Hypergraphs With a Unique Perfect Matching

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Under the direction of
Dr. John S. Caughman

February 26, 2012

Introduction

This presentation discusses the paper "On the maximum number of edges in a hypergraph with a unique perfect matching" written by:

Deepak Bal

Andrzej Dudek

Zelealem B. Yilma

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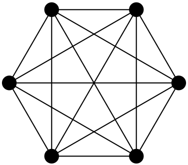

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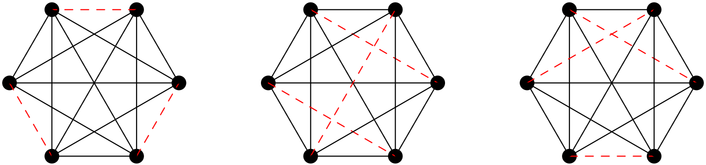
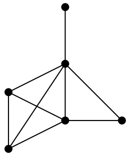


Definitions

A **graph** G is a finite set of vertices $\mathcal{V}$ along with a set of edges $\mathcal{E}$ where every edge is a set containing exactly two vertices.



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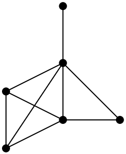
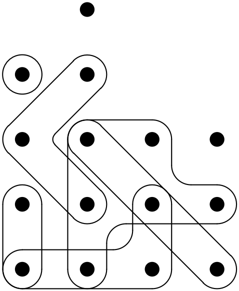


A **hypergraph** G is a finite set of vertices $\mathcal{V}$ along with a set of edges $\mathcal{E} \subseteq \mathcal{P}\mathcal{V} \setminus \{\emptyset\}$ (where $\mathcal{P}\mathcal{V}$ denotes the power set of $\mathcal{V}$) such that no two edges in $\mathcal{E}$ are equal as sets. A hypergraph is k-**uniform** if every $E \in \mathcal{E}$ has cardinality k.



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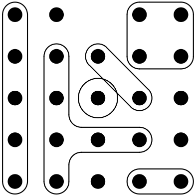
A **matching** in a hypergraph $G = (\mathcal{V}, \mathcal{E})$ is a set of pairwise disjoint edges $\{M_1, \ldots, M_m\}$



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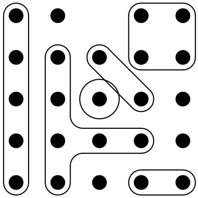
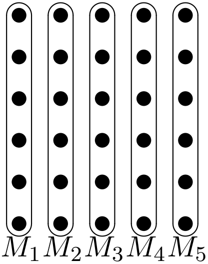


A **perfect matching** is a matching $\{M_1, \ldots, M_m\}$ such that $\mathcal{V} = \uplus_{i=1}^{m} M_i$. In other words, a perfect matching is a collection of edges that partition the vertex set.



For $k \geq 2$ and $m \geq 1$, let

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Theorem

Let $\mathcal{H}_m = (\mathcal{V}_m, \mathcal{E}_m)$ be a k-uniform hypergraph with km vertices and unique perfect matching. Then

$$|\mathcal{E}_m| \leq f(k, m)$$

where

$$f(k, m) = m + b_{k,2}\binom{m}{2} + b_{k,3}\binom{m}{3} + \cdots + b_{k,k}\binom{m}{k}.$$

Moreover, this bound is tight.

Values of $f(k, m)$

k \ m	1	2	3	4
2	1	4	9	16
3	1	11	48	130
4	1	36	297	1168
5	1	127	1878	10504

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- Add the edge that contains the k new vertices including the pendant.

Example $k = 2$

Start with $\mathcal{H}_1^*$.

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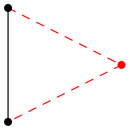
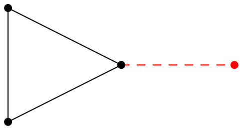
Add $k - 1 = 1$ new vertex and connect it to all previous vertices.



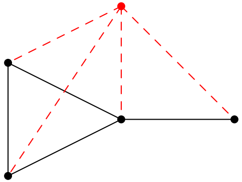
Example $k = 2$

Add a pendant vertex and an edge containing “new” vertices.
This is now $\mathcal{H}_2^*$.



Example $k = 2$

Repeat the process again to create the next hypergraph.
Add $k - 1 = 1$ new vertex and connect it to all previous vertices.



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Add a pendant vertex and an edge containing "new" vertices.
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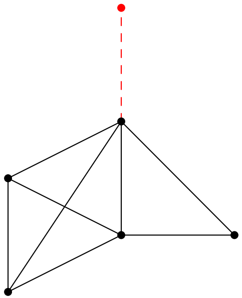

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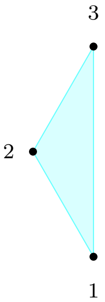

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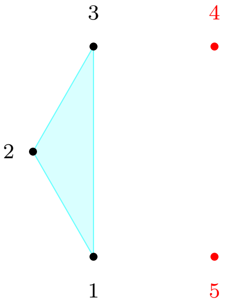

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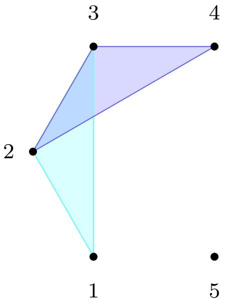
Add all edges that contain some “new” vertex.
Add edge $\{2, 3, 4\}$.



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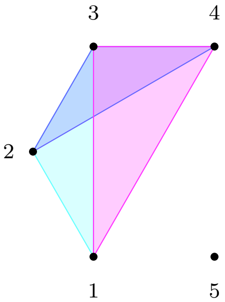

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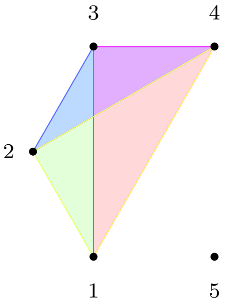

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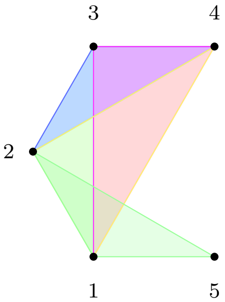

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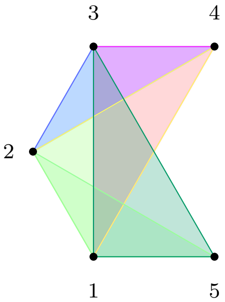

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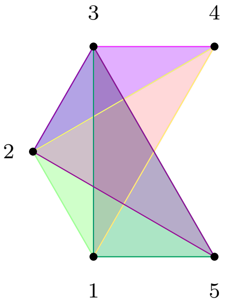

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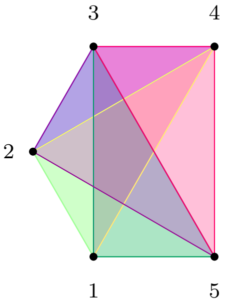

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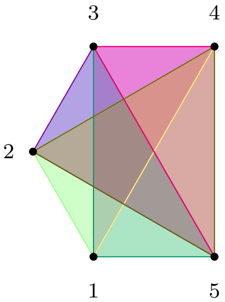

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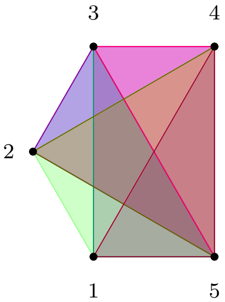

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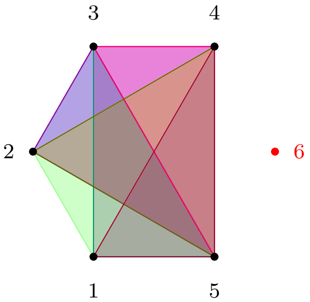
Add a pendant vertex.



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Add edge $\{4, 5, 6\}$ containing the “new” vertices.
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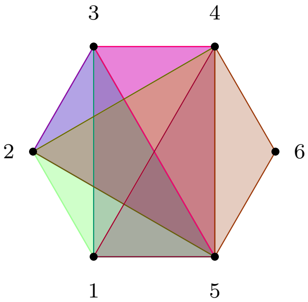

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[Pictures depict the proof in the 2-uniform case.]

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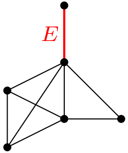

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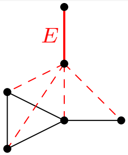

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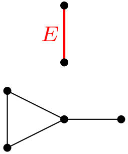

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- After eliminating such edges, we are left with a hypergraph isomorphic to $\mathcal{H}_{m-1}^*$ which has a unique perfect matching by induction hypothesis.

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- This equation can be solved by induction or by using recurrence relation solving strategies.

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- Unfortunately, the most obvious way to solve this equation yields

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- Additionally, induction-based proofs are rarely enlightening as to the true meaning of formulas.

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- Method 2: Count edges in $\mathcal{H}_m^*$ directly based upon the structure of the hypergraph without comparing it to $\mathcal{H}_{m-1}^*$.
- This requires some clever counting techniques such as the inclusion-exclusion principle. However, it does properly establish the correct formula for the number of edges in this hypergraph.

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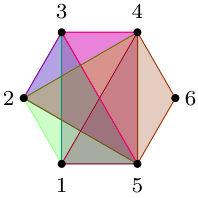
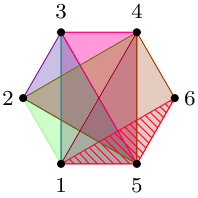
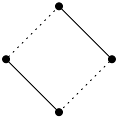




Outline

Two-Switch Example:



- Suppose the above depicts a portion of a graph that has a unique perfect matching. The solid edges represent matching edges.

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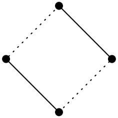

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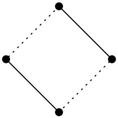

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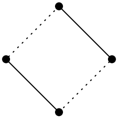
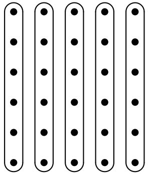
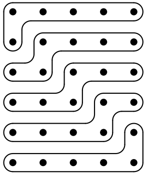


- Suppose the above depicts a portion of a graph that has a unique perfect matching. The solid edges represent matching edges.
- We cannot include both of the dashed edges in the graph. Otherwise, the perfect matching would not be unique:
- Start with the original perfect matching and discard the solid edges. Instead, trade them for the dashed edges to create a distinct perfect matching.
- Since we are not allowed to have both of the dashed edges in the graph, the total number of edges becomes constrained.

Hypergraph Generalization Example:



Matching Edges



Covering Edges

- The top left image depicts part of a hypergraph with a perfect matching. The edges shown are part of the perfect matching.

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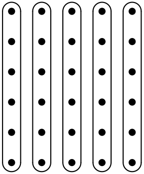

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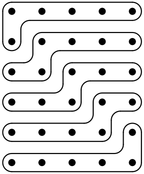

Covering Edges

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- The top right image depicts the same vertices. Suppose the edges in this image were also present in the hypergraph.

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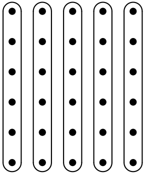
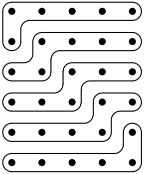


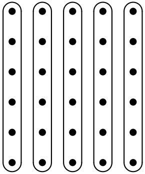
Matching Edges



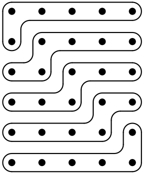
Covering Edges

- The top left image depicts part of a hypergraph with a perfect matching. The edges shown are part of the perfect matching.
- The top right image depicts the same vertices. Suppose the edges in this image were also present in the hypergraph.
- Start with the perfect matching. Remove the "matching edges" and include the "covering edges." This creates a distinct perfect matching.

Hypergraph Generalization Example:



Matching Edges



Covering Edges

- The top left image depicts part of a hypergraph with a perfect matching. The edges shown are part of the perfect matching.
- The top right image depicts the same vertices. Suppose the edges in this image were also present in the hypergraph.
- Start with the perfect matching. Remove the "matching edges" and include the "covering edges." This creates a distinct perfect matching.
- By uniqueness of the perfect matching, no such covering is allowed in the hypergraph, constraining the total number of possible edges.

Coverings

Definition

Suppose $\mathcal{L} = \{E_1, \ldots, E_\ell\}$ with $1 \le \ell \le k$ is a collection of disjoint edges. A collection of k-sets $\mathcal{C} = \{C_1, \ldots, C_\ell\}$ is a **covering** *of $\mathcal{L}$ if*

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- *Both* $\mathcal{L}$ *and* $\mathcal{C}$ *partition the same set of vertices.*

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Definition

Define $\mathcal{L}$ *as above and let* $F \subseteq \cup\mathcal{L}$ *be a* k*-set that intersects every edge of* $\mathcal{L}$*. The* **ordered type** *of* F *is* $\vec{b} = (b_1, \ldots, b_\ell)$ *where* $b_i = |F \bigcap E_i|$ *for* $1 \leq i \leq \ell$*.*

Coverings

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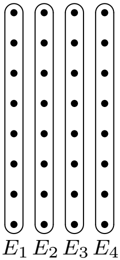
Suppose $\mathcal{L} = \{E_1, \ldots, E_\ell\}$ with $1 \leq \ell \leq k$ is a collection of disjoint edges. A collection of k-sets $\mathcal{C} = \{C_1, \ldots, C_\ell\}$ is a **covering** *of $\mathcal{L}$ if*

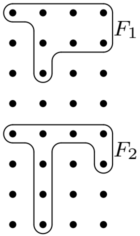
- *Every k-set in $\mathcal{C}$ nontrivially intersects every edge in $\mathcal{L}$ and*
- *Both $\mathcal{L}$ and $\mathcal{C}$ partition the same set of vertices.*

Definition

Define $\mathcal{L}$ as above and let $F \subseteq \cup\mathcal{L}$ be a k-set that intersects every edge of $\mathcal{L}$. The **ordered type** *of F is $\vec{b} = (b_1, \ldots, b_\ell)$ where $b_i = |F \bigcap E_i|$ for $1 \leq i \leq \ell$.The* **unordered type** *(abbreviated* **type***) of F is the unique rearrangement of the ordered type $(b_1, \ldots, b_\ell)$ of F such that the entries appear in nonincreasing order.*

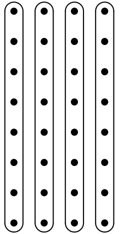
Coverings Example

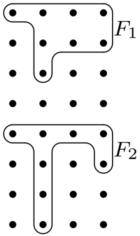




k-set	Ordered Type	(Unordered) Type
F_1		
F_2		

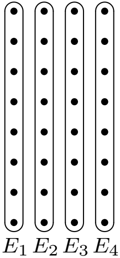
Coverings Example

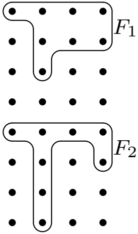




k-set	Ordered Type	(Unordered) Type
F_1	$(1,3,2,2)$	
F_2		

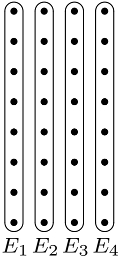
Coverings Example

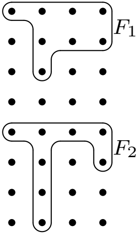




k-set	Ordered Type	(Unordered) Type
F_1	$(1, 3, 2, 2)$	$(3, 2, 2, 1)$
F_2		

Coverings Example





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F_1	$(1, 3, 2, 2)$	$(3, 2, 2, 1)$
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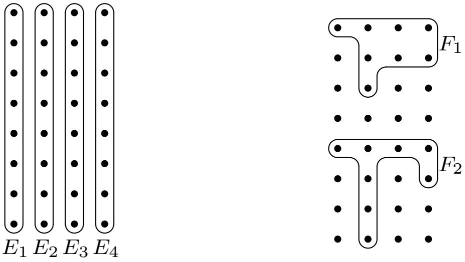
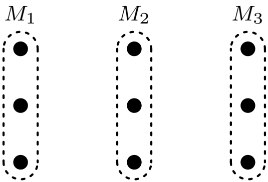
Coverings Example



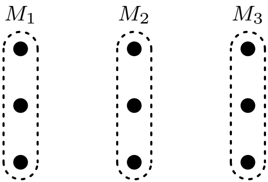
k-set	Ordered Type	(Unordered) Type
F_1	$(1, 3, 2, 2)$	$(3, 2, 2, 1)$
F_2	$(1, 4, 1, 2)$	$(4, 2, 1, 1)$

Main Theorem Proof Sketch



- Consider a 3-uniform complete hypergraph on 9 vertices. Let $\mathcal{M} = \{M_1, M_2, M_3\}$ be a perfect matching as depicted above.

Main Theorem Proof Sketch



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- In order to make $\mathcal{M}$ the unique perfect matching in the hypergraph, we must remove coverings of $\mathcal{M}$.

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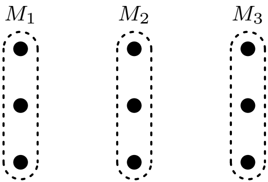

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- To organize our search, we start by considering the possible types of edges in the hypergraph:

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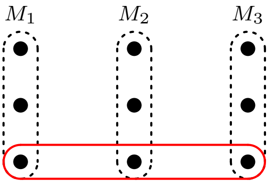

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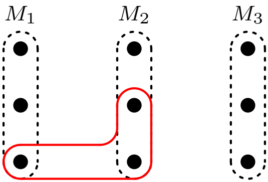

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(1,1,1)	(2,1,0)	

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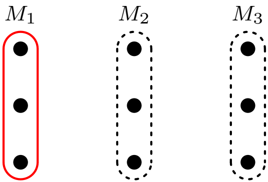

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(1,1,1)	(2,1,0)	(3,0,0)

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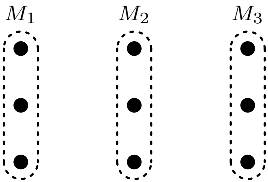

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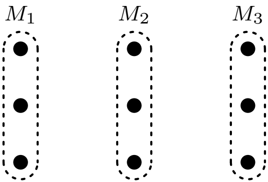
- To organize our search, we start by considering the possible types of edges in the hypergraph:

(1,1,1)	~~(2,1,0)~~	~~(3,0,0)~~

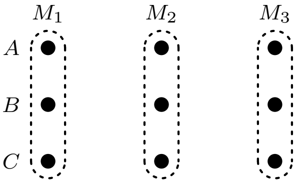
- Since edges in a covering must intersect every matching edge, we only consider edges of type (1,1,1).

Main Theorem Proof Sketch



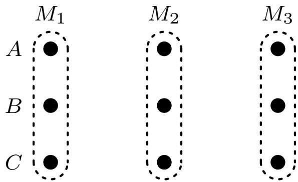
- We count the number of coverings of $\{M_1, M_2, M_3\}$ that only use edges of type $(1, 1, 1)$. Suppose $\{A, B, C\}$ is such a covering.

Main Theorem Proof Sketch



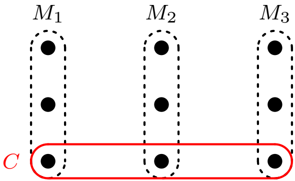
- We count the number of coverings of $\{M_1, M_2, M_3\}$ that only use edges of type $(1,1,1)$. Suppose $\{A, B, C\}$ is such a covering.
- We label each vertex by the covering edge that contains it. After possibly renaming the covering edges, we assume the vertices in M_1 are labeled as above.

Main Theorem Proof Sketch



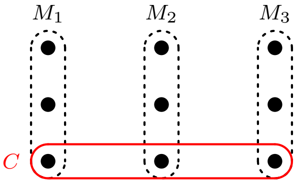
- We count the number of coverings of $\{M_1, M_2, M_3\}$ that only use edges of type $(1, 1, 1)$. Suppose $\{A, B, C\}$ is such a covering.
- We label each vertex by the covering edge that contains it. After possibly renaming the covering edges, we assume the vertices in M_1 are labeled as above.
- There are 6 ways to assign labels to M_2 and 6 ways to assign labels to M_3, giving a total of 36 coverings.

Main Theorem Proof Sketch



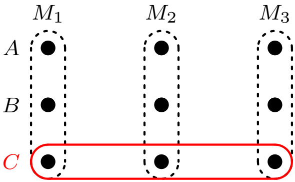
- Let C be a fixed edge of type $(1, 1, 1)$ as depicted above.

Main Theorem Proof Sketch



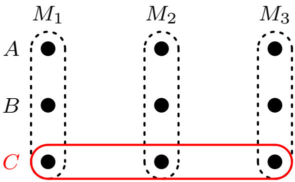
- Let C be a fixed edge of type $(1, 1, 1)$ as depicted above.
- We count the number of coverings $\{A, B, C\}$ that contain edge C and only use edges of type $(1, 1, 1)$.

Main Theorem Proof Sketch



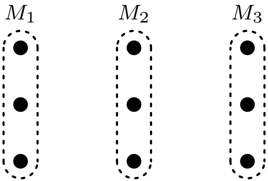
- Let C be a fixed edge of type $(1, 1, 1)$ as depicted above.
- We count the number of coverings $\{A, B, C\}$ that contain edge C and only use edges of type $(1, 1, 1)$.
- Again, we label each vertex by the covering edge that contains it. After possibly renaming the covering edges, we assume the vertices in M_1 are labeled as above.

Main Theorem Proof Sketch



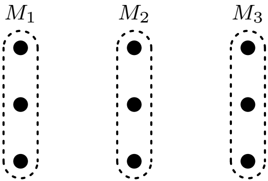
- Let C be a fixed edge of type $(1, 1, 1)$ as depicted above.
- We count the number of coverings $\{A, B, C\}$ that contain edge C and only use edges of type $(1, 1, 1)$.
- Again, we label each vertex by the covering edge that contains it. After possibly renaming the covering edges, we assume the vertices in M_1 are labeled as above.
- There are 2 ways to assign labels to M_2 and 2 ways to assign labels to M_3, giving a total of 4 coverings that contain edge C.

Main Theorem Proof Sketch



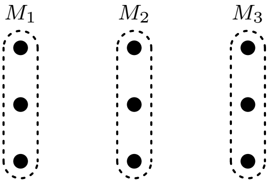
- By symmetry, every edge of type $(1, 1, 1)$ is contained in 4 coverings.

Main Theorem Proof Sketch



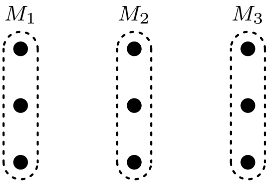
- By symmetry, every edge of type $(1, 1, 1)$ is contained in 4 coverings.
- Removing 1 edge from the hypergraph breaks 4 coverings.

Main Theorem Proof Sketch



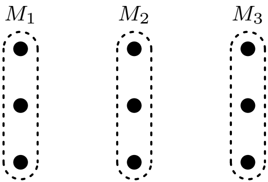
- By symmetry, every edge of type $(1, 1, 1)$ is contained in 4 coverings.
- Removing 1 edge from the hypergraph breaks 4 coverings.
- Removing 2 edges from the hypergraph breaks <u>at most</u> 8 coverings.

Main Theorem Proof Sketch



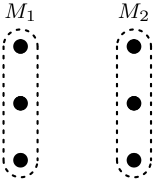
- By symmetry, every edge of type $(1,1,1)$ is contained in 4 coverings.
- Removing 1 edge from the hypergraph breaks 4 coverings.
- Removing 2 edges from the hypergraph breaks at most 8 coverings.
- Removing 9 edges from the hypergraph breaks at most 36 coverings.

Main Theorem Proof Sketch



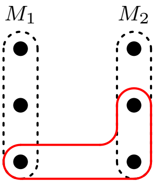
- By symmetry, every edge of type $(1, 1, 1)$ is contained in 4 coverings.
- Removing 1 edge from the hypergraph breaks 4 coverings.
- Removing 2 edges from the hypergraph breaks at most 8 coverings.
- Removing 9 edges from the hypergraph breaks at most 36 coverings.
- In order to remove all 36 coverings from the hypergraph, we must remove at least 9 edges of type $(1, 1, 1)$.

Main Theorem Proof Sketch



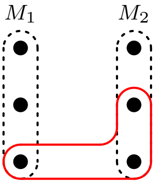
- We must also remove coverings of $\{M_1, M_2\}$.

Main Theorem Proof Sketch



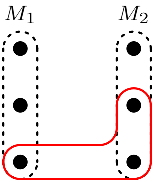
- We must also remove coverings of $\{M_1, M_2\}$.
- Every edge in a covering of $\{M_1, M_2\}$ is of type $(2, 1)$.

Main Theorem Proof Sketch



- We must also remove coverings of $\{M_1, M_2\}$.
- Every edge in a covering of $\{M_1, M_2\}$ is of type $(2, 1)$.
- To specify an edge E with $|E \bigcap M_1| = 1$ and $|E \bigcap M_2| = 2$, pick one vertex from M_1 and 2 vertices from M_2. There are $3 \cdot \binom{3}{2} = 9$ such edges.

Main Theorem Proof Sketch



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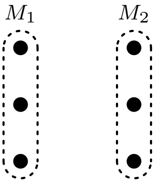
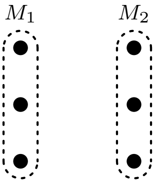
- All coverings are of the form $\{E, \overline{E}\}$ for an edge as previously described. Hence, there are also 9 coverings.

Main Theorem Proof Sketch



- Given any edge F of type $(2,1)$, F lies on exactly one covering $\{F, \overline{F}\}$. Caution: we may have $|F \bigcap M_1| = 1$ or $|F \bigcap M_1| = 2$.

Main Theorem Proof Sketch



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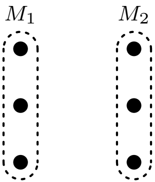
- In order to break all 9 coverings, we must remove at least 9 edges of type $(2,1)$.

Main Theorem Proof Sketch



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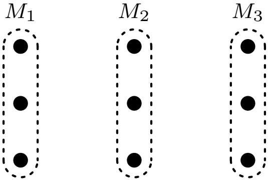
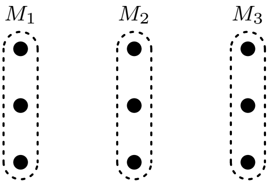
- In order to break all 9 coverings, we must remove at least 9 edges of type $(2,1)$.
- A symmetric situation occurs for any pair of 2 matching edges ($\{M_1, M_2\}, \{M_1, M_3\}$, or $\{M_2, M_3\}$). Hence we must remove at least $\binom{3}{2} \cdot 9 = 27$ edges of type $(2,1)$ from the hypergraph.

Main Theorem Proof Sketch



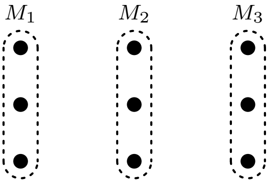
- The complete hypergraph has $\binom{9}{3} = 84$ edges.

Main Theorem Proof Sketch



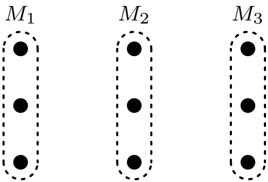
- The complete hypergraph has $\binom{9}{3} = 84$ edges.
- We must remove at least 9 edges of type $(1, 1, 1)$.

Main Theorem Proof Sketch



- The complete hypergraph has $\binom{9}{3} = 84$ edges.
- We must remove at least 9 edges of type $(1,1,1)$.
- We must remove at least 27 edges of type $(2,1)$.

Main Theorem Proof Sketch



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- We must remove at least 9 edges of type $(1, 1, 1)$.
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- There are at most $84 - 9 - 27 = 48$ edges remaining in the hypergraph.

For $k \geq 2$ and $m \geq 1$, let

$$b_{k,\ell} = \frac{\ell - 1}{\ell} \sum_{i=0}^{\ell-1} (-1)^i \binom{\ell}{i} \binom{k(\ell - i)}{k}.$$

Theorem

Let $\mathcal{H}_m = (\mathcal{V}_m, \mathcal{E}_m)$ be a k-uniform hypergraph with km vertices and unique perfect matching. Then

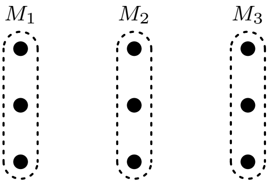
$$|\mathcal{E}_m| \leq f(k, m)$$

where

$$f(k, m) = m + b_{k,2}\binom{m}{2} + b_{k,3}\binom{m}{3} + \cdots + b_{k,k}\binom{m}{k}.$$

Moreover, this bound is tight.

Main Theorem Proof Sketch



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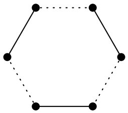
- We must remove at least 9 edges of type $(1, 1, 1)$.
- We must remove at least 27 edges of type $(2, 1)$.
- There are at most $84 - 9 - 27 = 48$ edges remaining in the hypergraph.
- $f(3, 3) = 3 + 9\binom{3}{2} + 18\binom{3}{3} = 48.$

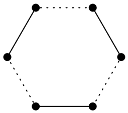
Outline

Electrons in Benzene Molecules



- Benzene consists of 6 carbon atoms arranged in a hexagon.

Electrons in Benzene Molecules



- Benzene consists of 6 carbon atoms arranged in a hexagon.
- Due to electron interactions along the hexagon, the carbon atoms tend to pair with a neighbor creating a diatomic molecule. This is the same as a perfect matching.

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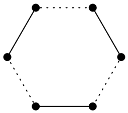

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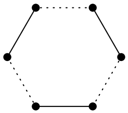
Electrons in Benzene Molecules



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- This graph has two perfect matchings. By symmetry, both of these have the same energy. Hence, the molecule resonates in between the two configurations.
- The resonance conjecture posits that resonating between states increases stability.

Thank You

Thanks for coming! Special thanks to Dr. John Caughman and Dr. Gerardo Lafferriere.