Multi-coloring and Mycielski's construction

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Fall 2010

Abstract

We consider a number of related results taken from two papers – one by W. Lin [1], and the other D. C. Fisher[2]. These articles treat various forms of graph colorings and the famous Mycielski construction. Recall that the Mycielskian $\mu(G)$ of a simple graph G is a graph whose chromatic number satisfies $\chi(\mu(G)) = \chi(G)+1$, but whose largest clique is no larger than the largest clique in G.

Extending the work of Mycielski, the results presented here investigate how the Mycielski construction affects a related parameter called the k^{th} chromatic number $\chi_k(G)$, developing an upper and lower bound for this parameter when applied to $\mu(G)$. We then prove that there are infinite families of graphs that realize both the upper and lower upper bounds.

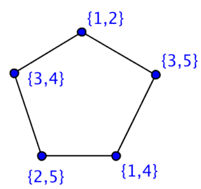
Alongside these main results, we also include a remarkable curiosity, first found by Fisher, that although the fractional chromatic number of a graph G might be expressible as a/b in lowest terms, that does not necessarily imply that there exists a proper coloring of G with the b-element subsets of an a-element set. We also demonstrate that the ratio $\chi_k(G)/k$, where $\chi_k(G)$ denotes the k-tuple chromatic number, is not a strictly decreasing function.

1. Introduction to Graph Colorings

A graph G consists of a finite set $V(G)$ of *vertices* together with a set $E(G)$ of 2-element subsets of $V(G)$. The elements of $E(G)$ are referred to as *edges*. Note that, with this definition, a graph is necessarily *simple*, meaning that there are no loops, multiple edges, or directed edges permitted. Two vertices u and v contained in an edge are said to be *adjacent*, and this is denoted $a \sim b$. We also say v is a *neighbor* of u.

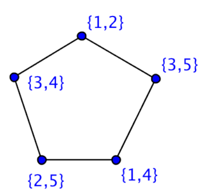
Let G denote a graph and let k denote a positive integer. A *k-coloring* of G is a mapping that sends each vertex in $V(G)$ to an element in the set $[k] := \{1, 2, \ldots, k\}$. We refer to the elements of $[k]$ as *colors*. A k-coloring is called *proper* if no two adjacent vertices are mapped to the same color. The *chromatic number* $\chi(G)$ is the smallest k for which a proper k-coloring exists.

There are many variations on the chromatic number of a graph; we mention two. A proper *k-tuple coloring* maps each vertex to a k-element set of positive integers, such that adjacent vertices are mapped to disjoint sets. The k^{th} *chromatic number* $\chi_k(G)$ is the smallest n such that proper k-tuple coloring exists that maps each vertex to a k-element subset of $[n]$. Notice that $\chi_1(G)$ is just the usual chromatic number $\chi(G)$. The reader may wish to verify, for example, that when G is a 5-cycle, we have $\chi_2(G) = 5$.



$\chi_2(C_5) = 5$

Now fix any $a, b \in \mathbb{N}$ with $a \geq b$. A proper *a/b-coloring* maps each vertex to some b-element subset of $[a]$, such that adjacent vertices are mapped to disjoint sets. Such colorings differ from k-tuple colorings only in the sense that both set sizes a and b are determined. So $\chi_k(G)$ can be viewed as the minimum a such that a proper a/k-coloring exists. By contrast, the *fractional chromatic number* $\chi_f(G)$ is the infimum of all the fractions a/b such that there exists a proper a/b-coloring. The fractional chromatic number is known to be always rational, but this is not trivial to prove [3].



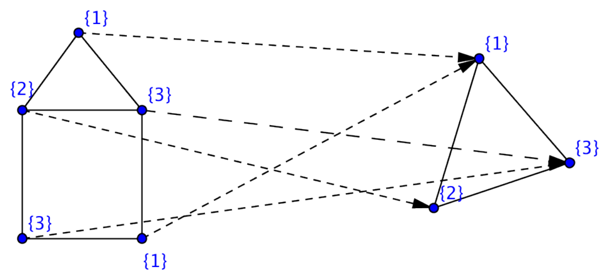
$\chi_f(C_5) = 5/2$

Curiously, if the fractional chromatic number of a graph is written a/b in lowest terms, there may not always exist a proper a/b-coloring. For example, a graph may have a proper 10/4-coloring and have $\chi_f(G) = 5/2$, but not have a proper 5/2-coloring. This was first shown by Fisher[2], and will be explored later in this paper.

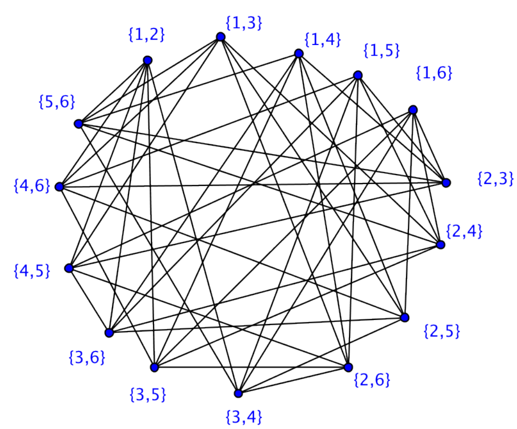
2. Homomorphisms and Colorings

Graph homomorphisms offer another way to view the proper colorings of a graph. Given two graphs X and Y, a mapping $f : V(X) \to V(Y)$ is a *homomorphism* if $f(x_1)$ is adjacent to $f(x_2)$ whenever x_1 and x_2 are adjacent

[3]. Graph homomorphisms are a bit different than many other notions of homomorphism, because the definition is just an "if" statement, rather than an "if and only if" statement. In particular, $x_1 \sim x_2$ implies $f(x_1) \sim f(x_2)$, but not vice-versa. The figure below illustrates a homomorphism from the graph on the left to the graph on the right.

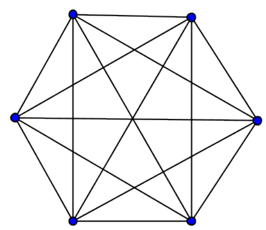


In order to connect homomorphisms with graph colorings, we need to define an important family of graphs. For any positive integers m and k, the *Kneser graph* $G_{m,k}$, is formed by defining $V(G_{m,k})$ to be the set of all k-element subsets of the set $[m] := \{1, 2, \ldots, m\}$. Two such subsets are adjacent if their intersection is empty.



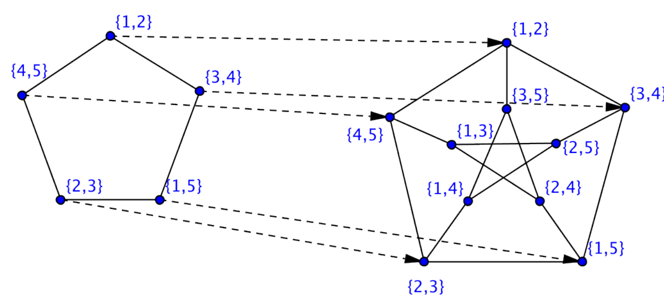
The graph $G_{6,2}$

The *complete graph* K_m can be defined as a special case of the Kneser graph in which k=1, so that $K_m = G_{m,1}$. Note that, in a complete graph, every pair of distinct vertices is adjacent.



The graph K_6

It is easy to see that there exists a homomorphism from a graph G to the Kneser graph $G_{m,k}$ if and only if there there exists a proper m/k-coloring of G. The figure below illustrates the homomorphism corresponding to the 5/2-coloring of the 5-cycle from above.



A homomorphism $C_5 \to G_{5,2}$

Similarly, a homomorphism to the complete graph K_m corresponds to a proper m-coloring of G. We note that an *isomorphism* is, by definition, a homomorphism that is bijective, and whose inverse is also a homomorphism.

3. Cliques and the Mycielski Construction

A set of k mutually adjacent vertices in a graph is called a k-clique. The clique number $\omega(G)$ of a graph is the size of the largest clique in G. Since the vertices of a clique must be assigned distinct colors, we have

$$\chi(G) \geq \omega(G).$$

In other words, one cause for a graph to have a large chromatic number is the existence of a large clique. It is natural to wonder if this is necessary.

The Mycielski construction gives us a way to increase the chromatic number by one without increasing the clique number. The Mycielskian $\mu(G)$ of a graph G is defined as follows. Let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$

and edge set E. Let V^1 be a copy of the vertex set, and u be a single vertex. Then the Mycielskian $\mu(G)$ has the vertex set

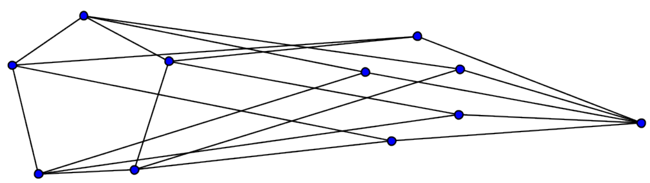
$$V^0 \cup V^1 \cup \{u\}.$$

The edge set of $\mu(G)$ is the set

$$\{v_j^0 v_{j'}^0 : v_j v_{j'} \in E\} \cup \{v_j^0 v_{j'}^1 : v_j v_{j'} \in E\} \cup \{v_j^1 u : \forall v_j^1 \in V^1\}$$

For example, the Mycielskian of the 5-cycle C_5 is indicated below:

The Mycielskian $\mu(C_5)$



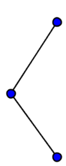
This construction can generalized. Let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E, and let p be any positive integer. For each integer i $(0 \leq i \leq p)$ let V^i be a copy of the vertices in V, so that $V^i = \{v_1^i, v_2^i, ..., v_n^i\}$. The p-Mycielskian $\mu_p(G)$ has the vertex set

$$V^0 \cup V^1 \cup V^2 \cup ... \cup V^p \cup \{u\},$$

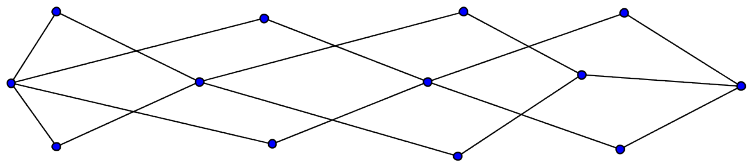
where u is a single vertex. The edge set of $\mu_p(G)$ is the set

$$\{v_j^0 v_{j'}^0 : v_j v_{j'} \in E\} \cup \left(\bigcup_{i=0}^{p-1} \left\{ v_j^i v_{j'}^{i+1} : v_j v_{j'} \in E \right\} \right) \cup \{v_j^p u : \forall v_j^p \in V^p\}$$

In other words, V^0 induces a subgraph isomorphic to G, while the other sets V^i contain no edges. Each vertex v_j^1 in V^1 is adjacent to every neighbor of v_j^0. The edges between V^0 and V^1 are "copied" between each V^i and V^{i+1}, and finally, vertex u is adjacent to all of V^p. An example of this construction is as follows. Let G be the following graph:



Then $\mu_3(G)$ would be the following:



We observe that $\mu_1(G)$ is just the original Mycielskian $\mu(G)$. Furthermore, we define the graph $\mu_0(G)$ to be the graph G plus a single new vertex u that is adjacent to all the vertices of G. Most of our main results concern $\mu(G)$.

4. Bounds on $\chi_k(\mu(G))$

Mycielski made the simple observation that $\chi(\mu_0(G)) = \chi(G) + 1$. Indeed, as we will now see, it is easy to show that, for any $k \geq 1$,

$$\chi_k(\mu_0(G)) = \chi_k(G) + k.$$

To see this, notice that $\mu_0(G)$ introduces one new vertex u which is adjacent to every vertex of G. Therefore, k new colors must be introduced so that u can be properly colored with a k-element set that is disjoint from all the colors assigned to vertices of G.

How do these results look if we replace $\mu_0(G)$ by $\mu(G)$? Wensong Lin discovered the following.

Theorem 4.1. *[1] For any graph G and any integer $k \geq 1$,*

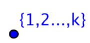
$$\chi_k(G) + 1 \leq \chi_k(\mu(G)) \leq \chi_k(G) + k.$$

Proof. Assume G has vertices $\{v_1, \ldots, v_n\}$. To prove the upper bound we let $m = \chi_k(G)$ and we construct a proper k-tuple coloring of $\mu(G)$ that uses at most $m + k$ colors. To this end, let c be a proper k-tuple coloring of G that assigns each vertex to some k-element subset of $[m]$. We now define a function h that assigns each vertex of $\mu(G)$ to a k-element subset of $[m + k]$. Define $h(v_i^0) = h(v_i^1) = c(v_i)$ for all $i \in [n]$ and define $h(u) = \{m+1, m+2, ..., m+k\}$. To see that this coloring h is proper, recall that every edge of $\mu(G)$ either: (i) has both endpoints in V^0, (ii) has one endpoint in V^0 and the other in V^1, or (iii) has one endpoint in V^1 and the other endpoint equal to u. Since h agrees with the proper coloring c when restricted to V^0, each of the edges in case (i) has disjoint k-tuples assigned to its endpoints. Since each $v_i^1 \in V^1$ is adjacent to the same vertices in V^0 as v_i^0, and since $h(v_i^1) = h(v_i^0)$, it follows that the edges in case (ii) have disjoint k-tuples assigned to their endpoints. Finally, observe that $h(u)$ consists of colors not appearing on any other vertex of $\mu(G)$. So the edges in case (iii) have disjoint k-tuples assigned to their endpoints. It follows that h is a proper k-tuple coloring of $\mu(G)$ that uses at most $m + k$ colors. Therefore $\chi_k(\mu(G)) \leq \chi_k(G) + k$.

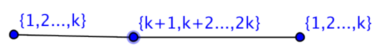
To prove the lower bound we let $t = \chi_k(\mu(G))$ and let g be a proper k-tuple coloring of $\mu(G)$ that uses at most t colors. Let k_0 be one of the colors in the k-element set $g(u)$. We will construct a proper k-tuple coloring f of G in which the color k_0 is not used. Define $f(v_j) = g(v_j^0)$ if $k_0 \notin g(v_j^0)$, and define $f(v_j) = g(v_j^1)$ if $k_0 \in g(v_j^0)$. Note that $k_0 \notin g(v_j^1)$ since $u \sim v_j^1$ and $k_0 \in g(u)$. It follows that for every vertex v_j of G, the k-tuple $f(v_j)$ does not contain k_0. To see that this coloring f is proper, assume $v_i \sim v_j$. Since both v_i^0 and v_i^1 are adjacent to v_j^0 in $\mu(G)$, it must be the case that both $g(v_i^0)$ and $g(v_i^1)$ are disjoint from $g(v_j^0)$. So if $k_0 \notin g(v_j^0)$, then $f(v_i)$ is disjoint from $f(v_j)$. Similarly, $f(v_i)$ is disjoint from $f(v_j)$ if $k_0 \notin g(v_i^0)$. Since $g(v_i^0)$ and $g(v_j^0)$ are disjoint, this exhausts all possibilities. Therefore f is a proper k-tuple coloring of G that uses at least 1 fewer colors than g. So $\chi_k(G) \leq \chi_k(\mu(G)) - 1$ as desired. □

Both bounds are easily achieved, as the following examples illustrate.

Example 4.2. (Upper bound) Let G be a graph consisting of a single isolated vertex.

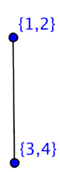


Then $\mu(G)$ appears as follows.

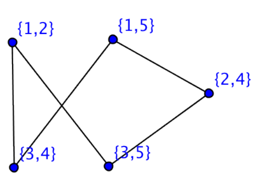


Notice that the edges in $\mu(G)$ force disjointness of the k-element sets assigned to their endpoints, so $\chi_k(\mu(G)) = \chi_k(G) + k$ and the upper bound is reached.

Example 4.3. (Lower bound) Let $G = K_2$ and consider the following proper 2-tuple coloring.



Then $\mu(G)$ is a 5-cycle C_5 and has a proper 2-tuple coloring that only uses 5 colors.



Since $\chi_k(G) + 1 \leq \chi_k(\mu(G))$ and $\chi_2(G) + 1 = 5$, this must be optimal. So $\chi_2(\mu(G)) = \chi_2(G) + 1$ and the lower bound is reached.

There are, however, infinite families of graphs that meet these bounds. We will present the results from [1] that show that the Kneser graphs of the form $G_{m,2}$ meet the upper bound, and the complete graphs K_n meet the lower bound when $n \geq k$.

5. Graphs That Meet the Upper Bound

In order to establish an infinite family of graphs which meet the upper bound, a proof by induction will be used. We will need two base cases to show that

$$\chi_2(\mu(G_{m,2})) = m + 2$$

for all integers $m \geq 5$. Therefore, we begin with two lemmas. These will establish that $G_{5,2}$ (the well-known Peterson graph) and $G_{6,2}$ meet the stated bound.

Lemma 5.1. *[1] The graph $G_{5,2}$ satisfies*

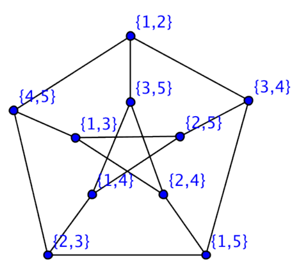
$$\chi_2(\mu(G_{5,2})) = 7$$

Before we present the proof of this result, we make a few observations. By Theorem 4.1, we know that

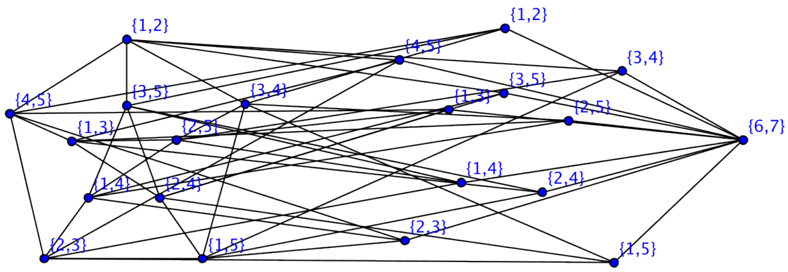
$$6 = \chi_2(G_{5,2}) + 1 \leq \chi_2(\mu(G_{5,2})) \leq \chi_2(G_{5,2}) + 2 = 7.$$

Therefore the k^{th} chromatic number is either 6 or 7.

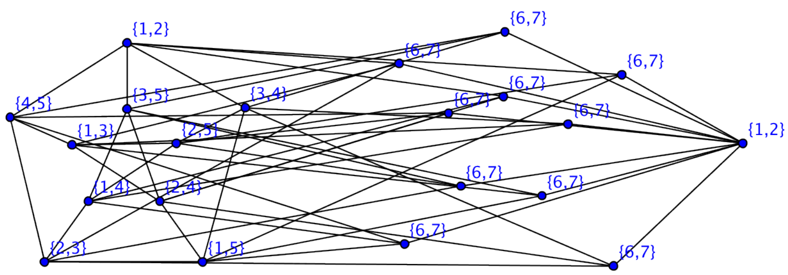
We would like to take this opportunity to illustrate the coloring described in the proof of Theorem 4.1. That construction, when applied to this graph, should produce a proper 2-tuple coloring of $\mu(G_{5,2})$ that uses 7 colors from any given proper 2-tuple coloring of $G_{5,2}$ that uses 5 colors. The following is a proper 2-tuple coloring of $G_{5,2}$.



Below we illustrate the associated proper 2-tuple coloring of $\mu(G_{5,2})$ given by the theorem.



This particular 2-tuple coloring is not by any means unique. For example, below is an equally valid proper 2-tuple coloring that is different from the one above.



In any case, it is clearly true from the above that $\chi_2(\mu(G_{5,2})) \leq 7$. To show that $\chi_2(\mu(G_{5,2})) = 7$, we will assume that there is a proper 2-tuple coloring of the graph that only uses 6 colors, and we will obtain a contradiction. Before embarking on this task, it is convenient to make two observations.

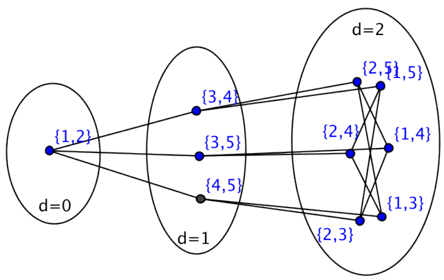
Observation 5.2. [3] The graphs $G_{m,k}$ are all vertex-transitive and $G_{5,2}$ has diameter 2.

Remark. Recall that a graph G is *vertex-transitive* if, for any pair of vertices x and y, there is an isomorphism from G to itself that maps x to y. The *diameter* of a graph is the maximum value of the path-length distance function when maximized over all pairs of vertices.

Proof. (of Observation 5.2) To see that $G_{m,k}$ is vertex-transitive, pick any two vertices x and y. Then x and y are k-element subsets of $[m]$. Let f be any permutation of m that sends x to y. Then $f(x)$ is disjoint from $f(y)$ iff x and y are disjoint. So f is an isomorphism.

Since the elements of $G_{5,2}$ are 2-element subsets of [5], any two vertices are either: (i) identical, (ii) disjoint, or (iii) intersect in 1 element. It follows that the two vertices are at distance 0, 1, or 2, respectively. (To see that vertices in case (iii) are distance 2 apart, let a, b, c, d, e be distinct elements of [5]. Notice that ab and bc share 1 element, but ab is disjoint from de and de is disjoint from bc. So $ab \sim de \sim bc$.) $\square$

Since the Petersen graph $P = G_{5,2}$ is vertex-transitive, each vertex will give rise to a distance-partition of the following general form.



It easy to verify, for example, that when the permutation (123) is applied to the vertices, the form of the distance partition will remain the same.

Observation 5.3. [1] Suppose h is a proper 2-tuple coloring of $\mu(G_{5,2})$ that uses at most 6 colors. Then every odd cycle C in V^0 will have at least one vertex x such that $h(x)$ is disjoint from $h(u)$.

Proof. Assume there is no $x \in C$ such that $h(x) \cap h(u) = \emptyset$. Then all of the vertices of C have one or two colors in common with $h(u)$. If a vertex has two colors in common with h(u) then any vertex in C that is adjacent to that vertex must be disjoint from $h(u)$, a contradiction. So all the vertices of C must have one color in common with $h(u)$. Without loss of generality, let $h(u) = \{1, 2\}$. Then the vertices in the cycle C will have either a one or a two in their coloring. But this would imply that C is bipartite, which is a contradiction since this is an odd cycle. Therefore, at least one vertex must have a coloring which is disjoint from $h(u)$. $\square$

We are now ready to prove Lemma 5.1. For clarity, we label the vertices of $G_{5,2}$, which are 2-element subsets of [5], as follows:

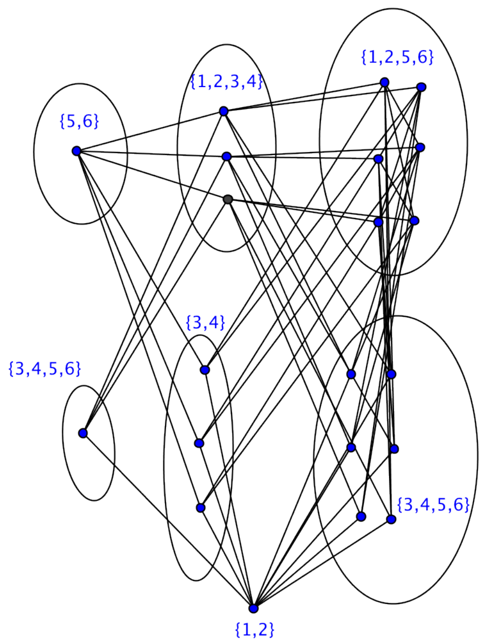
$$v_{ij} := \{i, j\}.$$

With this convention, the Mycielskian $\mu(G_{5,2})$ has vertices $\{v^0_{ij}\}$, $\{v^1_{ij}\}$, and u, where i and j are distinct elements of [5].

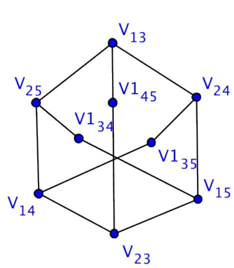
Proof. (of Lemma 5.1) By way of contradiction, assume h is a proper 2-tuple coloring of $\mu(G_{5,2})$ that uses at most 6 colors. By Observation 2, and since $G_{5,2}$ has odd cycles, there exists a vertex $v \in V^0$ such that $h(v) \cap h(u) = \emptyset$. By Observation 1, it doesn't matter which vertex, so we suppose it is v^0_{12}. Renaming the colors if necessary, we assume that $h(u) = \{1, 2\}$ and $h(v^0_{12}) = \{5, 6\}$. Since v^1_{35}, v^1_{34}, and v^1_{45} are adjacent to both v^0_{12} and u, it must be the case that

$$h(v^1_{35}) = h(v^1_{34}) = h(v^1_{45}) = \{3, 4\}.$$

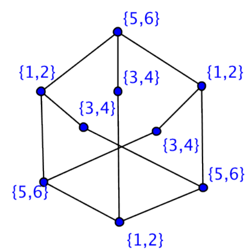
Consider the diagram of $\mu(G_{5,2})$ depicted below. In this diagram, the top 3 cells represent the distance partition of the vertices in V^0 according to their distance from v^0_{12}. (Recall that $G_{5,2}$ has diameter 2.) The 3 cells below this are the corresponding vertices in V^1, and the bottom vertex is u. The sets appearing by each cell describe the colors that are available to color those vertices. The proper 2-coloring h must assign to vertices in each cell only pairs of colors from the indicated subsets of [5].



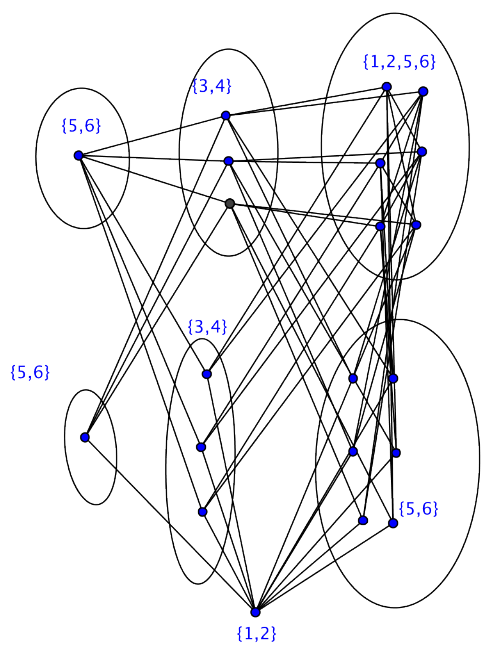
We can narrow the possibilities for how the vertices are colored in the top-rightmost cell of the above diagram. These are vertices in V^0 that are distance 2 from v^0_{12}. Below is the subgraph of $\mu(G_{5,2})$ that is induced on the vertices in the top-rightmost cell and the middle cell of the second row. Specifically, this is the subgraph induced on the vertices v^0_{13}, v^0_{24}, v^0_{15}, v^0_{23}, v^0_{14}, v^0_{25}, v^1_{45}, v^1_{34}, and v^1_{35}.



If $h(v_{13}^0) = \{5,6\}$ (a similar argument holds for $\{1,2\}$) the subgraph must be colored in the following way:



But if the central vertices v_{45}^1, v_{34}^1, and v_{34}^1 are replaced by v_{45}^0, v_{34}^0, and v_{34}^0, respectively, the induced subgraph is the same. So v_{45}^0, v_{34}^0, and v_{34}^0 must all be colored $\{3,4\}$. These are the vertices in the middle cell of the top row. This further forces the available colors in the rest for the graph in the following way:



To see that this is impossible, note that $v_{13}^0 \sim v_{24}^1$ and $v_{23}^0 \sim v_{15}^1$. Also, $h(v_{24}^1) = h(v_{15}^1) = \{56\}$, but one of v_{13}^0 and v_{24}^0 must be colored $\{5,6\}$, as they are opposite on the above 6-cycle. A similar argument shows that $h(v_{1,3}^0)$ cannot be $\{1,2\}$. It follows that $|h(v_{13}^0) \cap \{1,2\}| = |h(v_{13}^0) \cap \{5,6\}| = 1$ and, by symmetry, all vertices distance 2 away from v_{12}^0 in the set V^0 must satisfy the above condition.

Notice that if all the vertices in the middle cell of the top row are colored

$\{3,4\}$, in other words, if $h(v_{35}^0) = h(v_{34}^0) = h(v_{45}^0) = \{3,4\}$, then the vertices in the rightmost cell of the middle row must all be colored $\{5,6\}$. In particular, $h(v_{24}^1) = \{5,6\}$. This is impossible, however, since $v_{13}^0 \sim v_{24}^1$ and $|h(v_{13}^0) \cap \{5,6\}| = 1$. Therefore, one of the vertices v_{35}^0, v_{34}^0, v_{45}^0 must have a nonempty intersection with $h(u) = \{5,6\}$.

However, regardless of which of the vertices v_{35}^0, v_{34}^0, v_{45}^0 has a nonempty intersection with $h(u) = \{5,6\}$, there will be a 5-cycle containing that vertex and 4 of the vertices in the top-rightmost cell. For example, v_{35}^0, v_{14}^0, v_{23}^0, v_{15}^0, v_{24}^0 form a 5-cycle. But none of the vertices of such a cycle are disjoint from $\{5,6\}$, which contradicts Observation 2.

Therefore there is no proper 2-tuple coloring of $\mu(G_{5,2})$ that only uses 6 colors. It follows that $\chi_2(\mu(G_{5,2})) = 7$. □

In order to determine the 2-tuple number of $\mu(G_{m,2})$, we will need to cite two related results. It is well known (see [3] and [Johnson, Holroyd, Stahl]), although not trivial to show, that for any positive integers m and k,

$$\chi(G_{m,k}) = m - 2k + 2 \qquad \text{and} \qquad \chi_k(G_{m,k}) = m. \tag{1}$$

Using these two results, we are now in a position to prove the following.

Lemma 5.4. *[1] The graph $G_{6,2}$ satisfies*

$$\chi_2(\mu(G_{6,2})) = 8.$$

Proof. By (**??**), we know that $\chi(G_{6,2}) = 4$ and $\chi_2(G_{6,2}) = 6$. Now, by Theorem 4.1, we know that

$$7 = \chi_2(G) + 1 \leq \chi_2(\mu(G)) \leq \chi_2(G) + 2 = 8.$$

Assume, for a contradiction, that $\chi_2(\mu(G_{6,2})) = 7$ and let h be a proper 2-tuple coloring of $\mu(G_{6,2})$ that uses at most 7 colors. Without loss of generality let $h(u) = \{1,2\}$. Notice that any two adjacent vertices in $G_{6,2}$ have a unique common neighbor. So any v_i^0,$v_j^0 \in V^0$ of $\mu(G_{6,2})$ have a unique common neighbor in V^0, call it v_k^0. Notice that v_k^1 is adjacent to v_i^0, v_j^0 and u. Therefore either $h(v_i^0)$ or $h(v_j^0)$ must intersect $h(u)$ or else v_k^1 can not be properly colored with a 2-element set disjoint from all of v_i^0, v_j^0 and u. Therefore the set I_0 of all vertices $v \in V^0$ for which $h(v)$ is disjoint from $h(u)$ must form an independent set. Therefore V^0 can be partitioned into independent sets $I_0 = \{v \in V^0 : h(v) \cap h(u) = \emptyset\}$, $I_1 = \{v \in V^0 : 1 \in h(v) \cap h(u)\}$, and $I_2 = \{v \in V^0 : h(v) \cap h(u) = \{2\}\}$ meaning that the subgraph induced on V^0 is 3-colorable which is a contradiction that $\chi(G_{6,2}) = 4$. Therefore $\chi_2(\mu(G_{6,2})) = 8$. □

We now have completed the base cases for an inductive proof of the following.

Theorem 5.5. *[1] For any integer $m \geq 5$,*

$$\chi_2(\mu(G_{m,2})) = m + 2.$$

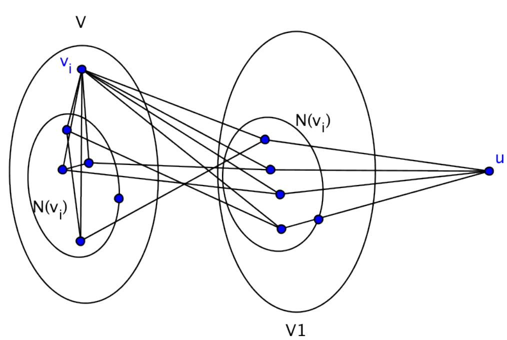
Proof. Proof by induction on m. Lemmas 2 and 3 verify the base cases. So by (**??**) and Theorem 4.1, we know that

$$m + 1 = \chi_2(G_{m,2}) + 1 \leq \chi_2(\mu(G_{m,2})) \leq \chi_2(G_{m,2}) + 2 = m + 2.$$

Assume, for a contradiction, that h is a proper 2-tuple coloring of $\mu(G_{m,2})$ that uses $m+1$ colors. Without loss of generality let $h(u) = \{1, 2\}$. If no vertex in V^0 were disjoint from $h(u)$ then the subgraph induced on V^0 would be bipartite, with bipartition

$$A = \{v \in V^0 : \{2\} = h(v) \cap h(u)\}$$
$$B = \{v \in V^0 : 1 \in h(v) \cap h(u)\}.$$

So there must exist a vertex $v_i \in V^0$ such that $h(v_i)$ is disjoint from $h(u)$. Without loss of generality we assume $h(v_i) = \{m, m+1\}$. Look at the following graph:



Note that the subgraph induced $N(v_i) \cup \{u\}$ is isomorphic to $\mu(G_{m-2,2})$. This is clear from the construction of $G_{m,2}$, since $N(v_i) \cap V^0$ corresponds to all 2-element subsets of an m-element set minus the 2-element subset that represents v_i. Since v_i is adjacent to all $N(v_i)$ and $h(v_i)$ is disjoint from u by selection of the vertex v_i, this subgraph must be colored with $m-1$ colors which is a contradiction of the induction hypothesis $\chi_2(\mu(G_{m-2,2}) = m$. Therefore $\chi_2(\mu(G_{m,2})) = m + 5$ for integers $m \geq 5$. □

6. Graphs That Meet the Lower Bound

The previous section established an infinite family of graphs that meet the upper bound for Theorem 4.1. In this section we consider the k-tuple colorings of the complete graphs on n vertices, where $n \geq k$. With these graphs, we will find that χ_k increases only by one when the Mycielskian is applied, so the lower bound is met. In fact, Lin [1] has proven a more general result about the p-Mycielskian which simplifies, when $p = 1$, to the results we are interested in for this paper.

Theorem 6.1. *[1] For any integers $p \geq 0, n \geq 3$, and $k \geq 1$, we have*

$$\chi_k(\mu_p(K_n)) = nk + \left\lceil \frac{(n-2)k}{(n-1)^{p+1} - 1} \right\rceil$$

Since this paper is primarily concerned with the case $p = 1$, the formula above simplifies to

$$\chi_k(\mu(K_n)) = nk + \left\lceil \frac{(n-2)k}{(n-1)^2 - 1} \right\rceil$$

Also note that if $k \leq n$ then $n^2 - 2n \geq (n-2)k$, meaning that $\left\lceil \frac{(n-2)k}{(n-1)^2-1} \right\rceil = 1$. For any k there will be an infinite number of K_n, where $n \geq k$ that will meet the lower bound of Theorem 4.1. This is our goal, and so it is in this form that we will prove the result.

Lemma 6.2. *For any integers $n \geq k \geq 1$, we have*

$$\chi_k(\mu(K_n)) = nk + 1.$$

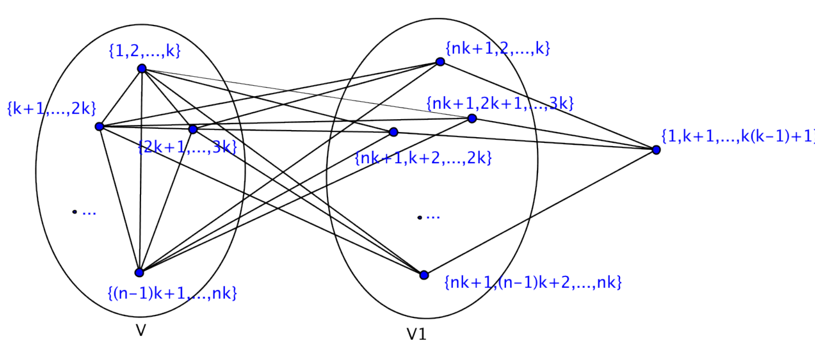
Proof. By Theorem 4.1 we know that

$$kn + 1 = \chi_k(K_n)) + 1 \leq \chi_k(\mu(K_n)).$$

To see that equality holds, it suffices to demonstrate that $\mu(K_n)$ has a proper k-tuple coloring with $nk+1$ colors. To this end, let $c(K_n)$ be a proper k-tuple coloring of K_n which uses kn colors. Define a k-tuple coloring of $\mu(K_n)$ by $h(v_i^0) = c(v_i)$ for all i $(1 \leq i \leq n)$ and let $h(v_i^1) = c(v_i)$, except replace the first color in this set with the color $nk+1$, and let

$$h(u) = \{1, k+1, 2k+1..., (k-1)k+1\}.$$

This a proper coloring of the graph $\mu(K_n)$ as shown in the following figure:



With the demonstration of a proper k-tuple coloring using $nk+1$ colors, we have completed our proof. □

We have seen in this section that there exists an infinite family of graphs (the complete graphs) that meet the lower bound of Theorem 4.1, and, in the previous section, we found an infinite family that meets the lower bound. Therefore the bounds established are the best possible.

7. Fractional Colorings of Mycielskians

Although the focus of this paper has concerned k-tuple colorings, rather than fractional colorings, it is important to understand some of the history that motivates these results. For example, it has been shown [1] that, for any graph G, the fractional chromatic number of the Mycielskian must satisfy

$$\chi_f(\mu(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}.$$

One consequence of this result is that, when combined with Mycielski's original result that $\chi(\mu(G)) = \chi(G) + 1$, it reveals that the gap between the chromatic number and the fractional chromatic number can be arbitrarily large.

The above result was later generalized [cite] to the p-Mycielskian, so that

$$\chi_f(\mu_p(G)) = \chi_f(G) + \frac{1}{\sum_{k=0}^{p}(\chi_f(G)-1)^k}.$$

Observe that, in the case $p = 1$, this equation reduces to the result above.

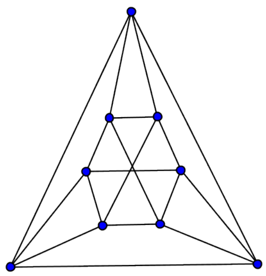
As mentioned earlier, however, the reduced fractions a/b that represent the fractional chromatic number of a graph might not be realized by a proper a/b-coloring of the graph [2]. This curious phenomenon led researchers to explore further the k^{th} chromatic number.

8. A Curiosity

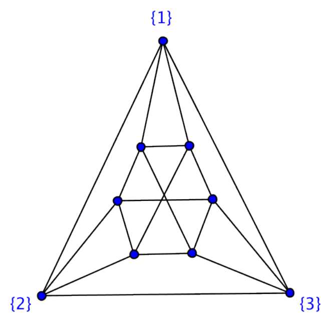
In this section, we expound upon the curious phenomenon described above. Notice that the chromatic numbers of the 5-cycle C_5 satisfy

$$\chi_2(C_5) = 5, \qquad \text{and} \qquad \chi_f(C_5) = 5/2.$$

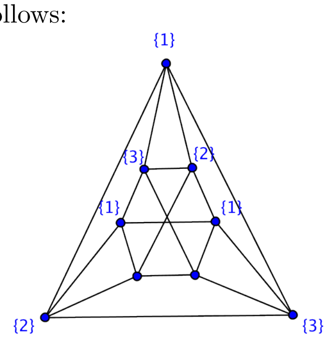
In this case, the fraction that is the fractional chromatic number does correspond to a proper a/b-coloring. However, this is not always the case. We consider the following graph whose fractional chromatic number is $\frac{3}{1}$, and yet there is no proper 1-tuple coloring that only uses 3 colors:



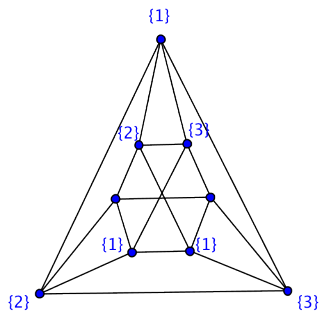
To show there is no proper 3/1-coloring, we will prove that all colorings using only 3 colors must be improper. To start, there is a 3-cycle around the outside. Without loss of generality assign it colors as follows:



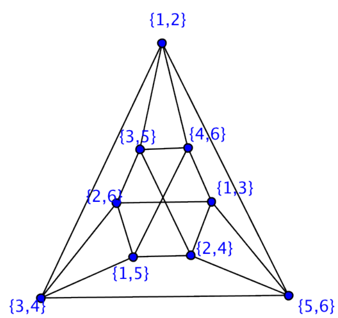
Looking at the vertex colored {1}, there is another 3-cycle containing it which can be colored in two ways, both leading to a contradiction. One way is as follows:



As you can see, after coloring the upper inner triangle, the colors of the two inner vertices are forced to be {1}. These vertices are adjacent, however. So this is an improper coloring, and the upper triangle must be colored differently.



This coloring of the upper triangle forces another pair of adjacent inner vertices to be colored $\{1\}$, therefore this is also an improper coloring. Therefore there is no way to 1-tuple color the graph only using 3 colors. However, there is a proper 2-tuple coloring of the graph that uses 6 colors as follows:



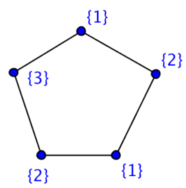
Therefore it is possible that the fractional chromatic number may, in lowest terms, be a/b without there existing a proper a/b-coloring of the graph!

9. The Value $\chi_k(G)/k$ Does Not Strictly Decrease

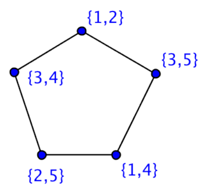
In this section we make an observation about the behavior of the ratio $\chi_k(G)/k$ as the value of k increases. In [2], Fisher points out that the fractional chromatic number is related to the k-tuple colorings as follows:

$$\chi_f = \lim_{k\to\infty} \chi_k(G)/k.$$

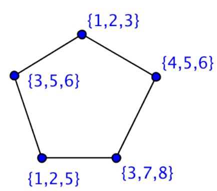
However, we would like to illustrate by example the fact that the sequence $\chi_k(G)/k$ is not strictly decreasing. Observe the following k-colorings of the 5-cycle C_5:



$\chi_1(C_5)/1 = 3$



$\chi_2(C_5)/2 = 5/2$



$\chi_3(C_5)/3 = 8/3$

We notice that

$$\chi_2(C_5) = 5/2 \leq 8/3 = \chi_3(C_5).$$

Therefore it is not a strictly decreasing sequence. In fact it is not difficult to show that $\chi_{2n}(C_5) = 5/2$ and $\chi_{2n+1}(C_5) = \frac{\lceil 2.5\cdot(2n+1)\rceil}{2n+1}$. This implies that the sequence of $\chi_k(C_5)/k$ will oscillate as it converges to 5/2.

References

[1] W. Lin, Multicoloring and Mycielski construction, Discrete Mathematics 308 (2008) 3565-3573.

[2] D.C. Fisher, Fractional colorings with large denominators, J. Graph Theory 20 (1995) 403-409.

[3] C. Godsil , G. Royle, Algebraic Graph Theory, Springer, New York, 2001.