A Sharp Upper Bound for the Largest Eigenvalue of the Laplacian Matrix of a Tree

By Tomohiro Kawasaki

Under the Direction of
Dr. John Caughman

With Faculty Reader
Dr. Paul Latiolais

In partial fulfillment of the requirement for the degree of
Masters of Science in Mathematics

Portland State University
Fariborz Maseeh Department of Mathematics + Statistics
July 28, 2011

Contents

1 Introduction **1**

2 Sharp upper bound **2**

2.1 Some preliminary lemmas 2

2.2 Main Results 8

2.3 Consequences 19

Acknowledgement **23**

References **23**

Abstract

This paper contains results in the field of algebraic graph theory and specifically concerns the spectral radius of the Laplacian matrix of a tree. Let $A(G)$ denote the adjacency matrix of a simple graph G. Then, the Laplacian matrix of G is given by $L(G) = D(G) - A(G)$ where D is the diagonal matrix whose diagonal entries are the vertex degrees. The main result provides an upper bound for the spectral radius of any tree with n vertices and k pendant vertices.

1 Introduction

In reading through several papers on spectral graph theory, I quickly came to understand the importance of knowing the spectral radius of the adjacency and/or Laplacian matrices (we will precisely define these later). Knowing the spectrum allows us to deduce important properties and structural parameters of a graph (e.g. the lowest eigenvalues tell us the algebraic connectivity, while the highest and lowest eigenvalues determine the spread of a graph). In this paper, we specifically focus on an upper bound for the spectrum of the Laplacian matrix of a tree. The results are from the recent paper of Honga and Zhang entitled "Sharp upper and lower bounds for largest eigenvalue of the Laplacian matrices of trees" [5].

Definitions and Objective

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix $A = A(G)$ is defined to be the $n \times n$ matrix (a_{ij}), where $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. Also, for each i, let d_i denote the degree of each vertex v_i in G. In this paper, we let $D = D(G)$ be the $n \times n$ diagonal matrix, where i^{th} diagonal entry is d_i.

We define the *spectral radius* of G to be the parameter $\rho(G) = \max_i(|\lambda_i|)$, where the maximum is taken over all the eigenvalues λ_i of the adjacency matrix $A(G)$. The adjacency matrix is a 0,1-symmetric matrix, so that every eigenvalue is real. By the Perron-Frobenius Theorem (cf. [1] or [3]), the spectral radius $\rho(G)$ is a simple eigenvalue (so the algebraic and geometric multiplicity is 1), and there is a unique positive (positive for each entry) unit eigenvector. We refer to such an eigenvector as the *Perron vector* of G.

We define the *Laplacian matrix* L to be the matrix $L(G) = D(G) - A(G)$ and the *signless Laplacian matrix* Q as $Q(G) = D(G) + A(G)$. We refer to the spectral radius of L as the *Laplacian spectral radius* of G and denote this by $\mu(G)$. Similarly, we use the term *signless Laplacian spectral radius* for the spectral radius of $Q(G)$ and denote this by $\nu(G)$. In this paper, we will provide numerous examples of these parameters that are relevant to our primary lemmas and theorems. In each of these cases, the spectrum of the adjacency, Laplacian, and signless Laplacian matrices are obtained by the software *Matlab*.

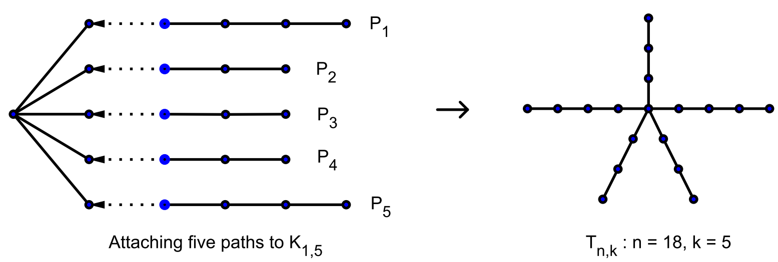
The Laplacian matrix is famously known to be a positive semidefinite matrix (cf. [4], for example). Therefore, we can place each eigenvalue of $L(G)$ in nonincreasing order as follows:

$$\mu(G) = \mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$$

Observe that the Laplacian spectral radius is simply the largest eigenvalue of $L(G)$.

By a *pendant vertex* of G, we mean a vertex whose degree is 1. We also use $N_G(v)$ to denote the set of vertices adjacent to a vertex v in G. We will let $\mathscr{T}_{n,k}$ denote the set of trees with n vertices and k pendant vertices for fixed n and k.

Now, for any fixed n and k, we define $T_{n,k}$ to be a tree graph obtained from a complete bipartite graph (we call this a *star* graph) $K_{1,k}$ and k paths of almost equal length, by joining each pendant vertex of $K_{1,k}$ to an end vertex of one path. To construct $T_{18,5}$, for example, we would start with the star $K_{1,5}$ and with five corresponding paths of almost equal length. Of course, we need to specify what we mean by "almost equal length" for the five paths we wish to attach to the star. Consider that, by the division algorithm, $n-1 = 18-1 = 17$ and $q \cdot k + r = 3 \cdot 5 + 2$. Besides a single, non-pendant vertex, we need 17 more vertices. Then, we need each path to have at least 3 vertices (this number is given by quotient, $q = 3$). Also, 2 paths (this number is given by the remainder $r = 2$) have one more extra vertex than the other paths. Using this construction, no two paths differ in length by more than one, thereby fulfilling the "almost equal length" condition.



To state it more generally, we construct $T_{n,k}$ by appending k paths to the pendant vertices of $K_{1,k}$. The number of vertices in $r < k$ of the paths is $q+1$, and the rest contain exactly q, where the nonnegative parameters are determined by the division algorithm:

$$n - 1 = qk + r.$$

The goal for this paper is to show that the Laplacian spectral radius for any tree $T \in \mathscr{T}_{n,k}$ is bounded by the Laplacian spectral radius of $T_{n,k}$. That is, the inequality

$$\mu(T) \leq \mu(T_{n,k})$$

holds, and the equality occurs if and only if T is isomorphic to $T_{n,k}$.

2 Sharp upper bound

We begin with a sequence of preliminary lemmas which we need in proving our main results.

2.1 Some preliminary lemmas

Lemma 2.1. *Let A be a nonnegative symmetric matrix and x be a unit vector of $\mathbb{R}^n$. If $\rho(A) = x^T Ax$, then $Ax = \rho(A)x$.*

Proof. Since A is a real symmetric matrix, every eigenvalue is real. Suppose that the multiplicity of $\rho(A)$ is a positive integer m, and order each eigenvalue in nonincreasing order. Then we have

$$\rho(A) = \lambda_1 = \lambda_2 = \cdots = \lambda_m > \lambda_{m+1} \geq \cdots \geq \lambda_n.$$

Since A is real symmetric, we know that $\mathbb{R}^n$ has an orthonormal basis of eigenvectors of A. Let α_i be a unit eigenvector associated with λ_i (so that $\alpha_i^T \alpha_j = 0$ for $i \neq j$). Thus, a unit vector $x \in \mathbb{R}^n$ can be written as a linear combination

$$x = a_1\alpha_1 + a_2\alpha_2 \cdots + a_n\alpha_n, \text{ where } a_i \in \mathbb{R} \text{ for } i = 1, 2, \ldots, n.$$

Given a unit vector x, the fact that $\alpha_i^T \alpha_j = 0$ $(i \neq j)$ implies the following:

$$1 = x^T \cdot x = (a_1\alpha_1^T + a_2\alpha_2^T \cdots + a_n\alpha_n^T) \cdot (a_1\alpha_1 + a_2\alpha_2 \cdots + a_n\alpha_n) = a_1^2 + a_2^2 \cdots + a_n^2.$$

By this result, we have

$$\begin{aligned}
\rho(A) = x^T Ax &= x^T(A(a_1\alpha_1 + a_2\alpha_2 \cdots + a_n\alpha_n)) \\
&= x^T(a_1A\alpha_1 + a_2A\alpha_2 + \cdots + a_nA\alpha_n) \\
&= x^T(a_1\lambda_1\alpha_1 + a_2\lambda_2\alpha_2 + \cdots + a_n\lambda_n\alpha_n) \\
&= (a_1\alpha_1 + a_2\alpha_2 \cdots + a_n\alpha_n)^T(a_1\lambda_1\alpha_1 + a_2\lambda_2\alpha_2 + \cdots + a_n\lambda_n\alpha_n) \\
&= a_1^2\lambda_1 + a_2^2\lambda_2 + \cdots + a_m^2\lambda_m + a_{m+1}^2\lambda_{m+1} \cdots + a_n^2\lambda_n \\
&\leq a_1^2\lambda_1 + a_2^2\lambda_1 + \cdots + a_m^2\lambda_1 + a_{m+1}^2\lambda_1 + \cdots + a_n^2\lambda_1 \\
&= \lambda_1(a_1^2 + a_2^2 + \cdots + a_n^2) \\
&= \lambda_1 \\
&= \rho(A).
\end{aligned}$$

We start and end with $\rho(A)$. Containing the inequality above, this only makes sense if $a_{m+1} = a_{m+2} = \cdots = a_n = 0$ holds (otherwise, the inequality is strict and a contradiction occurs). Thus, we may conclude that $x = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m$. Recall that each α_i $(1 \leq i \leq m)$ is an eigenvector associated with $\rho(A)$. Since x is expressible by a linear combination of such α_is, it lies in the eigenspace associated with $\rho(A)$. That is, $Ax = \rho(A)x$ must hold. □

Lemma 2.2. *If G is a bipartite graph, then $D(G) + A(G)$ and $D(G) - A(G)$ have the same spectrum.*

Proof. Suppose that G is a bipartite graph with n vertices. Let $V_1 = \{v_1, v_2, \ldots, v_k\}$ and $V_2 = \{v_{k+1}, v_{k+2}, \ldots, v_n\}$ denote the corresponding sets of vertices in each cell of the bipartition. We know that each cell is a independent set, so if we arrange the rows and columns of the adjacency matrix in the order $v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n$, then we see block matrices of zero entries, as shown on the left below:

$$\begin{array}{cc} A(G) & \begin{array}{ccc|ccc} v_1 & \cdots & v_i & v_{i+1} & \cdots & v_n \end{array} \\ \begin{array}{c} v_1 \\ \vdots \\ v_i \\ v_{i+1} \\ \vdots \\ v_n \end{array} & \left[\begin{array}{c|c} \mathbf{0} & \mathbf{B} \\ \hline \mathbf{B}^T & \mathbf{0} \end{array}\right] \end{array} \qquad \begin{array}{cc} D(G) & \begin{array}{ccc|ccc} v_1 & \cdots & v_i & v_{i+1} & \cdots & v_n \end{array} \\ \begin{array}{c} v_1 \\ \vdots \\ v_i \\ v_{i+1} \\ \vdots \\ v_n \end{array} & \left[\begin{array}{c|c} \mathbf{D}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{D}_2 \end{array}\right] \end{array}$$

At this point, the submatrix denoted $\mathbf{B}$ above will appear as an arbitrary 0,1-submatrix. Now, let us also represent the matrix $D(G)$ with corresponding block matrices as shown on the right above. Note that the dimension of these blocks matches the blocks in $A(G)$, so that the addition of block matrices makes sense. We now have a block matrix representation for $D + A$ and $D - A$.

$$L = D - A = \begin{bmatrix} D_1 & -B \\ -B^T & D_2 \end{bmatrix} \qquad Q = D + A = \begin{bmatrix} D_1 & B \\ B^T & D_2 \end{bmatrix}$$

Now, let

$$S = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$$

be the block matrix representation of an $n \times n$ matrix where I is an appropriate sized identity matrix. The dimension of each block in S is chosen to match the blocks in L and Q, so that the multiplication of block matrices makes sense. Now, we see that S is self-inverse, so $S = S^{-1}$ holds. Then,

$$SLS^{-1} = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D_1 & -B \\ -B^T & D_2 \end{bmatrix} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} D_1 & B \\ B^T & D_2 \end{bmatrix} = Q$$

which shows that the matrices L and Q are similar. Recall that similar matrices must have the same eigenvalues. Therefore, the statement holds. $\square$

Example. Let us see some examples of the above lemma. Figure 1 below is a non-bipartite graph, having an odd cycle.

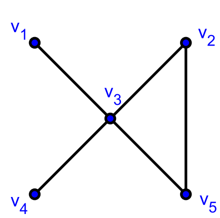


Figure 1 : a graph G

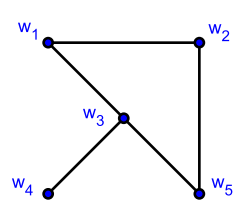


Figure 2: a bipartite graph B

Label vertices in G as it is shown, and notice that the Laplacian and signless Laplacian matrices of G are the following:

$$\begin{array}{c|c} L(G) & \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{bmatrix} \end{array} \qquad \begin{array}{c|c} Q(G) & \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix} \end{array}$$

The spectrum of $L(G)$ is given by (Matlab)

$$\{0.0000, 1.0000, 1.0000, 3.0000, 5.0000\},$$

whereas the spectrum of $Q(G)$ is given by

$$\{0.3187, 1.0000, 1.0000, 2.3579, 5.3234\},$$

which we see that L and Q have very different spectra. Now, let us consider Figure 2. This graph B contains no odd cycle, so B is a bipartite graph.

$$\begin{array}{c|c} L(B) & \begin{matrix} w_1 & w_2 & w_3 & w_4 & w_5 \end{matrix} \\ \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{matrix} & \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 3 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{bmatrix} \end{array} \qquad \begin{array}{c|c} Q(B) & \begin{matrix} w_1 & w_2 & w_3 & w_4 & w_5 \end{matrix} \\ \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{matrix} & \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 \\ 1 & 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix} \end{array}$$

The spectra of both $L(B)$ and $Q(B)$ are given by

$$\{0.0000, 0.8299, 2.0000, 2.6889, 4.4812\}$$

which is what is asserted by the lemma.

Lemma 2.3 (Li and Feng [6])**.** *Let u be a vertex of the connected graph G and for positive integers k and l, let $G_{k,l}$ denote the graph obtained from G by adding pendant paths of length k and l at u. If $k \geq l \geq 1$, then*

$$\rho(G_{k,l}) > \rho(G_{k+1,l-1}).$$

Lemma 2.4 (Li and Feng [6])**.** *Let u and v be two adjacent vertices of the connected graph G, and for nonnegative integers k and l, $G_{k,l}$ denote the graph obtained from G by adding pendant paths of length k and l at u and v, respectively. If $k \geq l \geq 1$, then*

$$\rho(G_{k,l}) > \rho(G_{k+1,l-1}).$$

These lemmas are similar except that, where Lemma 2.3 allows us to consider paths joined at a single vertex, Lemma 2.4 enables us to choose two distinct vertices (but they need to be adjacent). For an example, first take a "house graph" with labels as shown (see Figure 1 below). Then, let $u = v_1$, $k = 2$ and $l = 2$, and observe that $H_{2,2}(u)$ designates the new graph formed by following by the procedure of Lemma 2.3 (see Figure 2).

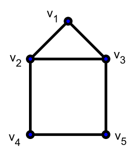


Figure1 : a house graph H

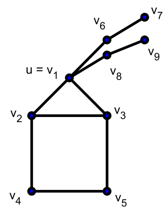


Figure2:$H_{2,2}(u)$

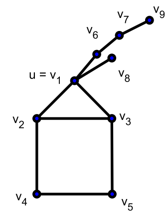


Figure3:$H_{3,1}(u)$

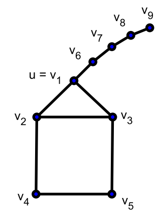


Figure4:$H_{4,0}(u)$

Following the lemma, we expect $H_{2,2}(u)$ to have the largest spectral radius of all three graphs.

$$\begin{array}{c} A(H_{2,2}(u)) \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} A(H_{3,1}(u)) \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

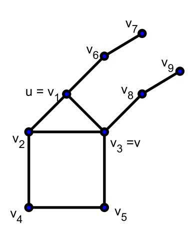
$$\begin{array}{c} A(H_{4,0}(u)) \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

According to the above matrices, their spectral radii are

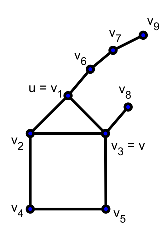
$$\rho(H_{2,2}(u)) = 2.6883, \rho(H_{3,1}(u)) = 2.6751, \text{ and } \rho(H_{4,0}(u)) = 2.5813$$

which is as desired.

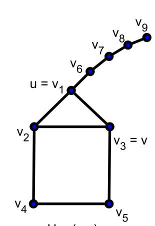
Next, take the house graph again, and let $u = v_1$ and $v = v_3$ (shown below).



$H_{2,2}(u,v)$



$H_{3,1}(u,v)$



$H_{4,0}(u,v)$

Similar to the previous lemma, we expect $H_{2,2}(u,v)$ to have the largest spectral radius. The two matrices below are the adjacency matrices of $H_{2,2}(u,v)$ and $H_{3,1}(u,v)$. Take the matrix $A(H_{4,0}(u))$ for $A(H_{4,0}(u,v))$ since $H_{4,0}(u)$ is the same graph as $H_{4,0}(u,v)$.

$$\begin{array}{cc} A(H_{2,2}(u,v)) & \begin{array}{ccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{array} & \left[\begin{array}{ccccccccc} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \begin{array}{cc} A(H_{3,1}(u,v)) & \begin{array}{ccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{array} & \left[\begin{array}{ccccccccc} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \end{array}$$

The spectral radii are given by

$$\rho(H_{2,2}(u,v)) = 2.6989, \rho(H_{3,1}(u,v)) = 2.6839, \text{ and } \rho(H_{4,0}(u,v)) = 2.5813.$$

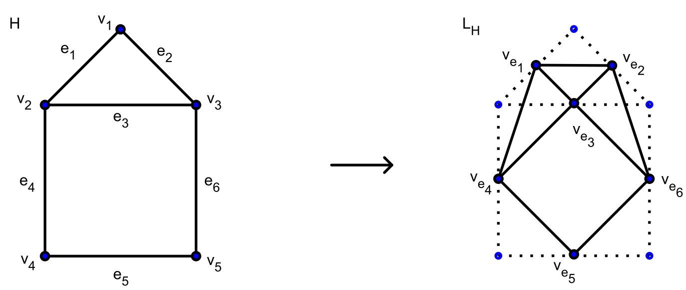
which is as desired.

Lemma 2.5 (Shu et al [8])**.** *Let G be a simple connected graph and L_G be the line graph of G. Then*

$$\mu(G) \leq 2 + \rho(L_G),$$

where equality holds if and only if G is a bipartite graph.

Example. Take again the house graph H with edge labels shown below. Let L_H denote its line graph.



The matrix on the left below is the Laplacian matrix for H, and the matrix on the right is the adjacency matrix for L_H.

$$\begin{array}{cc} L(H) & \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \left[\begin{array}{ccccc} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{array}\right] \end{array} \quad \begin{array}{cc} A(L_H) & \begin{array}{cccccc} v_{e_1} & v_{e_2} & v_{e_3} & v_{e_4} & v_{e_5} & v_{e_6} \end{array} \\ \begin{array}{c} v_{e_1} \\ v_{e_2} \\ v_{e_3} \\ v_{e_4} \\ v_{e_5} \\ v_{e_6} \end{array} & \left[\begin{array}{cccccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The above matrices give us the result that $\mu(H) = 4.6180$ and $\rho(L_H) = 6.4081$. Since $4.6180 \leq 2 + 6.4081$, the inequality $\mu(H) \leq 2 + \rho(L_H)$ holds as desired.

Example. Take a bipartite graph B, as shown in Figure 1. For convenience, we redraw this graph, and label the edges as shown in Figure 2. Finally, Figure 3 is its line graph.

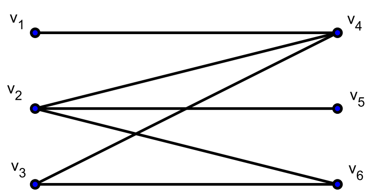


Figure 1: a bipartite graph: B

$\longrightarrow$

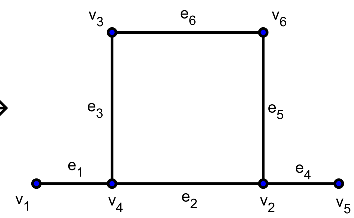


Figure 2 : redrawing B

$\longrightarrow$

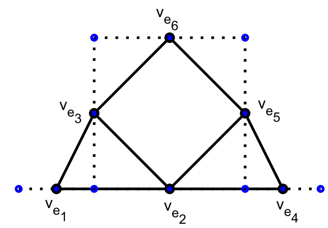


Figure 3 : a line graph: L_B

Their matrices are

$$
\begin{array}{c c}
L(B) & \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} &
\begin{bmatrix}
1 & 0 & 0 & -1 & 0 & 0 \\
0 & 3 & 0 & -1 & -1 & -1 \\
0 & 0 & 2 & -1 & 0 & -1 \\
-1 & -1 & -1 & 3 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 \\
0 & -1 & -1 & 0 & 0 & 2
\end{bmatrix}
\end{array}
\quad
\begin{array}{c c}
A(L_B) & \begin{array}{cccccc} v_{e_1} & v_{e_2} & v_{e_3} & v_{e_4} & v_{e_5} & v_{e_6} \end{array} \\
\begin{array}{c} v_{e_1} \\ v_{e_2} \\ v_{e_3} \\ v_{e_4} \\ v_{e_5} \\ v_{e_6} \end{array} &
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}
\end{array}.
$$

According to matrices above, we have $\mu(B) = 4.8136$ and $\rho(L_B) = 2.8136$. Since $\mu(B) = 4.8136 = 2 + 2.8136 = \rho(L_B)$, we see that the equality holds for a bipartite graph.

2.2 Main Results

To give a bound for the Laplacian spectral radius of $T \in \mathscr{T}_{n,k}$ and $T_{n,k}$, we might want to reconstruct (e.g., deleting and adding edges) $T_{n,k}$ from T, step by step, and watch how each step affects the Laplacian spectral radius.

If a certain inequality is guaranteed for each step of reconstruction, then we are much closer to our goal. The coming theorem gives a result for the signless Laplacian spectral radius. Because the construction might be complicated at first, let us begin with an example demonstrating each step.

Example. Take a graph H with labels (Figure 1). We call this the house graph.

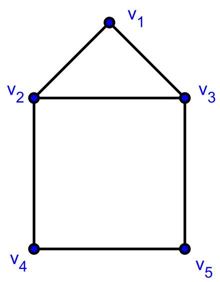


Figure 1: house graph H

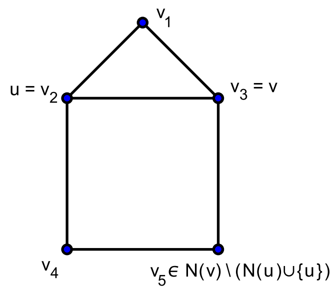


Figure 2: the neighbor of v but not u

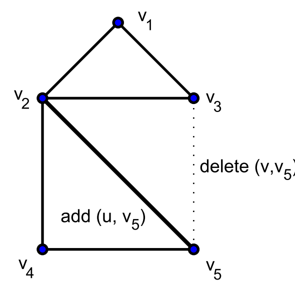


Figure 3 : reconstruction of H

First, we find the signless Laplacian matrix $Q(H)$ and its spectral radius $\nu(H)$. The matrix is shown below and we obtain $\nu(H) = 5.1149$.

$$\begin{array}{cc} Q(H) & \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \left[\begin{array}{ccccc} 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right] \end{array}$$

Recall that we have its associated positive unit eigenvector $\mathbf{x}$ (Perron vector):

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ 0.3797 & 0.5914 & 0.5914 & 0.2796 & 0.2796 \end{bmatrix}^T$$

Label each entry of $\mathbf{x}$ as x_i (for $0 \le i \le 5$) as shown. Then, let x_i correspond to v_i (e.g., x_3 corresponds to v_3 because they share the same index). Now, take two distinct vertices $u, v \in \{v_1, v_2, v_3, v_4, v_5\}$ such that $x_u \ge x_v$ holds. For example, let us choose $u = v_2$ and $v = v_3$, so that $x_u \ge x_v$ is preserved. Now, consider the set of vertices $N_H(v) \setminus (N_H(u) \cup \{u\})$. Looking at Figure 2 above, we see that the neighbors of v are v_1, v_2, and v_5. However, we do not count v_1 and v_2, since $v_1 \in N(u)$ and $v_2 = u$. We have $v_5 \in N(v) \setminus (N(u) \cup \{u\})$ but $v_1, v_2 \notin N(v) \setminus (N(u) \cup \{u\})$.

Now, we delete an edge (v, v_5) and add (u, v_5) (shown in Figure 3), and we denote this graph as H^*. The construction ends here. Then, the next theorem guarantees the inequality $\nu(H) < \nu(H^*)$.

Let us see this in our example. The matrix below is the signless Laplacian matrix for H^*.

$$\begin{array}{cc} Q(H^*) & \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \left[\begin{array}{ccccc} 2 & 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right] \end{array}$$

where its spectral radius is 5.5616, which we see $\nu(H) = 5.1149 < 5.5616 = \nu(H^*)$.

Now, we are ready to state the theorem which generalizes the example above.

Theorem 2.1. *Let u, v be two vertices of G and d_v be the degree of vertex v. Suppose $v_1, v_2, \ldots, v_s$ $(1 \leq s \leq d_v)$ are some vertices of $N_G(v)\backslash(N_G(u) \cup \{u\})$ and $x = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector of $D(G) + A(G)$, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges (v, v_i) and adding the edges (u, v_i) $(1 \leq i \leq s)$. If $x_u \geq x_v$, then $\nu(G) < \nu(G^*)$.*

Proof. Let G be an n-vertex graph and let $V(G) = \{v_1, \ldots, v_n\}$. Relabel if necessary, so that $\{v_1, \ldots, v_s\}$ are some vertices of $N_G(v)\backslash(N_G(u) \cup \{u\})$. We need to verify the inequality between $\nu(G)$ and $\nu(G)^*$, so let us first investigate the signless Laplacian matrices of G and G^*.

Without loss of generality, let $v_{s+1} = u$ and $v_{s+2} = v$.

$$\begin{array}{cc}
Q(G) & \begin{array}{ccc|c|c|ccc} v_1 & \cdots & v_s & u & v & v_{s+3} & \cdots & v_n \end{array} \\
\begin{array}{c} v_1 \\ \vdots \\ v_s \\ u \\ v \\ v_{s+3} \\ \vdots \\ v_n \end{array} &
\left[\begin{array}{ccc|c|c|ccc}
d_1 & & * & 0 & 1 & & & \\
& \ddots & & \vdots & \vdots & & \mathbf{A}_1 & \\
* & & d_s & 0 & 1 & & & \\ \hline
0 & \cdots & 0 & d_u & * & & \mathbf{r}_1 & \\ \hline
1 & \cdots & 1 & * & d_v & & \mathbf{r}_2 & \\ \hline
& & & & & d_{s+3} & & * \\
& \mathbf{A}_1^T & & \mathbf{r}_1^T & \mathbf{r}_2^T & & \ddots & \\
& & & & & * & & d_n
\end{array}\right]
\end{array}$$

The off-diagonal entries are either 0 or 1, which are determined by $A(G)$. However, the 1^{st} to s^{th} entries of the u^{th} and v^{th} columns (and also rows) are completely determined by the assumption that $v_i \in N_G(v)\backslash(N_G(u) \cup \{u\})$ for $1 \leq i \leq s$. Vertices u and v may be adjacent, so leave this entry undetermined as $*$. We denote the rest of the blocks by $\mathbf{A}_1$, $\mathbf{r}_1$, and $\mathbf{r}_2$.

We next describe $Q(G^*)$ (or Q^* as an abbreviation) for the new graph. Note that deleting edges (v, v_i) and adding (u, v_i) will not affect any entries other than the u^{th} and v^{th} columns (and also rows), and the degrees of u and v.

$$\begin{array}{cc}
Q(G^*) & \begin{array}{ccc|c|c|ccc} v_1 & \cdots & v_s & u & v & v_{s+3} & \cdots & v_n \end{array} \\
\begin{array}{c} v_1 \\ \vdots \\ v_s \\ u \\ v \\ v_{s+3} \\ \vdots \\ v_n \end{array} &
\left[\begin{array}{ccc|c|c|ccc}
d_1 & & * & 1 & 0 & & & \\
& \ddots & & \vdots & \vdots & & \mathbf{A}_1 & \\
* & & d_s & 1 & 0 & & & \\ \hline
1 & \cdots & 1 & d_u^* & * & & \mathbf{r}_1 & \\ \hline
0 & \cdots & 0 & * & d_v^* & & \mathbf{r}_2 & \\ \hline
& & & & & d_{s+3} & & * \\
& \mathbf{A}_1^T & & \mathbf{r}_1^T & \mathbf{r}_2^T & & \ddots & \\
& & & & & * & & d_n
\end{array}\right]
\end{array}$$

Let d_u^* and d_v^* denotes the degree of u and v in G^*. Deleting (v, v_i) for $1 \leq i \leq s$, we have $d_v^* = d_v - s$. Similarly, adding (u, v_i) for $1 \leq i \leq s$ would give $d_u^* = d_u + s$.

Having two matrix representations, now consider $Q^* - Q$ which is shown below.

$$
\begin{array}{cc}
Q^*-Q & \begin{array}{ccc|c|c|ccc} v_1 & \cdots & v_s & u & v & v_{s+3} & \cdots & v_n \end{array} \\
\begin{array}{c} v_1 \\ \vdots \\ v_s \\ u \\ v \\ v_{s+3} \\ \vdots \\ v_n \end{array} &
\left[\begin{array}{ccc|c|c|c}
 & & & 1 & -1 & \\
 & \mathbf{0} & & \vdots & \vdots & \mathbf{0} \\
 & & & 1 & -1 & \\ \hline
1 & \cdots & 1 & s & 0 & \mathbf{0} \\ \hline
-1 & \cdots & -1 & 0 & -s & \mathbf{0} \\ \hline
 & & & & & \\
 & \mathbf{0} & & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
 & & & & &
\end{array}\right]
\end{array}
$$

Most of the blocks have the same entries for Q and Q^*. This yields several 0 blocks in $Q^* - Q$. Now, let $x = \{x_1, x_2, \ldots, x_s, x_{s+1}, x_{s+2}, x_{s+3}, \ldots, x_n\}$ be the Perron vector given by the assumption, and consider $x^T Q^* x - x^T Qx$. (Note: we use $x_u = x_{s+1}$ and $x_v = x_{s+2}$ from now on.) Computation is now easy by looking at the above matrices.

$$x^T Q^* x - x^T Qx = x^T(Q^* - Q)x = 2\sum_{i=1}^{s} x_i(x_u - x_v) + s(x_u^2 - x_v^2)$$

The assumption $x_u \geq x_v$ yields $2\sum_{i=1}^{s} x_i(x_u - x_v) + s(x_u^2 - x_v^2) \geq 0$, so that $x^T Q^* x - x^T Qx \geq 0$. Thus, we have $x^T Q^* x \geq x^T Qx$. Then, the Rayleigh quotient with this result yields the following:

$$\nu(G^*) = \max_{\|y\|=1} y^T Q^* y \geq x^T Q^* x \geq x^T Qx = \nu(G).$$

Now, we show that this inequality is strict, by way of contradiction. If equality holds, then we must have $\nu(G^*) = x^T Q^* x = x^T Qx = \nu(G)$ by the above expression. Also, the preceding lemma implies that we have $\nu(G^*)x = Q^* x$ and $\nu(G)x = Qx$. Consider the v^{th} entry of the vector $\nu(G^*)x$.

$$(\nu(G^*)x)_v = (Q^* x)_v = d_v^* \cdot x_v + \sum_{v_i \in N_{G^*}(v)} x_i = (d_v - s)x_v + \sum_{v_i \in N_{G^*}(v)} x_i$$

Similarly, the v^{th} entry of the vector $\nu(G)x$ is

$$(\nu(G)x)_v = (Qx)_v = d_v \cdot x_v + \sum_{v_i \in N_G(v)} x_i$$

Then, refer to the v^{th} row of matrices Q and Q^*, and we get the following:

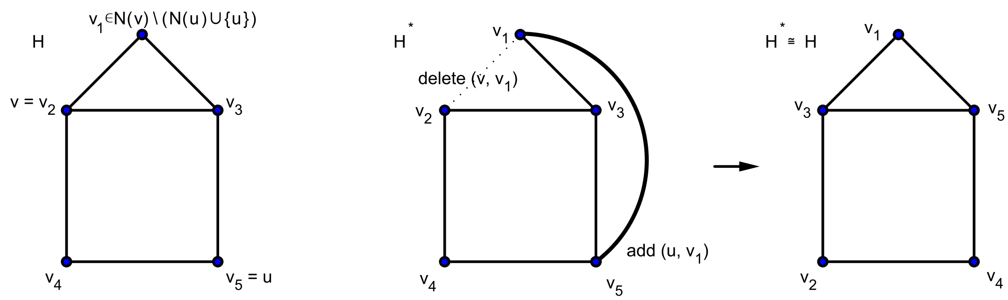
$$\sum_{v_i \in N_G(v)} x_i = \sum_{i=1}^{s} x_i + \sum_{v_i \in N_{G^*}(v)} x_i$$

which yields that

$$(\nu(G)x)_v = d_v \cdot x_v + \sum_{i=1}^{s} x_i + \sum_{v_i \in N_{G^*}(v)} x_i$$

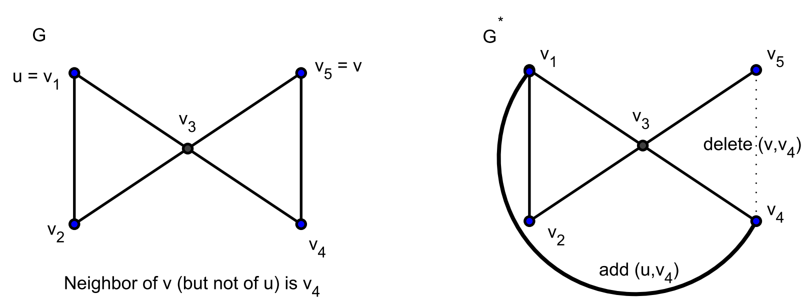
Now, compare $(\nu(G^*)x)_v$ and $(\nu(G)x)_v$. Since each entry in the Perron vector is positive, we must have $(\nu(G^*)x)_v < (\nu(G)x)_v$, from which we can deduce $\nu(G) < \nu(G^*)$. However, this contradicts our assumption, and therefore, the inequality must be strict. □

Example. Let us again consider the house graph, and this time, we take $u = v_5$ and $v = v_2$. Recall the Perron vector from the above example. We have $x_u < x_v$, which does not satisfy the assumption.



The above figure shows that applying the construction would derive a new graph H^* which is the same as H. Thus, $Q(H^*)$ must have the same spectral radius, and we see that the theorem does not hold.

Example. Now, let us consider the figure below.



The signless Laplacian matrix $Q(G)$ is shown on the left below:

$$\begin{array}{cc} Q(G) & \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 4 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \end{array} \qquad \begin{array}{cc} Q(G^*) & \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{bmatrix} 3 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 4 & 1 & 1 \\ 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix} \end{array}$$

Taking this matrix, we find that $\nu(G) = 5.5616$, and its Perron vector is given by

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ 0.3077 & 0.3077 & 0.7882 & 0.3077 & 0.3077 \end{bmatrix}^T$$

We take $v = v_5$ and $u = v_1$, so that $x_u \geq x_v$ is preserved. A graph G^* is shown on the right above, and its signless Laplacian matrix $Q(G^*)$ is also shown on the right above. The spectral radius of $Q(G^*)$ is 5.7785. The statement $\nu(G) < \nu(G^*)$ therefore holds.

Theorem 2.2. *Let T be a tree with n vertices and k pendant vertices. Then*

$$\mu(T) \leq \mu(T_{n,k}),$$

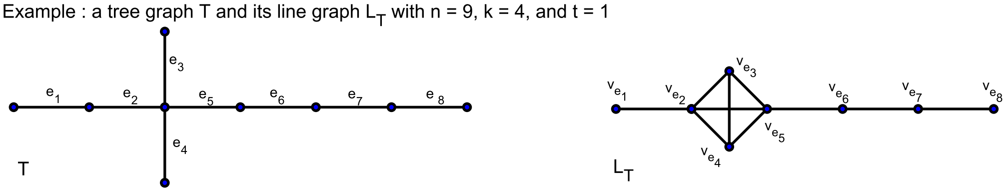
where equality holds if and only if T is isomorphic to $T_{n,k}$.

Proof. Let t be the number of vertices whose degrees are at least 3 (let us call such a vertex a *branch vertex*). We divide the argument into cases.

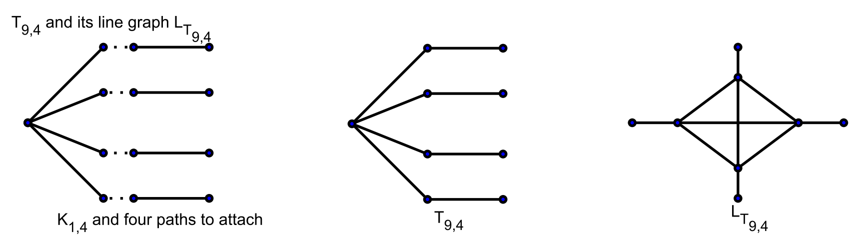
Case 1: If $t = 0$, then the tree does not have any branches. Each vertex has degree at most 2 so the graph must be a path. Notice that any path can be expressed as $T_{n,2}$, and thus, $T = T_{n,2}$ shows $\mu(T) = \mu(T_{n,2})$.

Case 2: If $t = 1$, then consider its line graph L_T. Note that the edges which are incident to the branch vertex would form a clique (complete subgraph) in the line graph.

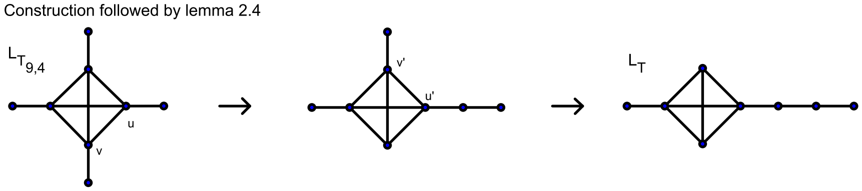
Example : a tree graph T and its line graph L_T with n = 9, k = 4, and t = 1



Let k be the degree of the branch vertex. Then, the line graph contains a complete graph K_k. Now, we see that this line graph can be obtained by adding paths $P_1, P_2, \ldots, P_k$ to each vertex of K_k. Then, notice that by applying Lemmas 2.3 and 2.4 (repeatedly if necessary), we obtain the line graph of $T_{n,k}$ with inequality $\rho(L_T) < \rho(L_{T_{n,k}})$.



Construction followed by lemma 2.4

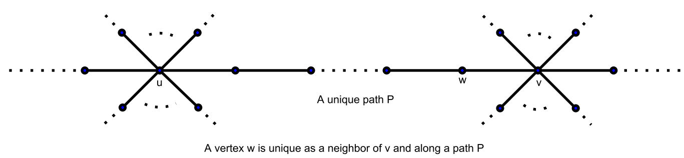


Since the trees T and $T_{n,k}$ are bipartite, $\mu(T) = 2 + \rho(L_T)$ and $\mu(T_{n,k}) = \rho(L_{T_{n,k}})$ by Lemma 2.5. Therefore, we have the following inequality:

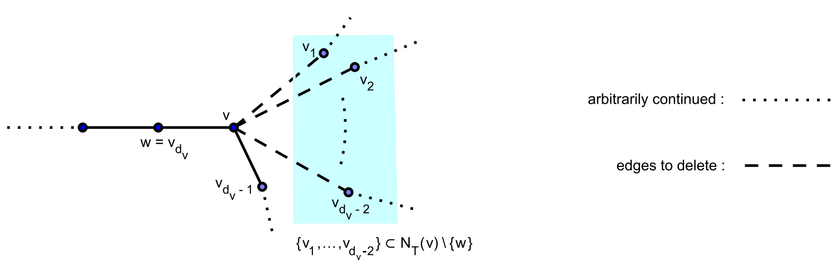
$$\mu(T) = 2 + \rho(L_T) < 2 + \rho(L_{T_{n,k}}) = \mu(T_{n,k}).$$

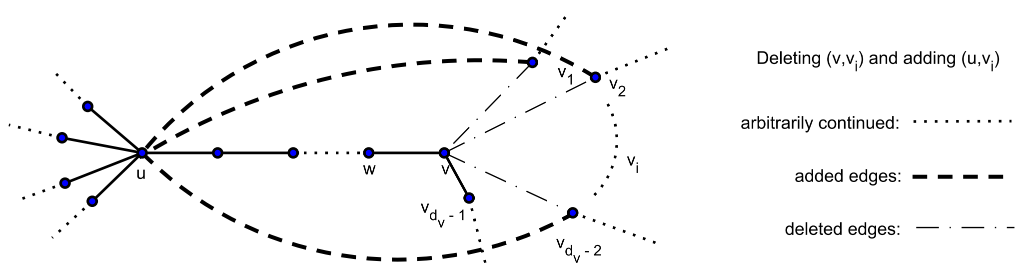
In addition, applying the constructions of Lemmas 2.3 and 2.4 would give the strict inequality. Thus, equality holds if and only if T is isomorphic to $T_{n,k}$.

Case 3: If $t > 1$, then let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of the Laplacian matrix of T. Similar to the proof of Theorem 2.1, each x_i corresponds to vertex v_i $(1 \leq i \leq n)$. Assuming $t > 1$, let u and v be two branch vertices. Then, without loss of generality, let $x_u \geq x_v$. Since T is a tree, there is a unique path P between u and v. Let w be the vertex which is the neighbor of v along the path P.



Now, let d_v be the degree of the vertex v, and consider the proper subset $\{v_1, v_2, \ldots, v_{d_v-2}\} \subset N_G(v) \backslash \{w\}$. We delete the edges (v, v_i) and add the edges (u, v_i) for $1 \leq i \leq d_v - 2$. Let T_1^* denote this new graph. Then, $\nu(T) < \nu(T_1^*)$ holds by Theorem 2.1, and thus, $\mu(T) < \mu(T_1^*)$ holds by Lemma 2.2.





Note that this construction preserves the number of pendant vertices, and the number of branch vertices becomes $t-1$. If $t - 1 > 1$, then take T_1^* and repeat this construction until the number of branches becomes 1. Then, we have the increasing sequence of inequalities

$$\mu(T_1^*) < \mu(T_2^*) < \cdots < \mu(T_{t-1}^*).$$

Note that the number of branch vertices is 1 in T_{t-1}^*, so that the argument in Case 2 can be applied. Thus, $\mu(T_{t-1}^*) < \mu(T_{n,k})$ must hold, and therefore, we conclude $\mu(T) \leq \mu(T_{n,k})$. $\square$

Example (Case 3 construction for Theorem 2.2)**.** Let us consider a tree T with $t = 3$ as depicted in the following figure with labels.

Figure 1: tree graph T with t = 3

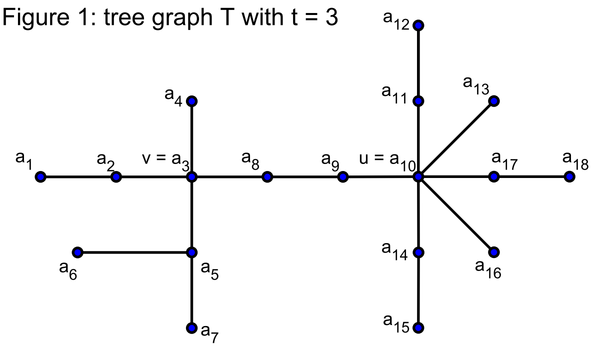
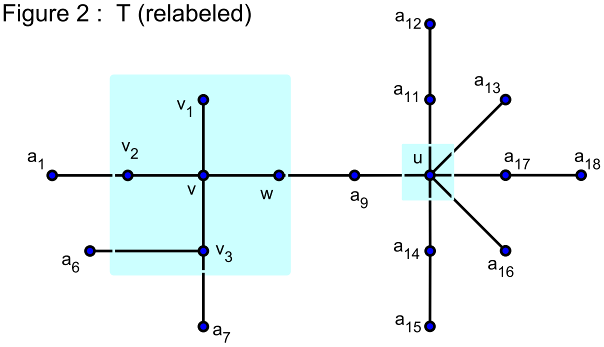


Figure 2 : T (relabeled)



Among three vertices, we need to find the Perron vector and specify u and v such that $x_u \geq x_v$ holds. First, the matrix below is the signless Laplacian matrix of T.

$$
\begin{array}{c}
Q(T) \\
a_1\\a_2\\a_3\\a_4\\a_5\\a_6\\a_7\\a_8\\a_9\\a_{10}\\a_{11}\\a_{12}\\a_{13}\\a_{14}\\a_{15}\\a_{16}\\a_{17}\\a_{18}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \end{array} \\
\begin{bmatrix}
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&2&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&4&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&3&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&2&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&2&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&6&1&0&1&1&0&1&1&0\\
0&0&0&0&0&0&0&0&0&1&2&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&2&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&2&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1
\end{bmatrix}
\end{array}
$$

This matrix gives $\nu(T) = 7.1329$, and its associated eigenvector (Perron vector) is

$$
\mathbf{x} = \left[\begin{array}{ccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \\ 0.0005 & 0.0031 & 0.0155 & 0.0025 & 0.0041 & 0.0007 & 0.0007 & 0.0388 & 0.1839 \end{array} \cdots \right.
$$

$$
\left. \cdots \begin{array}{ccccccccc} v_{10} & v_{11} & v_{12} & v_{13} & v_{14} & v_{15} & v_{16} & v_{17} & v_{18} \\ 0.9049 & 0.1821 & 0.0297 & 0.1475 & 0.1821 & 0.0297 & 0.1475 & 0.1821 & 0.0297 \end{array} \right]^T
$$

The vertex a_{10} (corresponding to x_{10}) is associated to the largest value of Perron vector. Thus, let $u = a_{10}$ and $v = a_3$, so that $x_u \geq x_v$ is preserved. The vertex $w = a_8$ is uniquely determined, and then relabel $\{v_1, v_2, v_3\} \subset N_G(v)\backslash\{w\}$ as in Figure 2. Now delete (v, v_i) and add (u, v_i) for $1 \leq i \leq d_v - 2$ where $d_v - 2 = d_{a_3} - 2 = 4 - 2 = 2$ (Figure 3).

Figure 3 : T_1^*

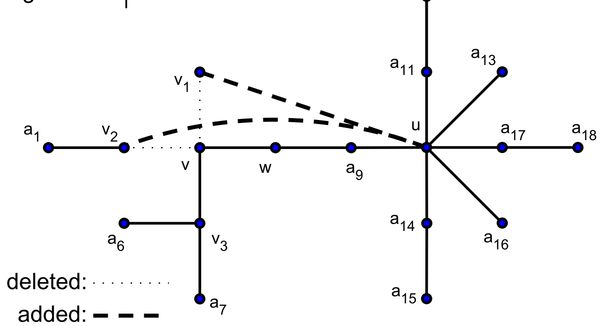
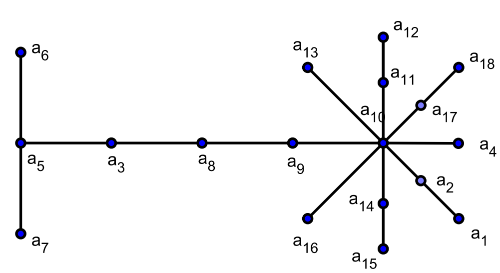


Figure 4 : rearranged T_1^*



Rearrange and relabel T_1^* (Figure 4). Now, this graph has two branch vertices, a_5 and a_{10}. Before we move on to the next construction, let us find the Laplacian spectral radius. The signless Laplacian matrix for T_1^* is shown below.

$$\begin{array}{r} Q(T_1^*) \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \\ a_{14} \\ a_{15} \\ a_{16} \\ a_{17} \\ a_{18} \end{array} \begin{array}{c} \begin{array}{cccccccccccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \end{array} \\ \left[\begin{array}{cccccccccccccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 8 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right] \end{array}$$

This matrix gives $\nu(T_1^*) = 9.0891$. Recall that Lemma 2.2 implies that the spectral radius of $D + A$ and $D - A$ are the same, so we have

$$\mu(T) = \nu(T) = 7.1329 < 9.0891 = \nu(T_1^*) = \mu(T_1^*)$$

as desired. Now, the Perron vector associated with $\nu(T_1^*)$ is the following:

$$\mathbf{x} = \left[\begin{array}{cccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & \cdots \\ 0.0165 & 0.1338 & 0.0028 & 0.1152 & 0.0005 & 0.0001 & 0.0001 & 0.0193 & 0.1342 & \cdots \end{array} \right.$$

$$\left. \begin{array}{cccccccccc} \cdots & v_{10} & v_{11} & v_{12} & v_{13} & v_{14} & v_{15} & v_{16} & v_{17} & v_{18} \\ \cdots & 0.9322 & 0.1338 & 0.0165 & 0.1152 & 0.1338 & 0.0165 & 0.1152 & 0.1338 & 0.0165 \end{array} \right]^T$$

Again, the vertex a_{10} is associated with the largest value in the Perron vector. We apply the same construction as previous which is illustrated in Figures 5 and 6 below.

Figure 5 : let $v = a_5$, $u = a_{10}$, $d_v - 2 = 3 - 2 = 1$

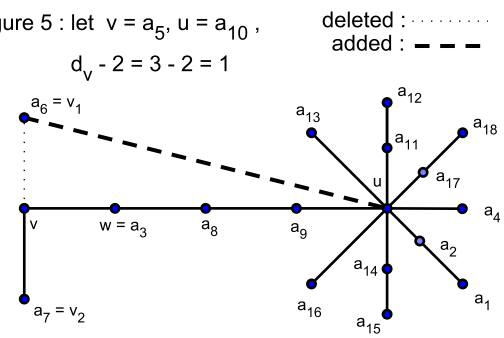
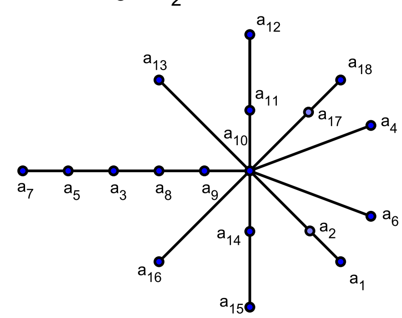


Figure 6 : rearranged T_2^*



The matrix below is the signless Laplacian matrix for T_2^*.

$$\begin{array}{cc}
\hat{L}(T_2^*) & \begin{array}{cccccccccccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \end{array} \\
\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \\ a_{14} \\ a_{15} \\ a_{16} \\ a_{17} \\ a_{18} \end{array} &
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 9 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
\end{array}$$

This matrix gives $\nu(T_2^*) = 10.0695$, so that we have $\mu(T) < \mu(T_1^*) < \mu(T_2^*)$ as desired.

Example. Let us consider Figure 1 below.

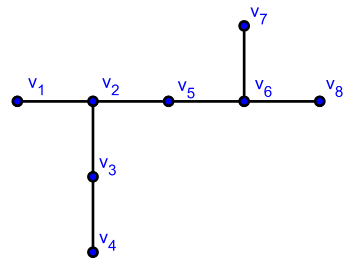


Figure 1 : T with n = 8, k = 4, and t = 2

Since T has $t = 2$ (branch vertices v_2 and v_6), we find the signless Laplacian spectral radius and its associated Perron vector.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

This matrix gives $\nu(T) = 4.4763$ and the Perron vector

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ 0.1780 & 0.6186 & 0.2826 & 0.0813 & 0.4527 & 0.5024 & 0.1445 & 0.1445 \end{bmatrix}^T.$$

We see that x_2 is the larger than x_6, so let $u = v_2$ and $v = v_6$. Let each vertex label be as in Figure 2, and then delete edge (v, v_1) and add (u, v_1). Let T_1^* denote the new graph shown in Figure 3.

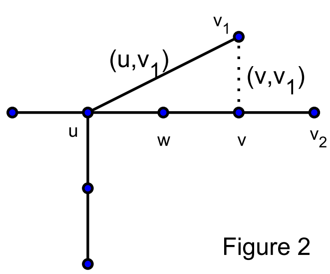


Figure 2

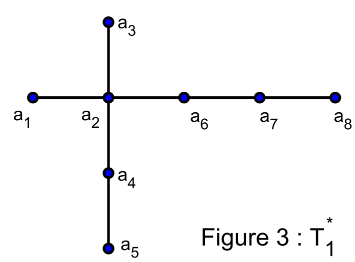


Figure 3 : T_1^*

Label T_1^* as shown, and we get $\mu(T_1^*) = 5.1732$. Now, T_1^* is a tree with $t = 1$. We then find its line graph (Figure 4). Note that this graph can be obtained by adding two paths to a complete graph K_4.

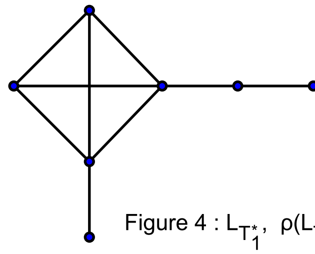


Figure 4 : $L_{T_1^*}$, $\rho(L_{T_1^*}) = 3.1732$

Now, the following figures are the graph $T_{8,4}$ and its line graph $L_{T_{8,4}}$.

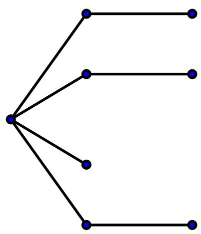


Figure 5 : $T_{8,4}$

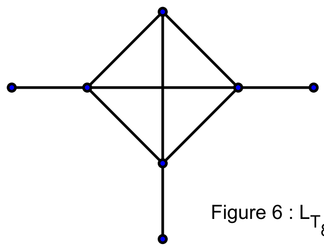


Figure 6 : $L_{T_{8,4}}$, $\rho(L_{T_{8,4}}) = 3.2361$

Notice that, by applying Lemma 2.4 to $L_{T_1^*}$, we obtain $L_{T_{8,4}}$. Now, with all results together, we have

$$\mu(T) < \mu(T_1^*) = 2 + \rho(L_{T_1^*}) < 2 + \rho(L_{T_{8,4}}) = \mu(T_{8,4})$$

which is the desired inequality.

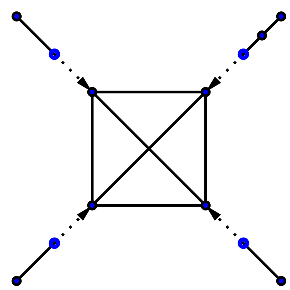
2.3 Consequences

The rest of paper is devoted to several consequences of our main result.

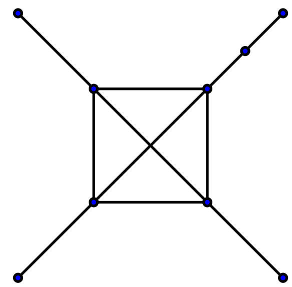
Let G be a connected graph. A *cut vertex* in G is a vertex whose deletion breaks G into at least two parts. Let $\Phi_{n,k}$ be the set of all connected graphs on n vertices that have exactly k cut vertices. The graph $G_{n,k}$ is obtained by adding paths $P_1, \ldots, P_{n-k}$ of almost equal length (by the length of a path, we mean the number of its vertices) to the vertices of the complete graph K_{n-k}.

The construction of $G_{n,k}$ is similar to the one for $T_{n,k}$. We are connecting $n-k$ paths to vertices of a complete graph, K_{n-k}. "Almost equal length" is again determined by the quotient q and the remainder r, given by the division algorithm with n and $n-k$. Thus, we are connecting $n-k$ paths of q vertices, and among these $n-k$ paths, there are r paths with an extra vertex.

Take $n = 9$ and $k = 5$, for example. Since $n - k = 9 - 5 = 4$, we start with a complete graph K_4. Since $9 = 2 \cdot 4 + 1$, the number of vertices in each of the paths is at least 2, and among the three paths, 1 path has an extra vertex.



K_4 and corresponding four paths



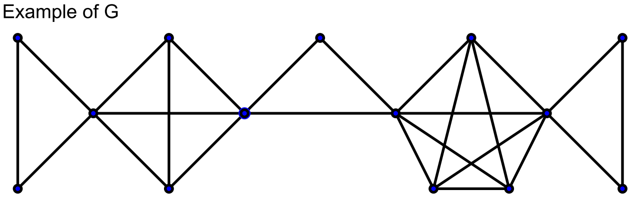
$G_{9,5}$

We see that $G_{9,5}$ really has 9 vertices and 5 cut vertices. Now, we can prove the following corollary as a consequence of our main results.

Corollary 2.1 (Berman and Zhang [2]). *Of all the connected graphs on n vertices and k cut vertices, the maximal spectral radius is obtained uniquely at $G_{n,k}$.*

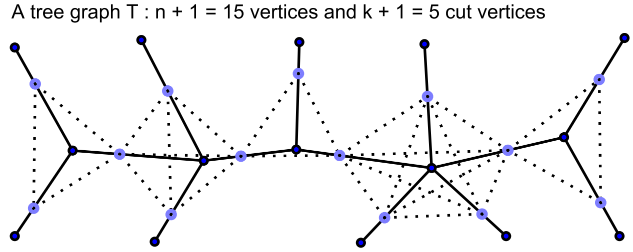
Proof. Take any $G \in \Phi_{n,k}$. We need to show that $\rho(G) \leq \rho(G_{n,k})$ with equality only when $G = G_{n,k}$. The adjacency matrix of a connected graph is irreducible. Thus, adding an edge e to G would increase the spectral radius (stated in [2], or we can see this holds by the Perron Frobenius theorem in [4]). Recall that G has n vertices and k cut vertices. If the above is true, then the spectral radius of $A(G)$ is maximized when each cut vertex connects exactly two blocks which are cliques. Thus, we assume that G is such a graph as just explained, and we will show that $\rho(G) \leq \rho(G_{n,k})$ is still preserved.

Example of G



Recall line graphs from Lemma 2.5 or Theorem 2.2. In a tree, we see that the edges which are incident to a branch vertex would form a complete subgraph in the line graph. That is, there exists a tree T whose line graph is isomorphic to G.

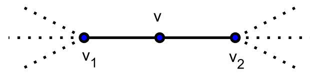
A tree graph T : n + 1 = 15 vertices and k + 1 = 5 cut vertices



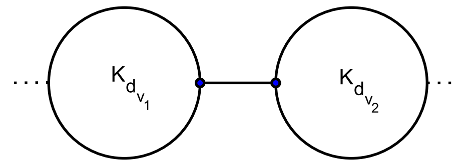
We see that the line graph of T is isomorphic to G. (represented by dotted lines)

There are n vertices in G and $G \cong L_T$. Thus, the number of edges in T must be n, and since T is a tree, there are exactly $n+1$ vertices. In addition, each cut vertex in G connects two blocks, So there are $k+1$ blocks, and thus, T has $k+1$ branch vertices.

Note that, in T, there are no vertices of degree 2. Otherwise, consider its line graph.



A tree graph T, containing a vertex of degree 2



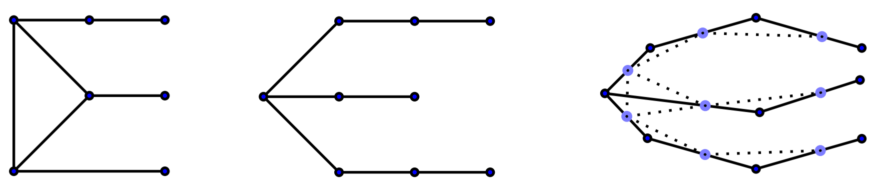
The line graph L_T

In L_T, blocks of cliques are connected by an edge, not a vertex. However, this line graph cannot be isomorphic to G as explained above. Therefore, the claim must hold, and thus, T must consist only of pendant vertices and branch vertices.

Recall that a non-cut vertex in a tree is exactly a pendant vertex. The number of pendant vertices in T is now given by $(n+1)-(k+1) = n-k$.

Now, consider a tree $T_{n+1,n-k}$. Recall that this graph consists of one branch vertex with $n-k$ paths of almost equal length, so that its line graph $L_{T_{n+1,n-k}}$ must be isomorphic to $G_{n,k}$.

Example: n = 8, k = 3



$G_{8,5}$ $T_{9,3}$ The line graph of $T_{8,3}$ is isomorphic to $G_{8,5}$

Then, applying Lemma 2.5 and Theorem 2.2 repeatedly, we get

$$\rho(G) = \rho(L_T) = \mu(T) - 2 \leq \mu(T_{n+1,n-k}) - 2 = 2 + \rho(L_{T_{n+1,n-k}}) - 2 = \rho(G_{n,k})$$

as desired. By the construction, the equality holds only when $G \cong G_{n,k}$. $\square$

Acknowledgement

This MTH 501 project is submitted in partial fulfillment of the requirements for the degree of Masters of Science in Mathematics from Portland State University. Special thanks to my advisor Dr. John Caughman and to my reader Dr. Paul Latiolais.

References

[1] A. Berman, R.J.Plemmons. *Nonnegative Matrices in the Mathematical Sciences*, Academic, New York, 1979 SIAM, Philadelphia, PA, 1994.

[2] A. Berman, X.D. Zhang, *On the spectral radius of graphs with cut vertices*, J. Combin. Theory B 83 (2001) 233-240.

[3] N. Biggs, *Algebraic to Graph Theory*, 2$^{\mathrm{nd}}$ ed. Cambridge University Press, Cambridge, 1995

[4] G. Chris, R. Gordon, *Algebraic Graph Theory*, Springer-Verlag, New York, Inc, 2001

[5] Y. Honga, X.D. Zhang, *Sharp upper and lower bounds for largest eigenvalue of the Laplacian matrices of trees*, Discrete Mathematics 296 (2005) 187197.

[6] Q. Li, K. Feng, *On the largest eigenvalue of a graph*, Acta Math Appl. Sinica 2(1979) 167-175(in Chinese).

[7] R. Merris, *Laplacian matrices of a graphs: a survey*, Linear Algebra Appl. 197-198 (1994) 143-176.

[8] J. Shu, Y.Hong, K. Wenren, *A sharp upper bound on the largest eigenvalue of the Laplacian matrix of a graph*, Linear Algebra Appl. 347 (2002) 123-129