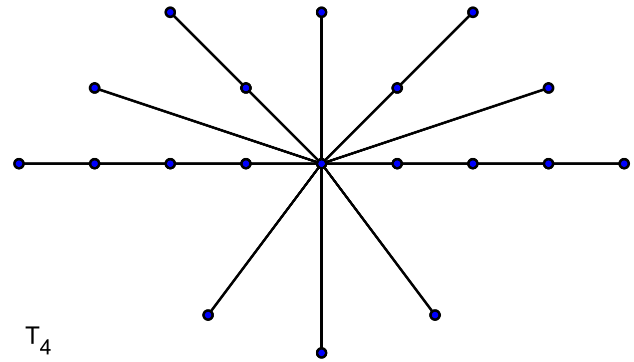
$\mu(T) = 6.1700 \;\; \mu(T_1) = 7.1074 \;\; \mu(T_2) = 8.0740 \;\; \mu(T_3) = 10.0426 \;\; \nu(T_4) = 11.0448$

Example

Example (case 2)

Example

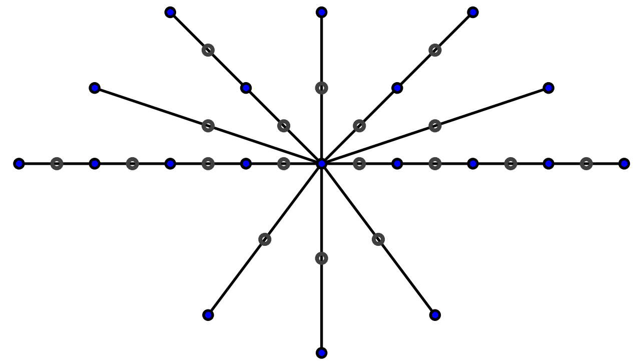
Example (case 2)



☐ Apply the argument of case 2 now.

Example

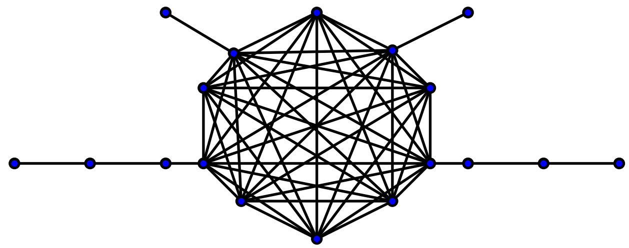
Example (case 2)



- ☐ Apply the argument of case 2 now.
- ☐ Construct the line graph of T_4, L_{T_4}

Example

Example (case 2)

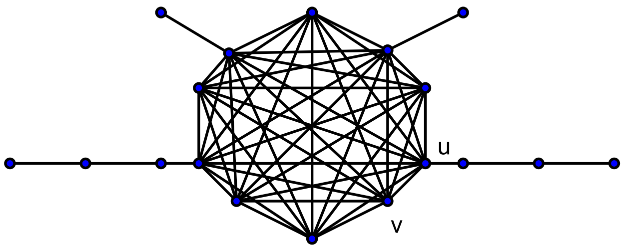


L_{T_4}

- ☐ Apply the argument of case 2 now.
- ☐ Construct the line graph of T_4, L_{T_4}
- ☐ Then, apply lemma 3 few times.

Example

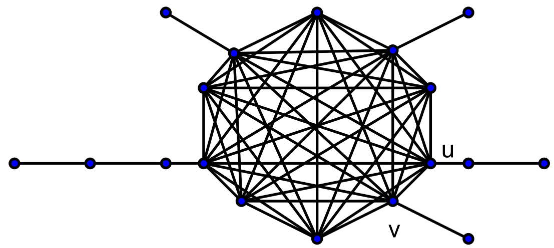
Example (case 2)



- ☐ Apply the argument of case 2 now.
- ☐ Construct the line graph of T_4, L_{T_4}
- ☐ Then, apply lemma 3 few times.

Example

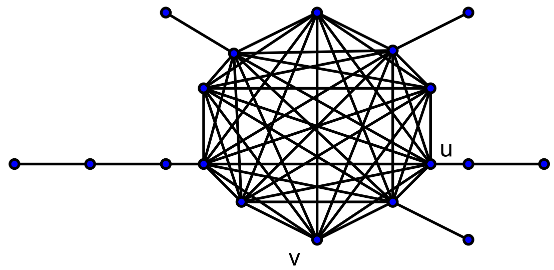
Example (case 2)



- ☐ Apply the argument of case 2 now.
- ☐ Construct the line graph of T_4, L_{T_4}
- ☐ Then, apply lemma 3 few times.

Example

Example (case 2)



- ☐ Apply the argument of case 2 now.
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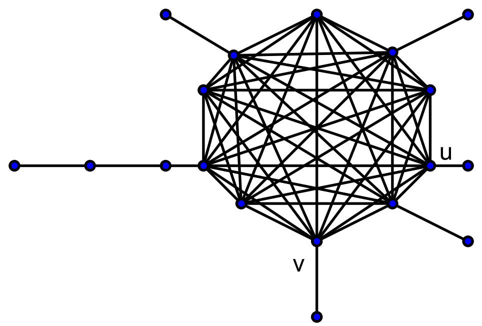
Example

Example (case 2)



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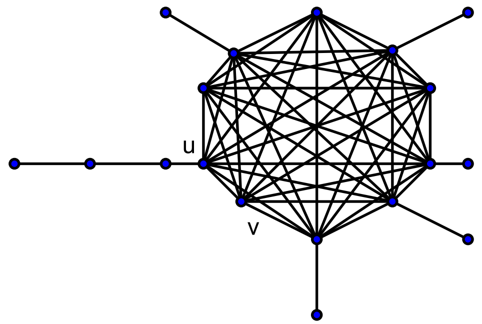

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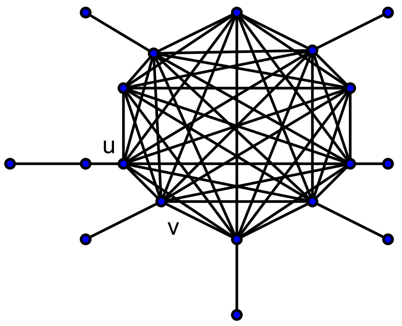

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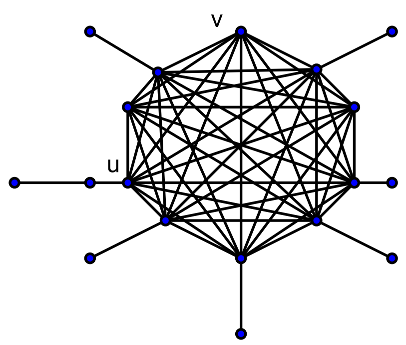

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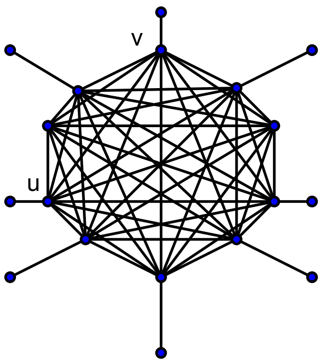

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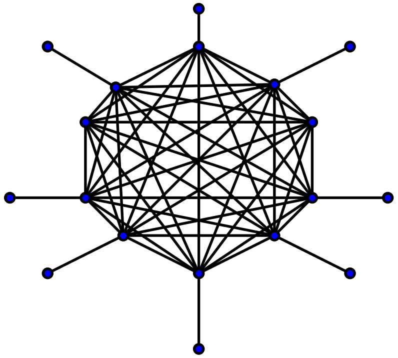

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- ☐ Notice that this is a complete graph K_{10} and 10 “almost equal length” paths.

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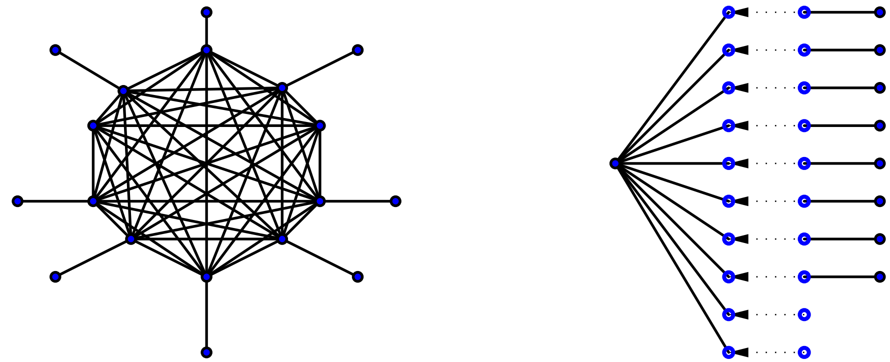

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- ☐ The line graph of $T_{19,10}$ is...

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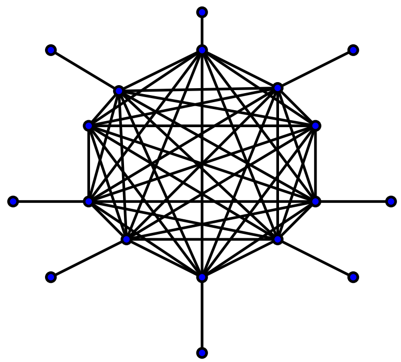

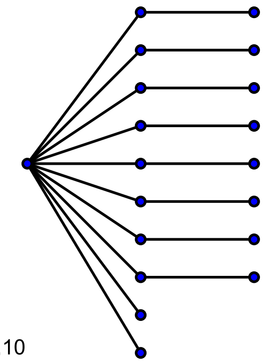

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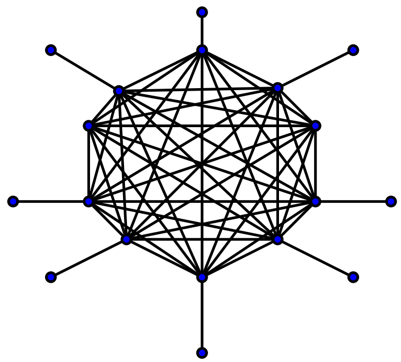

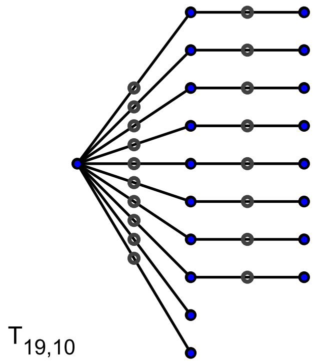

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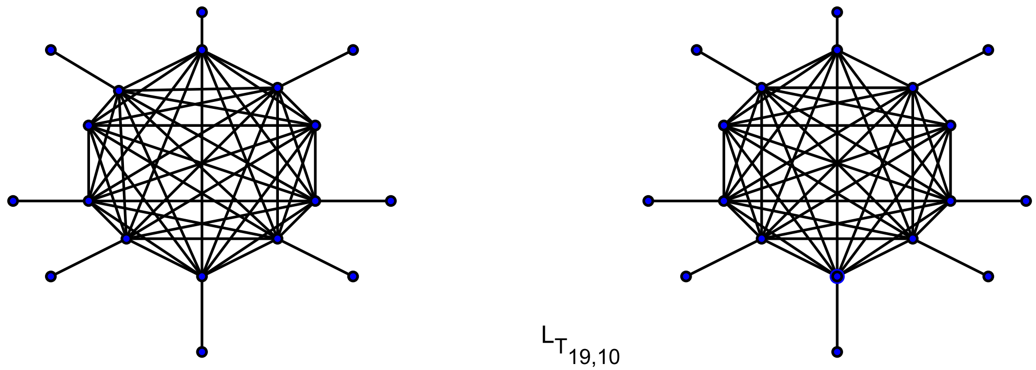

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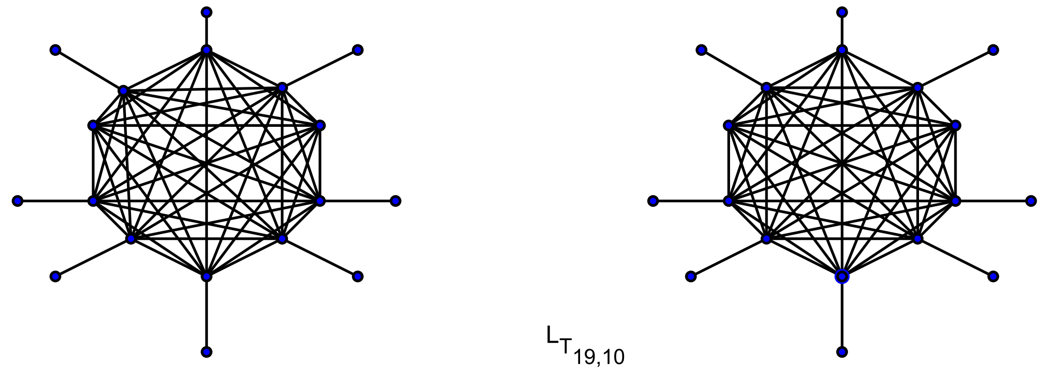
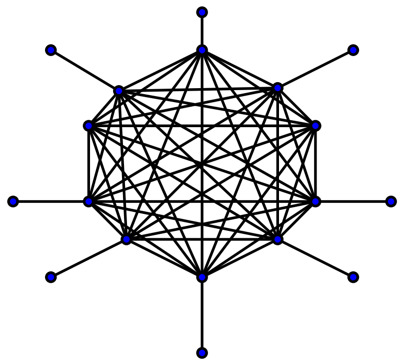


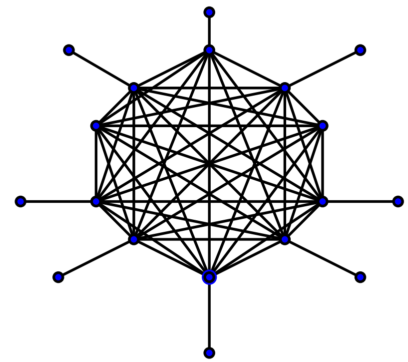
☐ Recall lemma 4, we have

Example

Example (case 2)



$L_{T_{19,10}}$



☐ Recall lemma 4, we have

$$\mu(T_4) = 2 + \rho(L_{T_4}) < 2 + \rho(L_{T_{19,10}}) = \mu(T_{19,10})$$

Example

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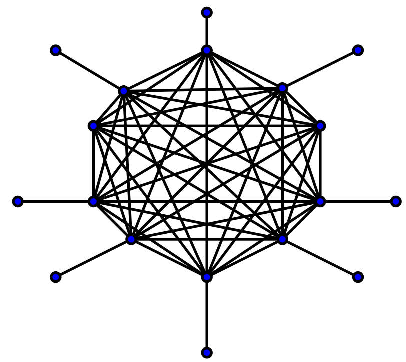

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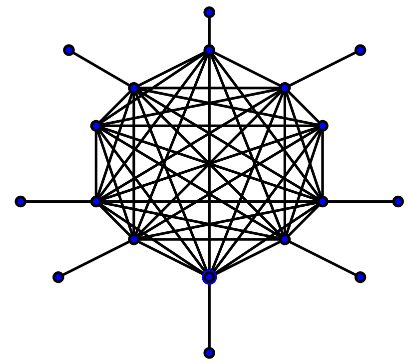


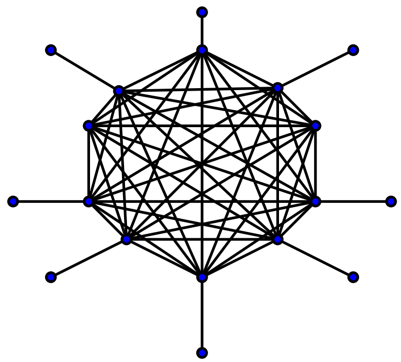
☐ Recall lemma 4, we have

$$\mu(T_4) = 2 + \rho(L_{T_4}) < 2 + \rho(L_{T_{19,10}}) = \mu(T_{19,10})$$

☐ In fact, $\mu(T_4) = 11.0448$ whereas $\mu(T_{19,10}) = 18.8615$.

Example

Example (case 2)



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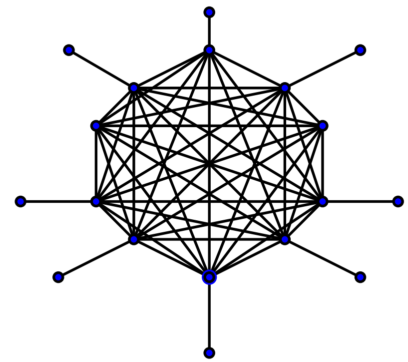
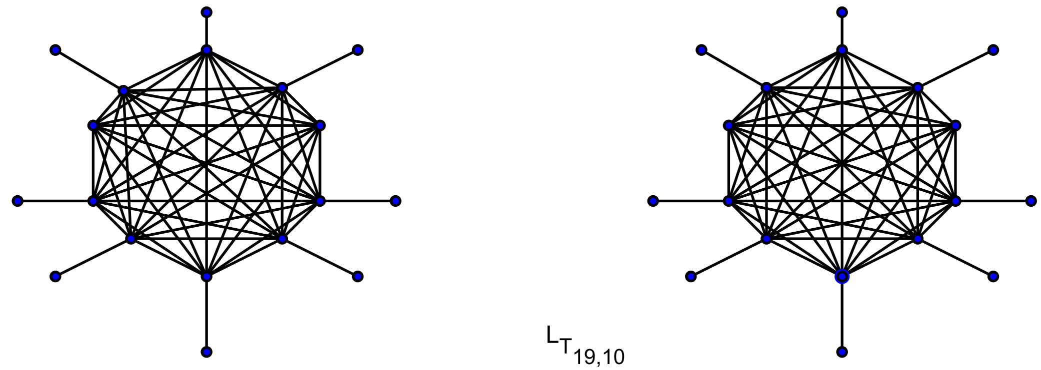


☐ Finally,

Example

Example (case 2)



☐ Finally,

Graph	T	T_1	T_2	T_3	T_4	$T_{19,10}$
μ	6.1700	7.1074	8.0740	10.0426	11.0448	18.8615

Example

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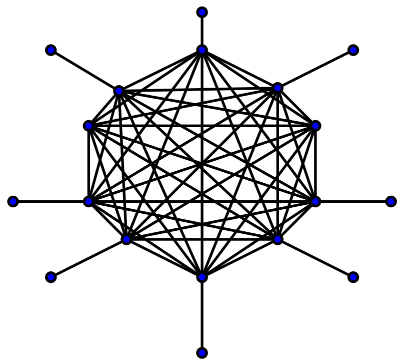
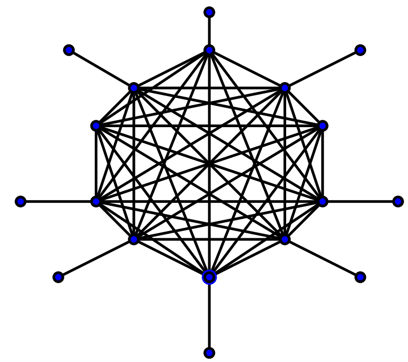


$L_{T_{19,10}}$



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Graph	T	T_1	T_2	T_3	T_4	$T_{19,10}$
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Therefore,

$$\mu(T) < \mu(T_{19,10})$$

as expected.

The End