Doubly Almost Bipartite Leonard Pairs

by

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Abstract

In Linear algebra, the concept of *Leonard pair* (LP) was motivated by the theory of Q-polynomial distance-regular graphs. In this dissertation, we will first give a brief introduction to LPs and to two closely-related classes of objects: (i) bipartite Leonard pairs (BLPs) and (ii) almost bipartite Leonard pairs (ABLPs). Taking these as departure points, we will introduce a new class of object - *doubly almost bipartite Leonard pairs* (DABLPs). The primary aim of our work is to fully classify (up to isomorphism) this new family. In addition, since there is known to be a natural correspondence between Leonard pairs and families of orthogonal polynomials, we reveal which families of orthogonal polynomials correspond to the DABLPs. Several related objects, such as Leonard triples, modular Leonard triples, spin Leonard pairs, and near-bipartite Leonard pairs have corresponding notions for the doubly almost bipartite case. These analogous objects are also defined and briefly explored.

Dedication

This dissertation is dedicated to my grandparents, *Morimichi Masuda* and *Marilyn Miles*, for making my academic journey in the United States possible. Also, to my mother, *Akiko Masuda*, whose support/encouragement/nagging has helped me complete this work.

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Glossary of Symbols

$\mathrm{End}(V)$	Algebra consisting of the $\mathbb{K}$-linear maps from V to V
$\mathbb{K}$	Algebraically closed field
$\mathscr{X}$	Association scheme on a non-emptry finite set X
M/M^*	Bose-Mesner/dual Bose-Mesner algebra
Γ	Distance-regular graph
$\mathbf{i}$	Imaginary unit $\sqrt{-1}$
$\mathcal{I}$	Intersection array $(\{b_i\}_{i=0}^{d-1}, \{c_i\}_{i=1}^{d})$
p_{ij}^h	Intersection number
q_{ij}^h	Krein parameter
$\mathrm{Mat}_{d+1}(\mathbb{K})$	$\mathbb{K}$-algebra consisting of all $(d+1)\times(d+1)$ matrices with entries in $\mathbb{K}$
δ_{ij}	Kronecker delta
Φ	Leonard system
$\mathcal{P}$	Parameter array $(\{\theta_i\}_{i=0}^{d}, \{\theta_i^*\}_{i=0}^{d}, \{\varphi_i\}_{i=1}^{d}, \{\phi_i\}_{i=1}^{d})$
∂	Path-length distance function for Γ
$T = T(x)$	Terwilliger (or subconstituent) algebra with respect to x
V	Vector space over $\mathbb{K}$
$\mathbb{K}^{d+1}$	Vector space consisting of the column vectors with $d+1$ rows and all entries in $\mathbb{K}$

1 Introduction

1.1 Overview

A finite, connected, undirected graph of diameter d is said to be *distance-regular* if, for any $0 \leq h, i, j \leq d$, and any vertices x and y that are distance h apart, there are exactly p^h_{ij} vertices z at distance i from x and j from y, for some constants p^h_{ij} (see Section 2.3). Examples include the 1-skeletons of the 5 Platonic solids, hypercube graphs of any dimension, cycles, complete graphs, any strongly regular graph, any distance transitive graph, and many other infinite families. These graphs are far from being completely classified, and since the 1970s, they have been researched actively due to their many connections with physics, combinatorics, algebra, error-correcting codes, knot theory, and more.

In 1982, Delsarte [14] explored a broad class of distance-regular graphs that are said to be *Q-polynomial* (see Section 2.3). For any Q-polynomial distance-regular graph, he showed that there are two special sequences of orthogonal polynomials that are related by what is now called *Askey-Wilson duality*. This was notable because, a few years prior (in 1972), Leonard had fully classified all pairs of orthogonal polynomial sequences that obey this duality. In particular, Leonard had found that all such sequences come from the terminating branch of the Askey scheme of orthogonal polynomials [33]. This branch consists of the q-Racah polynomials and their limits. Inspired by these results, Bannai and Ito published a thorough study of Q-polynomial distance-regular graphs [4], including a detailed reworking of Leonard's theorem. Shortly thereafter, the theory of *Leonard pairs* was introduced by Terwilliger in [44] to extend the work of Bannai and Ito. Leonard pairs situate the theory of orthogonal polynomials in a context of linear algebra and matrix theory. Specifically, this theory offers powerful tools to study any sequences of orthogonal polynomials with discrete support for which there is a dual sequence of orthogonal polynomials. Since their introduction over 20 years ago, Leonard pairs have proved very useful in the theory of algebraic combinatorics [26, 48], the theory of classical mechanics [51], and the representation theory of the Lie algebra $\mathfrak{sl}_2$ or the quantum group $U_q(\mathfrak{sl}_2)$ [23, 26, 27, 28, 29, 30, 32, 36, 50] just to name a few.

In this dissertation, we begin with a brief review of the basic theory of Leonard pairs and focus on two special classes: (i) bipartite Leonard pairs (BLPs) in Section 2.10 and (ii) almost bipartite Leonard pairs (ABLPs) in Section 2.11. Taking BLPs and ABLPs as departure points, we will introduce a new class of objects - the *doubly almost bipartite Leonard pairs* (DABLPs) in Section 3.1. The primary aim of our work is to fully classify (up to isomorphism) the doubly almost bipartite Leonard pairs. In addition, since there is generally known to be a natural correspondence between Leonard pairs and certain families of orthogonal polynomials, we aim to identify which families of orthogonal polynomials correspond to the Leonard pairs in this doubly almost bipartite case. Additionally, we explore some potential avenues of future research. Specifically, we introduce doubly almost bipartite analogues of several related objects, including Leonard triples, modular Leonard triples, and spin Leonard pairs, and a connection to near-bipartite Leonard pairs is introduced and explored.

1.2 Organization

This dissertation is organized as follows. After the general introduction given here in Chapter 1, the material provided in Chapter 2 will outline the basic definitions and background for the theory of Leonard pairs. Chapter 3 will introduce our primary object of interest, the doubly almost bipartite Leonard pairs. Here we will derive a number of important facts and preliminary results about these Leonard pairs that will be useful in our work. In Chapter 4, we present our main results. Specifically, we will classify the doubly almost bipartite Leonard pairs using Leonard's Theorem. In Chapter 5, we collect and discuss some of the potential future directions for this line of research as outlined above. Appendix A contains the detailed proof of Theorem 3.3.1. Appendix B contains some comments regarding generalizations of the all-ones DABITM (see (3.2.1)). Appendix C contains the list of both parameter and intersection arrays of all 13 families of Leonard pairs (see page 16).

2 Background

In this chapter we introduce some necessary background knowledge as we define a Leonard pair and provide some examples. By using these examples, we illustrate how Leonard pairs naturally arise in representation theory and the theory of orthogonal polynomials. Before we define the notion of Leonard pair, we first recall what it means for a square matrix to be *tridiagonal* and list several helpful lemmas regarding tridiagonal matrices.

2.1 Tridiagonal Matrices

Throughout this paper, V will denote a vector space over an algebraically closed field $\mathbb{K}$ with dimension $d+1$. Let $\mathrm{End}(V)$ denote the algebra consisting of the $\mathbb{K}$-linear maps from V to V (called the *endomorphism algebra*). Furthermore, for any nonnegative integer d, let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of all $(d+1) \times (d+1)$ matrices that have entries in $\mathbb{K}$. (We index the rows and columns by $0, 1, \ldots, d$.) The identity matrix and the matrix of $\mathrm{Mat}_{d+1}(\mathbb{K})$ whose entries are all one are denoted by I and J, respectively. Let $\mathbb{K}^{d+1}$ denote the vector space consisting of the column vectors with $d+1$ rows and all entries in $\mathbb{K}$.

Definition 2.1.1. A matrix T in $\mathrm{Mat}_{d+1}(\mathbb{K})$ is *tridiagonal* if the entries satisfy $T_{ij} = 0$ whenever $|i-j| > 1$ for any $0 \le i, j \le d$. Said another way, a tridiagonal matrix is a square matrix that has nonzero elements only on the main diagonal, the subdiagonal (the first diagonal below the main diagonal), and the superdiagonal (the first diagonal above the main diagonal).

$$\begin{pmatrix} a_0 & c_1 & & & 0 \\ b_0 & a_1 & c_2 & & \\ & b_1 & \ddots & \ddots & \\ & & \ddots & a_{d-1} & c_d \\ 0 & & & b_{d-1} & a_d \end{pmatrix}_{(d+1)\times(d+1)} \tag{2.1.1}$$

We will say a tridiagonal matrix is in:

- *standard form* if $b_0 = b_1 = \cdots = b_{d-1} = 1$ and
- *normalized form* if $c_1 = c_2 = \cdots = c_d = 1$.

Definition 2.1.2. A tridiagonal matrix as given in (2.1.1) is said to be *irreducible* if $b_i \neq 0$ and $c_{i+1} \neq 0$ for all $0 \leq i \leq d-1$. If at least one of b_i or c_{i+1} is 0 for some i $(0 \leq i \leq d-1)$, then it is said to be *reducible*.

For example, the following matrices are tridiagonal:

$$\begin{pmatrix} 2 & 3 & 0 & 0 \\ 1 & 4 & 2 & 0 \\ 0 & 5 & 3 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}.$$

Observe that the tridiagonal matrix given above on the left is irreducible and the one on the right is reducible.

Tridiagonal matrices are perhaps one of the most studied classes of matrices and much of the reason for this is that many algorithms in linear algebra require significantly less computational labor when they are applied to tridiagonal matrices.

Some examples include:

(1) finding eigenvalues,

(2) solving linear systems $A\vec{x} = \vec{b}$,

(3) finding LU factorizations, and

(4) evaluating determinants.

Next, we will state (and prove) two important lemmas about irreducible tridiagonal matrices that will be useful in developing the theory of Leonard pairs.

Lemma 2.1.1. *Every eigenspace of an irreducible tridiagonal matrix is 1-dimensional.*

Proof. Let $\vec{x} = \begin{pmatrix} x_0 & x_1 & \cdots & x_d \end{pmatrix}$ be a left-eigenvector of an irreducible tridiagonal matrix in (2.1.1) with eigenvalue θ. Then

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_d \end{pmatrix} \begin{pmatrix} a_0 & c_1 & & & 0 \\ b_0 & a_1 & c_2 & & \\ & b_1 & \ddots & \ddots & \\ & & \ddots & a_{d-1} & c_d \\ 0 & & & b_{d-1} & a_d \end{pmatrix} = \begin{pmatrix} \theta x_0 & \theta x_1 & \cdots & \theta x_d \end{pmatrix}. \tag{2.1.2}$$

Expanding the product on the left-hand side of (2.1.2) and equating the like-components on both sides, we see that

$$a_0x_0 + b_0x_1 = \theta x_0,$$

$$c_1x_0 + a_1x_1 + b_1x_2 = \theta x_1,$$

$$\vdots$$

$$c_dx_{d-1} + a_dx_d = \theta x_d.$$

Notice the three-term recurrence nature of these equations. Since $b_i \neq 0$ for all $i = 1, \ldots, d-1$, we may solve the above equations for x_i $(i = 1, \ldots, d)$ in terms of x_0, as follows:

$$
\begin{aligned}
x_1 &= \frac{(\theta - a_0)x_0}{b_0}, \\
x_2 &= \frac{[(\theta - a_0)(\theta - a_1) - b_0c_1]x_0}{b_0b_1}, \\
x_3 &= \frac{[(\theta - a_0)(\theta - a_1)(\theta - a_2) - b_0c_1(\theta - a_2) - b_1c_2(\theta - a_0)]x_0}{b_0b_1b_2}, \\
&\vdots
\end{aligned}
$$

This shows that the left-eigenvector $\vec{x}$ of a given irreducible tridiagonal matrix can be written in terms of one parameter x_0. $\square$

Lemma 2.1.2. *Assume $T \in Mat_{d+1}(\mathbb{K})$ is an irreducible tridiagonal matrix. Then T is similar to* (i) *a symmetric irreducible tridiagonal matrix,* (ii) *an irreducible tridiagonal matrix in standard form, and* (iii) *an irreducible tridiagonal matrix in normalized form.*

Proof. Let T be an arbitrary irreducible tridiagonal matrix as in (2.1.1).

(i) Define

$$\kappa_0 \equiv 1 \quad \text{and} \quad \kappa_i = \frac{\prod_{j=0}^{i-1} b_j}{\prod_{j=1}^{i} c_j}$$

and $K = \operatorname{diag}\bigl(\sqrt{\kappa_0}, \sqrt{\kappa_1}, \ldots, \sqrt{\kappa_d}\bigr)$. Then K is invertible and

$$K^{-1} = \operatorname{diag}\bigl(1/\sqrt{\kappa_0}, 1/\sqrt{\kappa_1}, \ldots, 1/\sqrt{\kappa_d}\bigr).$$

Furthermore,

$$K^{-1}TK = \begin{pmatrix} a_0 & \sqrt{b_0c_1} & & & 0 \\ \sqrt{b_0c_1} & a_1 & \sqrt{b_1c_2} & & \\ & \sqrt{b_1c_2} & \ddots & \ddots & \\ & & \ddots & a_{d-1} & \sqrt{b_{d-1}c_d} \\ 0 & & & \sqrt{b_{d-1}c_d} & a_d \end{pmatrix},$$

which is symmetric and irreducible tridiagonal.

(ii) Define $B = \operatorname{diag}(1, b_0, b_0b_1, \ldots, b_0b_1 \cdots b_{d-1})$. Then B is invertible and

$$B^{-1} = \operatorname{diag}(1, (b_0)^{-1}, (b_0b_1)^{-1}, \ldots, (b_0b_1 \cdots b_{d-1})^{-1}).$$

Furthermore,

$$B^{-1}TB = \begin{pmatrix} a_0 & b_0c_1 & & & 0 \\ 1 & a_1 & b_1c_2 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & a_{d-1} & b_{d-1}c_d \\ 0 & & & 1 & a_d \end{pmatrix},$$

which is an irreducible tridiagonal matrix in standard form, as claimed.

(iii) Define $C = \operatorname{diag}(1, c_1^{-1}, (c_1c_2)^{-1}, \ldots, (c_1c_2 \cdots c_d)^{-1})$. Then C is invertible and

$$C^{-1} = \operatorname{diag}(1, c_1, c_1c_2, \ldots, c_1c_2 \cdots c_d).$$

Furthermore,

$$C^{-1}TC = \begin{pmatrix} a_0 & 1 & & & 0 \\ b_0c_1 & a_1 & 1 & & \\ & b_1c_2 & \ddots & \ddots & \\ & & \ddots & a_{d-1} & 1 \\ 0 & & & b_{d-1}c_d & a_d \end{pmatrix},$$

which is an irreducible tridiagonal matrix in normalized form, as claimed.

$\square$

Corollary 2.1.1. *Assume $T \in Mat_{d+1}(\mathbb{R})$ is an irreducible tridiagonal matrix and $b_i c_{i+1} > 0$ for $0 \leq i \leq d-1$. Then T is similar to a real symmetric irreducible tridiagonal matrix.*

Proof. Immediate by the proof of (i) in the lemma above. $\square$

2.2 Association Schemes

A closely related structure to Leonard pairs and one of the primary structures of algebraic combinatorics and coding theory is the notion of an association scheme.

Definition 2.2.1. A *symmetric association scheme* $\mathscr{X}$ is a pair $(X, \{R_i\}_{i=0}^d)$, where X is a non-empty finite set and R_i is a non-empty relation on X for each i, with the following properties.

(i) $\{R_i\}_{i=0}^d$ is a partition of $X \times X$, that is, $\cup_{i=0}^d R_i = X \times X$ and $R_i \cap R_j = \varnothing$ for $i \neq j$;

(ii) $R_0 = \{(x,x) \mid x \in X\}$;

(iii) $R_i^t = R_i$ for $0 \leq i \leq d$, where $R_i^t = \{(y,x) \mid (x,y) \in R_i\}$;

(iv) there exist integers p_{ij}^h such that for any $0 \leq h, i, j \leq d$ and for any $x, y \in X$ with $(x,y) \in R_h$, the number of $z \in X$ with $(x,z) \in R_i$ and $(z,y) \in R_j$ is p_{ij}^h. (The p_{ij}^h are called the *intersection numbers* of $\mathscr{X}$.)

We often investigate association schemes by way of the following matrices.

Definition 2.2.2. Suppose $\mathscr{X}$ is a *symmetric association scheme*. For each i $(0 \leq i \leq d)$, define $A_i \in \mathrm{Mat}_X(\mathbb{K})$ with x, y entry given by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x,y) \in R_i, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.1}$$

We refer to the matrices $A_0, ..., A_d$ as the *associate matrices* of $\mathscr{X}$.

The associate matrices enjoy the following properties.

$$A_0 = I, \tag{2.2.2a}$$

$$\sum_{i=0}^{d} A_i = J, \tag{2.2.2b}$$

$$A_i^t = A_i \text{ for all } i \in \{0, \dots, d\}, \tag{2.2.2c}$$

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h \text{ for all } i, j. \tag{2.2.2d}$$

Note that the associate matrices commute, since for all i, j, $A_i A_j = A_j A_i$ holds and therefore $p_{ij}^h = p_{ji}^h$.

2.3 Distance-Regularity and Bose-Mesner Algebras

As mentioned in the overview (Section 1.1), distance-regular graphs (DRGs) give us many important examples of association schemes. Here we offer a brief introduction to the basic definitions. We will describe the *Bose-Mesner algebra*, the *dual Bose-Mesner algebra*, *P*- and *Q-polynomial property* (in Section 2.4), and the *subconstituent* or *Terwilliger algebra* (in Section 2.5).

For more information see [3, 6, 31, 18, 41, 42, 43].

Let $\Gamma = (X, R)$ be a finite, undirected, connected simple graph with vertex set X and edge set R. Furthermore, let $V = \mathbb{K}^X$ denote the vector space over $\mathbb{K}$ consisting of the column vectors with coordinates indexed by X and all entries in $\mathbb{K}$. Two vertices $x, y \in X$ are said to be *adjacent* (denoted $x \sim y$) whenever x and y form an edge. Let ∂ denote the *path-length distance function* for Γ, and define $d = \max\{\partial(x, y) \,|\, x, y \in X\}$. We call d the *diameter* of Γ. For $x \in X$ and an integer $i \geq 0$ define $\Gamma_i(x) = \{y \in X \,|\, \partial(x, y) = i\}$. For notational convenience we abbreviate $\Gamma(x) = \Gamma_1(x)$. For a nonnegative integer k we say that Γ is *regular with valency* k whenever $|\Gamma(x)| = k$ for all $x \in X$.

Definition 2.3.1. A graph Γ is said to be *distance-regular* whenever for all integers h, i, j $(0 \leq h, i, j \leq d)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, $p^h_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)|$ is independent of x and y. The numbers p^h_{ij} are called the *intersection numbers of* Γ. We often refer to Γ as a DRG.

For the remainder of this section we assume that Γ is distance-regular with $d \geq 3$.

By construction it is easy to see that $p^h_{ij} = p^h_{ji}$ for all $0 \leq h, i, j \leq d$. Let us abbreviate

$$a_j = p^j_{1j} \qquad (0 \leq j \leq d), \tag{2.3.1a}$$

$$b_j = p^j_{1,j+1} \quad (0 \leq j \leq d-1), \tag{2.3.1b}$$

$$c_j = p^j_{1,j-1} \quad (1 \leq j \leq d). \tag{2.3.1c}$$

Observe that $a_0 = 0$ and $c_1 = 1$ and furthermore, $b_j > 0$ $(0 \leq j \leq d-1)$ and $c_j > 0$ $(1 \leq j \leq d)$.

Now, if Γ is regular with valency k, then $k = b_0$ by (2.3.1b). Moreover,

$$a_j + b_j + c_j = k \tag{2.3.2}$$

for all $0 \leq j \leq d$, where $b_d = c_0 = 0$. For $0 \leq j \leq d$ define $k_j := p^0_{jj}$ and note that $k_j = |\Gamma_j(x)|$ for all $x \in X$. Observe that $k_0 = 1$ and $k_1 = k$. By a routine counting argument, we have $k_{j-1}b_{j-1} = k_j c_j$ for $1 \leq j \leq d$. Using this recursive relation, we have

$$k_j = \frac{b_0 b_1 \cdots b_{j-1}}{c_1 c_2 \cdots c_j} \qquad (0 \leq j \leq d). \tag{2.3.3}$$

By the triangle inequality and simple counting, one can easily derive the following well-known results (where δ_{ij} denotes the Kronecker delta function which is 1 when $i = j$ and 0 otherwise).

(i) $p^h_{ij} = 0$ if one of h, i, j is greater than the sum of the other two;

(ii) $p^h_{ij} \neq 0$ if one of h, i, j is equal to the sum of the other two;

(iii) $p^h_{0j} = \delta_{hj} \quad (0 \leq h, j \leq d)$;

(iv) $p^h_{i0} = \delta_{hi} \quad (0 \leq h, i \leq d)$;

(v) $p^0_{ij} = \delta_{ij} k_i \quad (0 \leq i, j \leq d)$;

(vi) $\displaystyle\sum_{i=0}^{d} p^h_{ij} = k_j \quad (0 \leq h, j \leq d)$.

Given any DRG Γ, the distance-i relations form a symmetric association scheme on the vertex set. Specifically, the distance matrices of Γ form the associate matrices for this scheme. Let us now elaborate the details of this connection.

Suppose Γ is a DRG of diameter d. With reference to Definition 2.2.2, for each $0 \leq i \leq d$ we may define a matrix $A_i \in \mathrm{Mat}_X(\mathbb{K})$ with $x, y \in X$ entry given by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i. \end{cases} \tag{2.3.4}$$

We call A_i the i^{th} *distance matrix of* Γ. We abbreviate $A = A_1$ and call this the *adjacency matrix of* Γ. Note that A_i satisfy the same properties as in (2.2.2a)-(2.2.2d). It follows immediately that we have a symmetric association scheme $\mathscr{X}_\Gamma$ that is associated with the DRG Γ.

Whenever we are given an association scheme $\mathscr{X}$, the associate matrices $A_0, \ldots, A_d$ form a basis for a subalgebra M of $\mathrm{Mat}_X(\mathbb{K})$. This leads to the following definition.

Definition 2.3.2. Given any symmetric association scheme $\mathscr{X}$, the subalgebra M of $\mathrm{Mat}_X(\mathbb{K})$ generated by the associate matrices $A_0, \ldots, A_d$ is called the *Bose-Mesner algebra* of $\mathscr{X}$.

Note that, since M has a basis of 0-1 matrices (the associate matrices), it follows that M is not only closed under ordinary matrix product, but also also under the entrywise (*Hadamard* or *Schur*) product, denoted by $\circ$.

By (2.3.4), it is clear that the associate matrices A_i are symmetric and mutually commute. As a result, they can be simultaneously diagonalized. Consequently M has a second basis $\{E_i\}_{i=0}^d$ such that

$$E_0 = |X|^{-1}J, \tag{2.3.5a}$$

$$\sum_{i=0}^{d} E_i = I, \tag{2.3.5b}$$

$$E_i^t = E_i \quad (0 \leq i \leq d), \tag{2.3.5c}$$

$$E_i E_j = \delta_{ij} E_i \quad (0 \leq i, j \leq d). \tag{2.3.5d}$$

We call $\{E_i\}_{i=0}^d$ the *primitive idempotents* of $\mathscr{X}$. Properties (2.3.5b) and (2.3.5d) imply that we may write

$$E_i \circ E_j = \sum_{h=0}^{d} q_{ij}^h E_h \qquad (0 \leq i, j \leq d) \tag{2.3.6}$$

for some constants q_{ij}^h, called the *Krein parameters* (or *dual intersection numbers*) of $\mathscr{X}$.

We now define the *dual Bose-Mesner algebra* of $\mathscr{X}$ relative to any given $x \in X$. To this end, fix a vertex $x \in X$ for the rest of this section. For each $0 \leq i \leq d$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{K})$ with

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i, \end{cases} \qquad (y \in X). \tag{2.3.7}$$

We call E_i^* the i^{th} *dual primitive idempotent of* $\mathscr{X}$ *with respect to* x. For $y \in X$,

$$E_i^* \hat{y} = \begin{cases} \hat{y} & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i, \end{cases} \tag{2.3.8}$$

where $\hat{y} \in V$ is a vector that has y-coordinate 1 and all other coordinates 0. (Note that $\{\hat{y}\}_{y \in X}$ form an orthonormal basis for V.) Observe that $\{E_i^*\}_{i=0}^d$ have similar properties as in (2.3.5b)-(2.3.5d):

$$E_0^* = \mathrm{diag}(0, \dots, 0, \overbrace{1}^{x}, 0, \dots, 0), \tag{2.3.9a}$$

$$\sum_{i=0}^{d} E_i^* = I, \tag{2.3.9b}$$

$$E_i^{*t} = E_i^* \quad (0 \leq i \leq d), \tag{2.3.9c}$$

$$E_i^* E_j^* = \delta_{ij} E_i^* \quad (0 \leq i, j \leq d). \tag{2.3.9d}$$

By these facts $\{E_i^*\}_{i=0}^d$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{K})$.

Definition 2.3.3. The commutative subalgebra M^* described above is called the *dual Bose-Mesner algebra of $\mathscr{X}$ with respect to x.*

2.4 P- and Q-Polynomial Property

Up to this point, we have not yet ordered the sequence $\{E_i\}_{i=0}^d$ of primitive idempotents in an association scheme $\mathscr{X}$. However, in an association scheme that arises from a distance-regular graph, the distance relations (and consequently the A_i matrices) are ordered naturally according to distance in the graph. And, since the eigenvalues of A_1 are all real in this case, we also obtain a natural ordering of the E_i matrices according to the ordering of these eigenvalues as real numbers. This leads to the following definition.

Definition 2.4.1. An association scheme $\mathscr{X}$ is said to be *P-polynomial* if the adjacency matrices $A_0, \ldots, A_d$ can be indexed so that each A_i is a polynomial of degree i in A_1. In this case, the Bose-Mesner algebra M is generated by A_1, that is $M = \langle A_1 \rangle$ and we think of $\mathscr{X}$ as the vertex set of a distance-regular graph.

It turns out that the corresponding association schemes of many important families of DRGs satisfy an important dual property which leads to the following definition.

Definition 2.4.2. An association scheme $\mathscr{X}$ is said to be *Q-polynomial* if the primitive idempotents $E_0, \ldots, E_d$ can be indexed so that each E_i is expressible as an entrywise polynomial of degree i in E_1.

The polynomials associated with a P- and Q-polynomial scheme are associated with certain orthogonal polynomials in the terminating branch of the *Askey-Wilson scheme* (and their limiting cases):

1. q-Racah
2. q-Hahn
3. Dual q-Hahn
4. Quantum q-Krawtchouk
5. q-Krawtchouk
6. Affine q-Krawtchouk
7. Dual q-Krawtchouk
8. Racah
9. Hahn
10. Dual Hahn
11. Krawtchouk
12. Bannai/Ito
13. Orphan

This leads to a dramatic reduction in the number of parameters when working with association schemes that belong to one of these families. As we will see later, the intersection numbers, and Krein parameters can be expressed in terms of at most 9 free parameters, organized into these 13 different cases.

2.5 Terwilliger Algebra, T- and Primary Module

In order to study P- and Q-polynomial schemes, Terwilliger introduced the idea of the *subconsituents of $\mathscr{X}$ with respect to $x \in X$*. This definition led to the notion of the *subconsituent* (or *Terwilliger*) *algebra* $T = T(x)$ of $\mathscr{X}$ relative to x. In this section we review these definitions, as well as the concept of a T-*module* and the so-called *primary module*.

From (2.3.7), we find that for $0 \leq i \leq d$,

$$E_i^* V = \operatorname{Span}\{\hat{y} \,|\, y \in \Gamma_i(x)\}. \tag{2.5.1}$$

By (2.5.1) and since $\{\hat{y}\}_{y \in X}$ is an orthonormal basis for V,

$$V = \bigoplus_{i=0}^{d} E_i^* V \quad \text{(orthogonal direct sum)}. \tag{2.5.2}$$

Definition 2.5.1. The span $E_i^* V$ defined in (2.5.1) is called the i^{th} *subconstituent of $\mathscr{X}$ with respect to x*.

Observe that:

(i) For $0 \leq i \leq d$, the subconstituent $E_i^* V$ is a common eigenspace for the dual Bose-Mesner algebra M^*.

(ii) $\dim(E_i^* V) = k_i$.

(iii) $E_0^* V = \mathbb{K}\hat{x}$.

(iv) $AE_i^* V \subseteq E_{i-1}^* V + E_i^* V + E_{i+1}^* V$, where $E_{-1}^* = E_{d+1}^* = 0$.

Now we define the *subconstituent* (or *Terwilliger*) *algebra* and the related objects called the T-*modules*.

Definition 2.5.2. Fix any x in a given P- and Q-polynomial scheme $\mathscr{X}$. The *Terwilliger algebra* of $\mathscr{X}$ (with respect to x) is the subalgebra $T = T(x)$ of $\mathrm{Mat}_X(\mathbb{K})$ generated by $A = A_1$ and $A^* = A^*(x) = \mathrm{diag}(E_1)_x$, that is,

$$T = T(x) = \langle A, A^* \rangle. \tag{2.5.3}$$

Definition 2.5.3. Given a Terwilliger algebra T (with respect to $x \in X$), by a *T-module* we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$. A T-module W is called *irreducible* whenever $W \neq 0$ and W does not contain a T-module besides 0 and itself.

Notice that T acts on $V = \mathbb{K}^X$ by left-multiplication, and V is a direct sum of irreducible T-modules. For more information on the Terwilliger algebra of an association scheme, see [9, 10, 13, 16, 18, 22, 38, 41, 42, 43].

Let us finally define the *primary module* for T.

Definition 2.5.4. The *primary module* for the Terwilliger algebra T is

$$V_0 = \mathrm{Span}\{v_0, \ldots, v_d\}, \tag{2.5.4}$$

where, for each $i = 0, \ldots, d$,

$$v_i = \sum_{\partial(x,y)=i} \hat{y}. \tag{2.5.5}$$

(Recall that for any $y \in X$, $\hat{y} \in V = \mathbb{K}^X$ is the vector that has y-coordinate 1 and all other coordinates 0. See page 14, immediately below (2.3.8).)

The primary module T_0 for T also has a basis consisting of eigenvectors for A:

$$V_0 = \mathrm{Span}\{w_0, \ldots, w_d\}, \tag{2.5.6}$$

where $w_0 = \vec{1}$ (i.e., all ones vector) and, for each $0 \leq i \leq d$, $Aw_i = AE_iw_i = \theta_i w_i$ (θ_i is the eigenvalue corresponding to w_i).

The following two propositions motivate the definition of a *Leonard pair* given in Definition 2.6.1.

Proposition 2.5.1. For the ordered basis $v_0, \ldots, v_d$, the generators A and A^* of the Terwilliger algebra T are irreducible tridiagonal and diagonal, respectively. That is,

$$A = \begin{pmatrix} a_0 & c_1 & & & 0 \\ b_0 & a_1 & c_2 & & \\ & b_1 & \ddots & \ddots & \\ & & \ddots & \ddots & c_d \\ 0 & & & b_{d-1} & a_d \end{pmatrix}, \quad A^* = \begin{pmatrix} \theta_0^* & & & & 0 \\ & \theta_1^* & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \theta_d^* \end{pmatrix}, \qquad (2.5.7)$$

for some scalars $a_i, b_i, c_i, \theta_i^* \in \mathbb{K}$ with $b_{i-1}c_i \neq 0$ for $1 \leq i \leq d$ to ensure the irreducibility of A.

Proposition 2.5.2. For the ordered basis $w_0, \ldots, w_d$, the generators A and A^* of the Terwilliger algebra T are diagonal and irreducible tridiagonal, respectively. That is,

$$A = \begin{pmatrix} \theta_0 & & & & 0 \\ & \theta_1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \theta_d \end{pmatrix}, \quad A^* = \begin{pmatrix} a_0^* & c_1^* & & & 0 \\ b_0^* & a_1^* & c_2^* & & \\ & b_1^* & \ddots & \ddots & \\ & & \ddots & \ddots & c_d^* \\ 0 & & & b_{d-1}^* & a_d^* \end{pmatrix}, \qquad (2.5.8)$$

for some scalars $a_i^*, b_i^*, c_i^* \in \mathbb{K}$ with $b_{i-1}^* c_i^* \neq 0$ for $1 \leq i \leq d$ to ensure the irreducibility of A^* and θ_i's are the eigenvalues of A.

Therefore, Propositions 2.5.1 and 2.5.2 indicate that the action of T on V_0 can be easily described and understood. For many families of P- and Q-polynomial schemes, all of the irredicible modules have a similar form (these are called *thin schemes*).

2.6 Leonard Pairs (LPs)

We are finally ready to define a Leonard pair.

Definition 2.6.1. A *Leonard pair* (LP) on V is an ordered pair (A, A^*) of linear transformations $A : V \to V$ and $A^* : V \to V$ in $\mathrm{End}(V)$ that satisfy conditions (i) and (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.

(ii) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal.

(See Propositions 2.5.1 and 2.5.2.) The *diameter* of the Leonard pair (A, A^*) is defined to be one less than the dimension of V. We refer to V as the vector space *underlying* the Leonard pair (A, A^*).

Note. In common notational convention, A^* often denotes the conjugate-transpose of A. However, we are not using this convention. In a Leonard pair (A, A^*) the elements A and A^* are arbitrary subject to (i) and (ii) above.

Leonard pairs were first introduced by Terwilliger [44] to extend the algebraic approach of Bannai and Ito [3] to a result of D. Leonard concerning the sequences of orthogonal polynomials with discrete support for which there is a dual sequence of orthogonal polynomials. By classifying LPs, Terwilliger has given an elegant reframing and generalization of Leonard's classification of P- and Q-polynomial schemes. In [45] Terwilliger classified the LPs up to isomorphism. By that classification, the isomorphism classes of LPs fall naturally into the thirteen families given on page 16. (For each integer $d \geq 3$ these families partition the isomorphism classes of LPs that have diameter d.)

Since a matrix $A \in \mathrm{Mat}_{d+1}(\mathbb{K})$ can be viewed as a linear transformation from $\mathbb{K}^{d+1}$ to $\mathbb{K}^{d+1}$, we have the following useful lemma to characterize a Leonard pair.

Lemma 2.6.1. *An ordered pair (A, A^*) of matrices in $Mat_{d+1}(\mathbb{K})$ is a Leonard pair on $\mathbb{K}^{d+1}$ if and only if the following hold.*

(i) *There exists a non-singular matrix Q_1 such that $Q_1^{-1}AQ_1$ is irreducible tridiagonal and $Q_1^{-1}A^*Q_1$ is diagonal.*

(ii) *There exists a non-singular matrix Q_2 such that $Q_2^{-1}AQ_2$ is diagonal and $Q_2^{-1}A^*Q_2$ is irreducible tridiagonal.*

When (i) *and* (ii) *hold we say that A and A^* form a Leonard pair via conjugating matrices Q_1 and Q_2.*

As an example of a Leonard pair (see [39]), set $V = \mathbb{K}^4$ and

$$A = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix} \qquad \text{and} \qquad A^* = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

and view A and A^* as linear transformations on V. We assume that the characteristic of $\mathbb{K}$ is not 2 or 3 to ensure A is irreducible. We claim that (A, A^*) is a Leonard pair on V. Notice that condition (i) in Definition 2.6.1 (or equivalently in Lemma 2.6.1) is satisfied by letting $Q_1 = I$, where I is the 4×4 identity matrix.

On the other hand, if we set

$$Q_2 = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix},$$

then we can easily check that $Q_2^{-1} A Q_2 = A^*$ and $Q_2^{-1} A^* Q_2 = A$, which are diagonal and irreducible tridiagonal, respectively, verifying condition (ii) in Lemma 2.6.1.

Note. The diagonal entries of A^* are the eigenvalues of A and the columns of Q_2 consist of the eigenbasis for A. The above pair (A, A^*) is called a *self-dual Leonard pair* (i.e., there exists an automorphism of $\mathrm{End}(V)$ that swaps A and A^*).

The above example turns out to be a member of the following infinite family of Leonard pairs: For any nonnegative integer d, the pair

$$A = \begin{pmatrix} 0 & d & & & & 0 \\ 1 & \ddots & d-1 & & & \\ & 2 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \\ 0 & & & & d & 0 \end{pmatrix} \qquad A^* = \begin{pmatrix} d & & & & 0 \\ & d-2 & & & \\ & & d-4 & & \\ & & & \ddots & \\ 0 & & & & -d \end{pmatrix} \tag{2.6.1}$$

is a Leonard pair on $V = \mathbb{K}^{d+1}$, provided the characteristic of $\mathbb{K}$ is zero or an odd prime greater than d to ensure that A is irreducible. Terwilliger showed that Definition 2.6.1 is satisfied by choosing $Q_1 = I_{d+1}$ and by letting the ij-entry of Q_2 be given by the following expression (see [39], Equation (3))

$$(Q)_{ij} = \binom{d}{j} {}_2F_1\left(\begin{matrix} -i, -j \\ -d \end{matrix} \middle| \, 2 \right), \tag{2.6.2}$$

where

$${}_2F_1\left(\begin{matrix} -i, -j \\ -d \end{matrix} \middle| \, 2 \right) := \sum_{n=0}^{d} \frac{(-i)_n(-j)_n 2^n}{(-d)_n n!} \qquad (0 \le i, j \le d) \tag{2.6.3}$$

is called a *hypergeometric function* and

$$(a)_n = \begin{cases} 1 & \text{if } n = 0 \\ a(a+1)(a+2)\cdots(a+n-1) & \text{if } n > 0 \end{cases} \tag{2.6.4}$$

is called the (rising) *Pochhammer symbol*. (The details of the above calculation can be found in [39].)

Leonard pairs naturally occur in the theory of orthogonal polynomials, occurring in families such as (see the list given on page 16):

- Racah
- Hahn, Dual Hahn
- Krawtchouk
- q-Racah
- q-Hahn, Dual q-Hahn
- q-Krawtchouk (classical, affine, quantum, dual)

Since Leonard pairs are linear-algebraic in nature, it is reasonable to define the notion of *isomorphic Leonard pairs*. See the following definition.

Definition 2.6.2. Let V and W be vector spaces over $\mathbb{K}$. Let (A, A^*) and (B, B^*) denote Leonard pairs on V and W, respectively. By an *isomorphism of Leonard pairs* we mean an isomorphism of vector spaces $\iota : V \to W$ such that $\iota A \iota^{-1} = B$ and $\iota A^* \iota^{-1} = B^*$. We say that (A, A^*) and (B, B^*) are *isomorphic* if there is an isomorphism of Leonard pairs from (A, A^*) to (B, B^*).

An isomorphism of Leonard pairs can also be seen from the following point of view. By the *Skolem-Noether Theorem*[1] (see also [44], Corollary 7.125), a map $\sigma : \mathrm{End}(V) \to \mathrm{End}(W)$ is a $\mathbb{K}$-algebra isomorphism if and only if there exists a $\mathbb{K}$-linear bijection $K : V \to W$ such that $X^\sigma = KXK^{-1}$ for all $X \in \mathrm{End}(V)$. In this case, we say that K *gives* σ. Assume that K gives σ. Then a $\mathbb{K}$-linear map $\widetilde{K} : V \to W$ gives σ if and only if there exists a nonzero $\alpha \in \mathbb{K}$ such that $\widetilde{K} = \alpha K$.

2.7 Leonard Systems (LSs)

When working with a Leonard pair, it is sometimes convenient to consider a closely related and more abstract object called a *Leonard system.* To define this we first make several observations about LPs. Most of the information in this section can be found in [40].

[1]Let R and S be simple unitary rings, and let c be the center of S, which is a field. If the dimension of S over c is finite (i.e., if S is a central simple algebra of finite dimension), and R is also a c-algebra, then given c-algebra homomorphisms $\phi, \psi : R \to S$, there exists a unit u in S such that for all r in R

$$\psi(r) = u \cdot \phi(r) \cdot u^{-1}.$$

In particular, every automorphism of a central simple c-algebra is an inner automorphism.

Lemma 2.7.1. *Let (A, A^*) be a LP on V. Then the eivenvalues of both A and A^* are distinct and contained in $\mathbb{K}$.*

Proof. By Definition 2.6.1(ii), there exists a basis for V consisting of eigenvectors of A. So the eivenvalues of A are clearly all in $\mathbb{K}$. To show that the eivenvalues of A are distinct, we show the minimal polynomial of A has degree equal to $\dim(V)$. To this end, by Definition 2.6.1(i), there exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and we denote this matrix T. Lemma 2.6.1 implies that A and T have the same minimal polynomial. On the other hand, the tridiagonal shape of T tells us that $I, T, T^2, \ldots, T^d$ are linearly independent, where $d + 1 = \dim(V)$ and therefore, the minimal polynomial of T has degree $d+1$. This shows that the minimal polynomial of A has degree $d + 1$ also and hence the eigenvalues of A are distinct. The case of A^* is similar. $\square$

We now define a Leonard system.

Definition 2.7.1. By a *Leonard system* (LS) on V we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \tag{2.7.1}$$

of elements in $\mathrm{End}(V)$ that satisfy:

(i) A, A^* are both multiplicity-free[2] elements of $\mathrm{End}(V)$.

(ii) $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of A.

(iii) $\{E_i^*\}_{i=0}^d$ is an ordering of the primitive idempotents of A^*.

[2] A and A^* are diagonalizable and their eigenspaces all have dimension one.

(iv) $E_i^* A E_j^* \begin{cases} = 0, & \text{if } |i-j| > 1; \\ \neq 0, & \text{if } |i-j| = 1; \end{cases} \qquad (0 \leq i, j \leq d).$

(v) $E_i A^* E_j \begin{cases} = 0, & \text{if } |i-j| > 1; \\ \neq 0, & \text{if } |i-j| = 1; \end{cases} \qquad (0 \leq i, j \leq d).$

The Leonard system Φ is said to be *over* $\mathbb{K}$ and have *diameter* d.

LPs and LSs are related as follows: Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a LS on V. Then (A, A^*) is a LP on V. Conversely, let (A, A^*) denote a LP on V. Then each of A, A^* is multiplicity-free. Moreover there exists an ordering of the primitive idempotents $\{E_i\}_{i=0}^d$ and $\{E_i^*\}_{i=0}^d$ of A and A^*, respectively such that Φ is a LS on V. This leads to the following definition.

Definition 2.7.2. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a LS on V. Then the pair (A, A^*) forms a LP on V. We say this pair is *associated* with Φ. Observe each LS is associated with a unique LP.

Using Definition 2.6.2, we may define the notion of an isomorphism of LS.

Definition 2.7.3. Let V and V' be vector spaces over $\mathbb{K}$. Let Φ and Φ' denote LSs on V and V', respectively. By an *isomorphism of LSs* from Φ to Φ', we mean an isomorphism of vector spaces $\iota : V \mapsto V'$ such that $\iota\Phi\iota^{-1} = \Phi'$ and $\iota\Phi'\iota^{-1} = \Phi$. We say that Φ and Φ' are *isomorphic* if there is an isomorphism of LSs from Φ to Φ'.

LSs can be modified in several different ways to get a new LS. Let Φ denote a LS. Then each of the following three sequences is also a LS on V:

$$\Phi^* = (A^*, \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d), \tag{2.7.2a}$$

$$\Phi^{\downarrow} = (A, \{E_i\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d), \tag{2.7.2b}$$

$$\Phi^{\Downarrow} = (A, \{E_{d-i}^*\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d). \tag{2.7.2c}$$

We refer to Φ^* (resp. $\Phi^{\downarrow}$ and $\Phi^{\Downarrow}$) as the *dual* (resp. *first inversion* and *second inversion*) of Φ. If we view $*, \downarrow, \Downarrow$ as permutations on the set of all LSs, then it is easy to verify that

$$*^2 = \downarrow^2 = \Downarrow^2 = 1, \tag{2.7.3a}$$

$$\Downarrow * = * \downarrow, \qquad \downarrow * = * \Downarrow, \qquad \downarrow \Downarrow = \Downarrow \downarrow. \tag{2.7.3b}$$

It is also easy to see that the group generated by the symbols $\{*, \downarrow, \Downarrow\}$ subject to the relations (2.7.3a) and (2.7.3b) is the dihedral group D^4 and $\{*, \downarrow, \Downarrow\}$ induce an action of D_4 on the set of all LSs.

We end this section by recalling some parameters that will help us characterize a given LS.

Definition 2.7.4. Let Φ denote the LS. For $0 \leq i \leq d$, we let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with the primitive idempotents E_i (resp. E_i^*). We refer to $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) as the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of Φ.

2.8 The Standard Basis and the Split Basis

Let Φ denote the LS on V. Using Φ we define three bases for V, called the Φ-*standard basis*, the Φ-*split basis*, and Φ-*inverted split basis*. In each of the three cases, the basis is defined up to multiplication of each element by the same nonzero scalar in $\mathbb{K}$. The information in this section can be found in [40].

In order to define a *standard basis*, we need the following lemma.

Lemma 2.8.1. [40, Lemma 5.1] *Let Φ be a LS on V. Let u be a nonzero element of E_0V. Then for $0 \leq i \leq d$, the element E_i^*u is nonzero and hence a basis for E_i^*V.*

Moreover, the sequence

$$E_0^*u, E_1^*u, \dots, E_d^*u \tag{2.8.1}$$

is a basis for V and $u = \sum_{i=0}^{d} E_i^*u$.

Similarly, let u^* be a nonzero element of E_0^*V. Then for $0 \le i \le d$, the element E_iu^* is nonzero and hence a basis for E_iV. Moreover, the sequence

$$E_0u^*, E_1u^*, \dots, E_du^* \tag{2.8.2}$$

is a basis for V and $u^* = \sum_{i=0}^{d} E_iu^*$.

We now define a *standard basis* for V.

Definition 2.8.1. [40, Definition 5.2] Let Φ be a LS on V. By a Φ-*standard basis* for V, we mean a sequence (2.8.1), where u is a nonzero vector in E_0^*V.

Remark Given a LP (A, A^*), by Definition 2.6.1 it is natural to represent one of A, A^* by an irreducible tridiagonal matrix and the other by a diagonal matrix. With respect to a Φ-standard basis for V, the matrices representing A and A^* can be written as

$$A = \begin{pmatrix} a_0 & c_1 & & & 0 \\ b_0 & a_1 & c_2 & & \\ & b_1 & \ddots & \ddots & \\ & & \ddots & \ddots & c_d \\ 0 & & & b_{d-1} & a_d \end{pmatrix}, \qquad A = \begin{pmatrix} \theta_0^* & & & & 0 \\ & \theta_1^* & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \theta_d^* \end{pmatrix}, \tag{2.8.3}$$

for some scalars $a_i, b_i, c_i, \theta_i^* \in \mathbb{K}$ with $b_{i-1}c_i \ne 0$ for $1 \le i \le d$. (Recall (2.5.7) on page 20.) We call the scalars $\{a_i\}_{i=0}^{d}, \{b_i\}_{i=0}^{d-1}, \{c_i\}_{i=1}^{d}$ the *intersection numbers* of Φ. Since $u = \sum_{i=0}^{d} E_i^*u$ and $Au = \theta_0 u$,

$$a_i + b_i + c_i = \theta_0 \qquad (0 \le i \le d), \tag{2.8.4}$$

where $c_0 = b_d = 0$.

Next, we define the notion of a *split basis*. We will first recall two sequences of scalars which we will find useful. These sequences are called the *first split sequence* and the *second split sequence* of a LS Φ.

To this end, let Φ denote a LS on V. For $0 \leq i \leq d$, define

$$U_i = (E_0^*V + E_1^*V + \cdots + E_i^*V) \cap (E_iV + E_{i+1}V + +E_dV). \tag{2.8.5}$$

Each of $U_0, U_1, \ldots, U_d$ has dimension 1, and that

$$V = \bigoplus_{i=0}^{d} U_i \qquad \text{(direct sum)}. \tag{2.8.6}$$

Moreover,

$$U_0 + U_1 + \cdots + U_i = E_0^*V + E_1^*V + \cdots + E_i^*V, \tag{2.8.7a}$$

$$U_i + U_{i+1} + \cdots + U_d = E_iV + E_{i+1}V + \cdots + E_dV \tag{2.8.7b}$$

for $0 \leq i \leq d$. The elements A and A^* act on the U_i in the following way:

$$(A - \theta_i I)U_i = U_{i+1} \qquad (0 \leq i \leq d-1), \tag{2.8.8a}$$

$$(A - \theta_d I)U_d = 0, \tag{2.8.8b}$$

$$(A^* - \theta_i^* I)U_i = U_{i-1} \qquad (1 \leq i \leq d), \tag{2.8.8c}$$

$$(A^* - \theta_0^* I)U_0 = 0, \tag{2.8.8d}$$

where θ_i, θ_i^* are from Definition 2.7.4. By (2.8.8a), $(A - \theta_{i-1}I)U_{i-1} = U_i$ and combining this result with (2.8.8c), we see that

$$(A^* - \theta_0^* I)(A - \theta_{i-1} I)U_i = U_i, \tag{2.8.9}$$

implying that U_i is an eigenspace for $(A^* - \theta_0^* I)(A - \theta_{i-1} I)$ and the corresponding eigenvalue is a nonzero element of $\mathbb{K}$. We denote this eigenvalue by φ_i. We refer to the sequence $\{\varphi_i\}_{i=1}^d$ as the *first split sequence* of Φ.

We let $\{\phi_i\}_{i=1}^d$ denote the first split sequence of $\Phi^{\Downarrow}$, and call this the *second split sequence* of Φ. For notational convenience, we define $\varphi_0 = \varphi_{d+1} = \phi_0 = \phi_{d+1} = 0$.

We are finally ready to obtain the *split basis* for V as follows. Set $i = 0$ in (2.8.8a) to get $U_0 = E_0^* V$. Combining this with (2.8.8a), we find

$$U_i = (A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I)E_0^* V \qquad (0 \leq i \leq d). \tag{2.8.10}$$

Let $u \in E_0^* V$ be a nonzero vector. From (2.8.10) we find that the vector $(A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I)u$ is a basis for U_i for $0 \leq i \leq d$. Combining this fact with (2.8.6) we see that the sequence

$$(A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I)u \qquad (0 \leq i \leq d) \tag{2.8.11}$$

is a basis for V.

Definition 2.8.2. Let Φ denote a LS on V. By a Φ-*split basis* for V, we mean a sequence (2.8.11), where u is a nonzero vector in $E_0^* V$.

Remark Given a LP (A, A^*), by Definition 2.6.1 it is natural to represent one of A, A^* by an irreducible tridiagonal matrix and the other by a diagonal matrix (as in 2.8.3). However, with respect to any Φ-split basis for V, the matrices representing A and A^* can be written as

$$A = \begin{pmatrix} \theta_0 & & & & 0 \\ 1 & \theta_1 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & \theta_d \end{pmatrix}, \quad A = \begin{pmatrix} \theta_0^* & \varphi_1 & & & 0 \\ & \theta_1^* & \varphi_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \varphi_d \\ 0 & & & & \theta_d^* \end{pmatrix}. \qquad (2.8.12)$$

We call this the *split representation*. (The matrix A and A^* in (2.8.12) are said to be in *lower* and *upper bidiagonal*, respectively.)

2.9 Parameter Array

We now introduce sequences of parameters that will be used to described a given LP/LS and classify LPs in Chapter 4. The information in this section can be found in [44, 47].

Recall that in Section 2.7, we defined the eigenvalue and dual eigenvalue sequences $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ (see Definition 2.7.4) as well as the first and second split sequence of a LS $\{\varphi_i\}_{i=1}^d$ and $\{\phi_i\}_{i=1}^d$ in the preceeding Section 2.8. These four sequences form a *parameter array*. See the following definition.

Definition 2.9.1. Let Φ be a LS on V. By the *parameter array* (denoted by $\mathcal{P}$) of Φ we mean the sequence $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$.

The next four lemmas mention several characteristics of parameter arrays.

Lemma 2.9.1. [44, Theorem 1.9] *Two LPs over $\mathbb{K}$ are isomorphic if and only if they have a parameter array in common.*

Lemma 2.9.2. [44, Theorem 1.9] *Consider a sequence*

$$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d) \qquad (2.9.1)$$

of scalars in $\mathbb{K}$. There exists a LS Φ on V with parameter array (2.9.1) if and only if the following conditions hold:

(i) $\theta_i \neq \theta_j, \qquad \theta^*_i \neq \theta^*_j \qquad$ *if* $\quad i \neq j \qquad (0 \leq i, j \leq d)$;

(ii) $\varphi_i \neq 0, \qquad \phi_i \neq 0 \qquad (1 \leq i \leq d)$;

(iii) $\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{i-1} - \theta_d) \qquad (1 \leq i \leq d)$;

(iv) $\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0) \qquad (1 \leq i \leq d)$;

(v) *The expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \tag{2.9.2}$$

are equal and independent of i for $2 \leq i \leq d-1$. (Both the eigenvalue and dual eigenvalue sequences $\{\theta_i\}_{i=0}^d$ and $\{\theta^*_i\}_{i=0}^d$ are said to be recurrent.)

Moreover, if (i)-(v) *hold then Φ is unique up to isomorphism of LSs.*

Lemma 2.9.3. [47, Lemma 10.3] *For $d \geq 1$, a parameter array $\mathcal{P}$ is uniquely determined by $\varphi_i, \{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d$.*

Lemma 2.9.4. [49, Theorem 1.11] *Let $A, A^* \in \mathrm{Mat}_{d+1}(\mathbb{K})$. Assume that A and A^* are lower bidiagonal and upper bidiagonal, respectively. Then the following* (i), (ii) *are equivalent.*

(i) *The pair (A, A^*) is a LP on $\mathbb{K}^{d+1}$.*

(ii) *There exists a parameter array $\mathcal{P} = (\{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$ over $\mathbb{K}$ such that*

$$A_{ii} = \theta_i, \qquad A^*_{ii} = \theta^*_i \qquad (0 \leq i \leq d), \tag{2.9.3a}$$

$$A_{i,i-1}A^*_{i-1,i} = \varphi_i \qquad (1 \leq i \leq d). \qquad (2.9.3b)$$

Suppose (i), (ii) *hold. For* $0 \leq i \leq d$ *let* E_i *(resp.* E^*_i*) denote the primitive idempotent of* A *(resp.* A^**) associated with* θ_i *(resp.* θ^*_i*). Then* $(A; \{E_i\}_{i=0}^d; A^*, \{E^*_i\}_{i=0}^d)$ *is a LS on* $\mathbb{K}^{d+1}$ *with parameter array* $\mathcal{P}$*.*

We end this section with the following two definitions that will be used in Chapter 4.

Definition 2.9.2. Let $\mathcal{P}$ denote a parameter array over $\mathbb{K}$ with $d \geq 3$. Define $\beta \in \mathbb{K}$ such that $\beta + 1$ is equal to the common value of the two fractions in (2.9.2). We call β the *fundamental constant* of $\mathcal{P}$.

Definition 2.9.3. Let (A, A^*) denote a LP over $\mathbb{K}$ with diameter $d \geq 3$. The parameter arrays of (A, A^*) have the same fundamental constant β; we call β the *fundamental constant* of (A, A^*).

Before we consider our primary object of interest, we consider two closely-related classes of objects: *bipartite Leonard pairs* (BLPs) and *almost bipartite Leonard pairs* (ABLPs) that are inspired by DRG families.

2.10 Bipartite Leonard Pairs (BLPs)

Given a tridiagonal matrix in (2.1.1), it is said to be *bipartite* whenever all entries on the main diagonal are zero

$$\begin{pmatrix} 0 & c_1 & & & 0 \\ b_0 & \ddots & c_2 & & \\ & b_1 & \ddots & \ddots & \\ & & \ddots & \ddots & c_d \\ 0 & & & b_{d-1} & 0 \end{pmatrix}. \tag{2.10.1}$$

This leads to the following definition.

Definition 2.10.1. A Leonard pair (A, A^*) is said to be:

(i) *bipartite* whenever the matrix representing A from Definition 2.6.1(i) is bipartite.

(ii) *dual bipartite* (or DB) whenever the matrix representing A^* from Definition 2.6.1(ii) is bipartite.

(iii) *totally bipartite* (or TB) whenever it is both bipartite and dual bipartite.

Brown classified up to isomorphism the totally bipartite Leonard pairs of Bannai/Ito type in [7]. Motivated by [7], Hou, Wang, and Gao classified up to isomorphism the totally bipartite Leonard pairs in [37]. The classification reveals that these Leonard pairs are of the q-Racah, Krawtchouk, or Bannai/Ito type.

2.11 Almost Bipartite Leonard Pairs (ABLPs)

Given a tridiagonal matrix in (2.1.1), it is said to be *almost bipartite* whenever exactly one of a_0, a_d is nonzero and $a_i = 0$ for $1 \leq i \leq d-1$.

$$\begin{pmatrix} 0 & c_1 & & & 0 \\ b_0 & \ddots & c_2 & & \\ & b_1 & \ddots & \ddots & \\ & & \ddots & 0 & c_d \\ 0 & & & b_{d-1} & a_d \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a_0 & c_1 & & & 0 \\ b_0 & 0 & c_2 & & \\ & b_1 & \ddots & \ddots & \\ & & \ddots & \ddots & c_d \\ 0 & & & b_{d-1} & 0 \end{pmatrix}. \tag{2.11.1}$$

This leads to the following definition.

Definition 2.11.1. A Leonard pair (A, A^*) is said to be:

(i) *almost bipartite* (AB) whenever the matrix representing A from Definition 2.6.1(i) is almost bipartite.

(ii) *dual almost bipartite* (or DAB) whenever the matrix representing A^* from Definition 2.6.1(ii) is almost bipartite.

(iii) *totally almost bipartite* (or TAB) whenever it is both AB and DAB.

Brown classifed up to isomorphism the totally almost bipartite Leonard pairs of Bannai/Ito type in [7]. Motivated by [7], Gao, Hou, and Wang classified up to isomorphism the totally almost bipartite Leonard pairs of q-Racah type.

3 Doubly Almost Bipartite Leonard Pairs

Now, taking BLPs and ABLPs as departure points, we introduce a new class of object - *doubly almost bipartite Leonard pairs* (*DABLPs*).

3.1 Definition and Motivation

Given a tridiagonal matrix in (2.1.1), it is said to be *doubly almost bipartite* whenever $a_0 \neq 0, a_d \neq 0$ and $a_i = 0$ for $1 \leq i \leq d-1$

$$\begin{pmatrix} a_0 & c_1 & & & 0 \\ b_0 & 0 & c_2 & & \\ & b_1 & \ddots & \ddots & \\ & & \ddots & 0 & c_d \\ 0 & & & b_{d-1} & a_d \end{pmatrix}. \tag{3.1.1}$$

(Note the intersection array of a DRG could never have this form since a_0 counts the neighbors of a vertex x distance 0 from x, thus $a_0 = 0$ for any DRG. See page 12, immediately below (2.3.1c).)

This leads to the following definition.

Definition 3.1.1. A Leonard pair (A, A^*) is said to be:

(i) *doubly almost bipartite* (DAB) whenever the matrix representing A from Definition 2.6.1(i) is doubly almost bipartite.

(ii) *dual doubly almost bipartite* (or DDAB) whenever the matrix representing A^* from Definition 2.6.1(ii) is doubly almost bipartite.

(iii) *totally doubly almost bipartite* (or TDAB) whenever it is both DAB and DDAB.

To somewhat motivate this doubly almost bipartite structure of A, we consider the following series of lemmas given in [15, 43].

Lemma 3.1.1. [43, Lemma 5.4] *Let (A, A^*) be a Leonard pair. Then*

$$A^3A^* - (\beta+1)A^2A^*A + (\beta+1)AA^*A^2 - A^*A^3 \\ -\gamma(A^2A^* - A^*A^2) - \varrho(AA^* - A^*A) = 0, \qquad (3.1.2a)$$

$$A^{*3}A - (\beta+1)A^{*2}AA^* + (\beta+1)A^*AA^{*2} - AA^{*3} \\ -\gamma^*(A^{*2}A - AA^{*2}) - \varrho^*(A^*A - AA^*) = 0, \qquad (3.1.2b)$$

where

$$\beta = \frac{\theta_i - \theta_{i+1} + \theta_{i+2} - \theta_{i+3}}{\theta_{i+1} - \theta_{i+2}} = \frac{\theta_i^* - \theta_{i+1}^* + \theta_{i+2}^* - \theta_{i+3}^*}{\theta_{i+1}^* - \theta_{i+2}^*} \qquad (0 \le i \le d-3), \qquad (3.1.3a)$$

$$\gamma = \theta_i - \beta\theta_{i+1} + \theta_{i+2} \qquad (0 \le i \le d-2), \qquad (3.1.3b)$$

$$\gamma^* = \theta_i^* - \beta\theta_{i+1}^* + \theta_{i+2}^* \qquad (0 \le i \le d-2), \qquad (3.1.3c)$$

$$\varrho = \theta_i^2 - \beta\theta_i\theta_{i+1}\theta_{i+1}^2 - \gamma(\theta_i + \theta_{i+1}) \qquad (0 \le i \le d-1), \qquad (3.1.3d)$$

$$\varrho^* = \theta_i^{*2} - \beta\theta_i^*\theta_{i+1}^*\theta_{i+1}^{*2} - \gamma^*(\theta_i^* + \theta_{i+1}^*) \qquad (0 \le i \le d-1). \qquad (3.1.3e)$$

Lemma 3.1.2. [43, Lemma 5.5] *Let $(A, A^*), \beta, \gamma, \gamma^*, \varrho, \varrho^*$ be as in Lemma 3.1.1. Let E_i^* $(0 \le i \le d)$ be the i^{th} dual primitive idempotent. Then*

(i) $[E_i^*AE_i^*, E_i^*AE_{i+1}^*AE_i^*] = h_i[E_i^*AE_i^*, E_i^*AE_{i-1}^*AE_i^*] \qquad (0 \le i \le d-1),$ (3.1.4)

where $$h_i = \frac{\theta_{i-1}^* - \theta_i^*}{\theta_i^* - \theta_{i+1}^*} \qquad (1 \le i \le d-1), \qquad (3.1.5)$$

h_0, h_d are indeterminates, and $[s,t] := st - ts$ denotes the Lie bracket.

(ii) $e_i^- E_{i-1}^* A E_{i-2}^* A^2 E_i^* + (\beta+2) E_{i-1}^* A E_i^* A E_{i-1}^* A E_i^*$

$$+ e_i^+ E_{i-1}^* A^2 E_{i+1}^* A E_i^* + E_{i-1}^* (A E_i^*)^3 - \beta E_{i-1}^* A E_{i-1}^* (A E_i^*)^2 + (E_{i-1}^* A)^3 E_i^*$$

$$= \gamma\big(E_{i-1}^* (A E_i^*)^2 + (E_{i-1}^* A)^2 E_i^*\big) \qquad (1 \le i \le d) \tag{3.1.6}$$

where

$$e_i^+ = \frac{\theta_{i-1}^* - (\beta+2)\theta_i^* + (\beta+1)\theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \qquad (1 \le i \le d-1), \tag{3.1.7a}$$

$$e_i^- = \frac{-(\beta+1)\theta_{i-2}^* + (\beta+2)\theta_{i-}^* - \theta_i^*}{\theta_{i-1}^* - \theta_i^*} \qquad (2 \le i \le d-1), \tag{3.1.7b}$$

and e_d^+, e_1^- *are indeterminants.*

(iii) *For* $2 \le i \le d$,

$$g_i^- E_{i-2}^* A E_{i-2}^* A^2 E_i^* + E_{i-2}^* (A E_{i-1}^*)^2 A E_i^* + g_i^+ E_{i-2}^* A (A E_i^*)^2 = \gamma E_{i-2}^* A^2 E_i^*, \tag{3.1.8}$$

where

$$g_i^+ = \frac{\theta_{i-2}^* - (\beta+1)\theta_{i-1}^* + \beta\theta_i^*}{\theta_{i-2}^* - \theta_i^*} \qquad (2 \le i \le d), \tag{3.1.9a}$$

$$g_i^- = \frac{-\beta\theta_{i-2}^* + (\beta+1)\theta_{i-1}^* - \theta_i^*}{\theta_{i-2}^* - \theta_i^*} \qquad (2 \le i \le d). \tag{3.1.9b}$$

(iv) *Let* $h_i^*, e_i^{*+}, e_i^{*-}, g_i^{*+}, g_i^{*-}$ *denote the constants obtained from (3.1.5), (3.1.7a), (3.1.7b), (3.1.9a), (3.1.9b) by replacing* θ_j^* *by* θ_j $(0 \le j \le d)$. *Then the equations (3.1.5), (3.1.6), (3.1.8) still hold after replacing* $\gamma, \varrho, A, h_i, e_i^\pm, g_i^\pm$, *and* E_j^* $(0 \le j \le d)$ *by* $\gamma^*, \varrho^*, A^*, h_i^*, e_i^{*\pm}, g_i^{*\pm}$, *and* E_j $(0 \le j \le d)$, *respectively.*

Lemma 3.1.3. [43, Lemma 5.6] *Let $h_i^*, e_i^{\pm}, e_i^{*\pm}, g_i^{\pm}, g_i^{*\pm}$ be as in Lemma 3.1.2. Then*

$$
\begin{aligned}
e_i^{+} &= \frac{\theta_i^* - \theta_{i+2}^*}{\theta_i^* - \theta_{i-1}^*} \qquad (1 \le i \le d-2), && (3.1.10a)\\
e_i^{-} &= \frac{\theta_{i-1}^* - \theta_{i-3}^*}{\theta_{i-1}^* - \theta_i^*} \qquad (3 \le i \le d), && (3.1.10b)\\
g_i^{+} &= \frac{\theta_i^* - \theta_{i+1}^*}{\theta_i^* - \theta_{i-2}^*} \qquad (2 \le i \le d-1), && (3.1.10c)\\
g_i^{-} &= \frac{\theta_{i-2}^* - \theta_{i-3}^*}{\theta_{i-2}^* - \theta_i^*} \qquad (3 \le i \le d). && (3.1.10d)
\end{aligned}
$$

To get $e_i^{*\pm}, g_i^{*\pm}$, replace θ_j^* by θ_j $(0 \le j \le d)$ in the above formulae. In particular, $h_i^*, e_i^{\pm}, e_i^{*\pm}, g_i^{\pm}, g_i^{*\pm}$ are all nonzero due to the fact that both A and A^* are multiplicity-free and thus θ_j, θ_j^* are distinct.

Lemma 3.1.4. [15, Lemma 2.3(i), (ii)] *Let $\Gamma = (X, R)$ be a DRG of diameter $d \ge 3$. Suppose that Γ is Q-polynomial with respect to an eigenvalue θ, and suppose that the intersection number a_2 is zero. Then:*

(i) *there exists real numbers $\gamma(\theta), g_i^-(\theta)$, and $g_i^+(\theta)$ such that*

$$
\begin{aligned}
g_i^-(\theta)a_{i-2} + a_{i-1} + g_i^+(\theta)a_i = \gamma(\theta), &\qquad 2 \le i \le d; && (3.1.11a)\\
g_i^+(\theta) \ne 0, &\qquad 2 \le i \le d-1. && (3.1.11b)
\end{aligned}
$$

(ii) *the intersection numbers $a_1, \ldots, a_{d-1}$ are all zero.*

The following is the variation of Lemma 3.1.4 above that gives a motivation to why one might be interested in the doubly almost bipartite structure of A.

Lemma 3.1.5. *Let*

$$A = \begin{pmatrix} a_0 & b_0 & & & 0 \\ c_1 & a_1 & b_1 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & a_{d-1} & b_{d-1} \\ 0 & & & c_d & a_d \end{pmatrix} \quad \textit{and} \quad A^* = \mathrm{diag}(\theta_0^*, \ldots, \theta_d^*)$$

be $(d+1) \times (d+1)$ irreducible tridiagonal and diagonal matrices (with $d \geq 3$), respectively. Suppose $a_i = 0$ for $i = 1, 2, 3$. Then

(i) *there exist real numbers γ, g_i^+, and g_i^- (as in (3.1.3b), (3.1.10c), and (3.1.10d), respectively) such that*

$$g_i^- a_{i-2} + a_{i-1} + g_i^+ a_i = \gamma, \qquad 2 \leq i \leq d, \tag{3.1.12a}$$

$$g_i^+ \neq 0, \qquad 2 \leq i \leq d-1; \tag{3.1.12b}$$

(ii) *$a_i = 0$ for all $i = 4, \ldots, d-1$.*

Proof. (i) Note that (3.1.12b) follows directly from Lemma 3.1.3.

To prove (3.1.12a), let us define the following three matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$:

$$L = \sum_{i=1}^{d} E_{i-1}^* A E_i^*, \tag{3.1.13a}$$

$$F = \sum_{i=0}^{d} E_i^* A E_i^*, \tag{3.1.13b}$$

$$R = \sum_{i=0}^{d-1} E_{i+1}^* A E_i^*. \tag{3.1.13c}$$

(These three matrices are referred to as *lowering, flat*, and *raising operators*, respectively.) With these three operators, together with the fact that $A = L + F + R$, we can easily show that

$$E^*_{i-2}AE^*_{i-2}A^2E^*_i = FL^2E^*_i,$$

$$E^*_{i-2}(AE^*_{i-1})^2AE^*_i = LFLE^*_i,$$

$$E^*_{i-2}A(AE^*_i)^2 = L^2FE^*_i,$$

$$E^*_{i-2}A^2E^*_i = L^2E^*_i.$$

Hence (3.1.8) in Lemma 3.1.2(iii) can be rewritten in terms of L, F, and R as follows:

$$g_i^- FL^2E^*_i + LFLE^*_i + g_i^+ L^2FE^*_i = \gamma L^2E^*_i$$

or equivalently,

$$(g_i^- FL^2 + LFL + g_i^+ L^2F - \gamma L^2)E^*_i = 0. \qquad (3.1.14)$$

Let $\vec{1}$ denote the $(d+1) \times 1$ all ones vector and observe by (3.1.13a) and (3.1.13b) that

$$LE^*_i\vec{1} = b_{i-1}E^*_{i-1}\vec{1}, \qquad (1 \leq i \leq d), \qquad (3.1.15a)$$

$$FE^*_i\vec{1} = a_iE^*_i\vec{1}. \qquad (1 \leq i \leq d), \qquad (3.1.15b)$$

Applying (3.1.14) to $\vec{1}$ and using (3.1.15a) and (3.1.15b) yields

$$b_{i-2}b_{i-1}(g_i^- a_{i-2} + a_{i-1} + g_i^+ a_i - \gamma)E^*_{i-2}\vec{1} = 0 \tag{3.1.16}$$

for all $2 \le i \le d$ and since $b_{i-2}, b_{i-1}, E^*_{i-2}$, and $\vec{1}$ are all nonzero, (3.1.12a) follows immediately.

(ii) Setting $i = 3$ in (3.1.12a), we find that $\gamma = 0$ and thus it becomes

$$g_i^- a_{i-2} + a_{i-1} + g_i^+ a_i = 0 \qquad (2 \le i \le d). \tag{3.1.17}$$

By (3.1.12b), $g_i^+ \neq 0$ for $2 \le i \le d-1$ and a simple induction shows that $a_4, \dots, a_{d-1}$ are zero, as claimed.

□

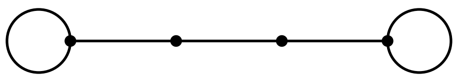
3.2 All Ones DABLPs

Ultimately, we would like to classify up to isomorphism the DABLPs. However, in order to simplify the problem, we are going to first consider the following situation.

Fix an integer $d \ge 1$ and consider a pair of $(d+1) \times (d+1)$ matrices (A, A^*) over $\mathbb{K}$ that have the following form:

$$A = \begin{pmatrix} 1 & 1 & & & 0 \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ 0 & & & 1 & 1 \end{pmatrix} \qquad A^* = \begin{pmatrix} \theta_0^* & & & & 0 \\ & \theta_1^* & & & \\ & & \ddots & & \\ & & & \theta_{d-1}^* & \\ 0 & & & & \theta_d^* \end{pmatrix} \tag{3.2.1}$$

We call the matrix A above the *all ones doubly almost bipartite irreducible tridiagonal matrix (DABITM)*. Note that the matrix A given in (3.2.1) arises as the adjacency matrix of a path with a loop at each leaf. See Figure 3.2.1 and the corresponding adjacency matrix below.



$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Figure 3.2.1: P_4 with a loop & its corresp. adjacency matrix.

In seemingly unrelated research, Willenbring, Bourn, and Erickson set out to study something of unexpected value: the expected value of the *Earth Mover's Distance* - a metric used to compare histograms [5, 17]. One aspect of their work is related to properties of the matrix A given in (3.2.1).

Our primary goal is to find attractive necessary and sufficient conditions for the pair (A, A^*) in (3.2.1) to form an all ones DABLP. Observe that:

- A and A^* are already doubly almost bipartite and diagonal, respectively. To satisfy condition (i) in Definition 2.6.1 (or Lemma 2.6.1), we simply need to choose $Q_1 = I$ so that $I^{-1}AI = A$ and $I^{-1}A^*I = A^*$ are doubly almost bipartite and diagonal, respectively.
- The first half of condition (ii) in Definition 2.6.1 (or Lemma 2.6.1) can be satisfied by considering the diagonalization of A so that $Q_2^{-1}AQ_2 = \Lambda$ where Λ is the diagonal matrix consisting of the eigenvalues of A and Q_2 is the conjugating matrix whose columns consist of the eigenvectors of A.

At this point, we simply need to determine A^* so that the matrix representing it is irreducible tridiagonal.

3.3 Eigenvalues/Eigenvectors of All Ones DABITM

The following theorem tells us the eigenvalues and eigenvectors of all ones DABITM, which will be used to identify the companion matrix A^* to A so that (A, A^*) forms an all ones DABLP. The proof is elementary (but lengthy) and thus given in Appendix A.

Theorem 3.3.1. *The matrix A in (3.2.1) has the eigenvalues*

$$\theta_i = q^i + q^{-i}, \qquad 0 \leq i \leq d \tag{3.3.1}$$

and the k^{th}-entry x_k of the corresponding eigenvector is

$$x_k = C\big(q^{ik} + q^{i(1-k)}\big), \qquad 1 \leq k \leq d+1 \tag{3.3.2}$$

where C is an arbitrary constant and $q = e^{\mathbf{i}\pi/(d+1)}$. (Here $\mathbf{i} = \sqrt{-1}$.)

Proof. See Appendix A. $\square$

3.4 Characterization of A^*

By Theorem 3.3.1, the k^{th}-entry x_k of the eigenvector of all ones DABITM A given in (3.2.1) corresponding to the i^{th} eigenvalue θ_i is given by $x_k = C\big(q^{ik} + q^{i(1-k)}\big)$ where $1 \leq k \leq d+1$ and $q = e^{\mathbf{i}\pi/(d+1)}$. Since C is arbitrary, let $C \equiv 1$.

Let Q_2 (mentioned on page 43 - second bullet point) be the conjugating matrix whose columns consist of the eigenvectors of A. Then

$$Q_2 = \begin{pmatrix} 2 & q+1 & q^2+1 & \cdots & q^d+1 \\ 2 & q^2+q^{-1} & q^4+q^{-2} & \cdots & q^{2d}+q^{-d} \\ 2 & q^3+q^{-2} & q^6+q^{-4} & \cdots & q^{3d}+q^{-2d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & q^{d+1}+q^{-d} & q^{2(d+1)}+q^{-2d} & \cdots & q^{(d+1)d}+q^{-d^2} \end{pmatrix}. \tag{3.4.1}$$

Let $D = \mathrm{diag}\big(1/2, q^{-1/2}, q^{-1}, \cdots, q^{-d/2}\big)$ and define

$$\begin{aligned} \widetilde{Q_2} &\equiv Q_2 D \\ &= \begin{pmatrix} 1 & q^{1/2}+q^{-1/2} & q+q^{-1} & \cdots & q^{d/2}+q^{-d/2} \\ 1 & q^{3/2}+q^{-3/2} & q^3+q^{-3} & \cdots & q^{3d/2}+q^{-3d/2} \\ 1 & q^{5/2}+q^{-5/2} & q^5+q^{-5} & \cdots & q^{5d/2}+q^{-5d/2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & q^{(2d+1)/2}+q^{-(2d+1)/2} & q^{2d+1}+q^{-(2d+1)} & \cdots & q^{(2d+1)d/2}+q^{-(2d+1)d/2} \end{pmatrix}. \end{aligned} \tag{3.4.2}$$

Observe that post-multiplying Q_2 by D scales the the i^{th} column (i.e., the i^{th} eigenvector of A) of Q_2 by $q^{-i/2}$ for $0 \le i \le d$ and this is more of a "cosmetic" reason in order to make the exponents of the q terms symmetric. Then it is clear that $\widetilde{Q_2}$ still diagonalizes A, that is, $\big(\widetilde{Q_2}\big)^{-1} A \widetilde{Q_2} = \Lambda$ where Λ is the diagonal matrix consisting of the eigenvalues of A. The following theorem identifies the companion matrix A^* to A so that (A, A^*) form a DABLP.

Theorem 3.4.1. *Let*

$$A = \begin{pmatrix} 1 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ & & & 1 & 1 \end{pmatrix} \qquad A^* = \begin{pmatrix} \theta_0^* & & & & \\ & \theta_1^* & & & \\ & & \theta_2^* & & \\ & & & \ddots & \\ & & & & \theta_d^* \end{pmatrix},$$

where $\theta_i^* = q^{(2i+1)/2} + q^{-(2i+1)/2}$ *with* $q = e^{\mathbf{i}\pi/(d+1)}$ $(0 \le i \le d)$. *Then* (A, A^*) *form an all ones DABLP on* $\mathbb{K}^{d+1}$ *via the identity matrix* I *and* $\widetilde{Q_2}$ *in (3.4.2).*

Proof. Referring to the bullet points on page 43, we simply need to show that $(\widetilde{Q_2})^{-1}A^*\widetilde{Q_2}$ is irreducible tridiagonal. To this end, we claim that the matrix representing A^* is the following $(d+1)\times(d+1)$ irreducible bipartite tridiagonal matrix T:

$$T = \begin{pmatrix} 0 & 2 & & & 0 \\ 1 & \ddots & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & 0 \end{pmatrix}. \tag{3.4.3}$$

That is, we wish to show that $(\widetilde{Q_2})^{-1}A^*\widetilde{Q_2} = T$ or equivalently, $A^*\widetilde{Q_2} = \widetilde{Q_2}T$. For notational convenience, let us introduce the following notation:

$$\langle n\rangle_q := q^n + q^{-n} \quad \text{for } n \in \mathbb{Q}.$$

Using the above new notation, the left-hand side of the matrix equation $A^*\widetilde{Q_2} = \widetilde{Q_2}T$ simplifies to

$$A^*\widetilde{Q_2} = \begin{pmatrix} \left\langle\frac{1}{2}\right\rangle_q & & & & 0 \\ & \left\langle\frac{3}{2}\right\rangle_q & & & \\ & & \left\langle\frac{5}{2}\right\rangle_q & & \\ & & & \ddots & \\ 0 & & & & \left\langle\frac{2d+1}{2}\right\rangle_q \end{pmatrix} \begin{pmatrix} 1 & \left\langle\frac{1}{2}\right\rangle_q & \langle 1\rangle_q & \cdots & \left\langle\frac{d}{2}\right\rangle_q \\ 1 & \left\langle\frac{3}{2}\right\rangle_q & \langle 3\rangle_q & \cdots & \left\langle\frac{3d}{2}\right\rangle_q \\ 1 & \left\langle\frac{5}{2}\right\rangle_q & \langle 5\rangle_q & \cdots & \left\langle\frac{5d}{2}\right\rangle_q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \left\langle\frac{2d+1}{2}\right\rangle_q & \langle 2d+1\rangle_q & \cdots & \left\langle\frac{(2d+1)d}{2}\right\rangle_q \end{pmatrix}$$

$$= \begin{pmatrix} \left\langle\frac{1}{2}\right\rangle_q & \left\langle\frac{1}{2}\right\rangle_q^2 & \left\langle\frac{1}{2}\right\rangle_q\langle 1\rangle_q & \cdots & \left\langle\frac{1}{2}\right\rangle_q\left\langle\frac{d}{2}\right\rangle_q \\ \left\langle\frac{3}{2}\right\rangle_q & \left\langle\frac{3}{2}\right\rangle_q^2 & \left\langle\frac{3}{2}\right\rangle_q\langle 3\rangle_q & \cdots & \left\langle\frac{3}{2}\right\rangle_q\left\langle\frac{3d}{2}\right\rangle_q \\ \left\langle\frac{5}{2}\right\rangle_q & \left\langle\frac{5}{2}\right\rangle_q^2 & \left\langle\frac{5}{2}\right\rangle_q\langle 5\rangle_q & \cdots & \left\langle\frac{5}{2}\right\rangle_q\left\langle\frac{5d}{2}\right\rangle_q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left\langle\frac{2d+1}{2}\right\rangle_q & \left\langle\frac{2d+1}{2}\right\rangle_q^2 & \left\langle\frac{2d+1}{2}\right\rangle_q\langle 2d+1\rangle_q & \cdots & \left\langle\frac{2d+1}{2}\right\rangle_q\left\langle\frac{(2d+1)d}{2}\right\rangle_q \end{pmatrix}. \tag{3.4.4}$$

Note that for any $n \in \mathbb{Q}$, there are several helpful identities involving $\langle \cdot \rangle_q$:

(i) $$\begin{aligned}\langle n\rangle_q^2 &= \left(q^n + q^{-n}\right)^2\\ &= q^{2n} + q^{-2n} + 2\\ &= \langle 2n\rangle_q + 2\end{aligned}$$

(ii) $$\begin{aligned}\left\langle \frac{n}{2}\right\rangle_q \langle n\rangle_q &= \left(q^{n/2} + q^{-n/2}\right)\left(q^n + q^{-n}\right)\\ &= q^{3n/2} + q^{-3n/2} + q^{n/2} + q^{-n/2}\\ &= \left\langle \frac{3n}{2}\right\rangle_q + \left\langle \frac{n}{2}\right\rangle_q\end{aligned}$$

(iii) $$\begin{aligned}\langle -n\rangle_q &= q^{-n} + q^{-(-n)}\\ &= q^n + q^{-n}\\ &= \langle n\rangle_q\end{aligned}$$

(iv) Assume n is an odd integer. Then

$$\begin{aligned}\left\langle \frac{n}{2}\right\rangle_q \left\langle \frac{nd}{2}\right\rangle_q &= \left(q^{n/2} + q^{-n/2}\right)\left(q^{nd/2} + q^{-nd/2}\right)\\ &= \underbrace{\left(q^{n(1+d)/2} + q^{-n(1+d)/2}\right)}_{\text{(I)}} + \underbrace{\left(q^{n(1-d)/2} + q^{-n(1-d)/2}\right)}_{\text{(II)}}.\end{aligned}$$

Since $q = e^{\mathbf{i}\pi/(d+1)}$, the expression (I) simplifies to

$$\begin{aligned}q^{n(1+d)/2} + q^{-n(1+d)/2} &= \left(e^{\mathbf{i}\pi/(d+1)}\right)^{n(1+d)/2} + \left(e^{\mathbf{i}\pi/(d+1)}\right)^{-n(1+d)/2}\\ &= e^{\mathbf{i}n\pi/2} + e^{-\mathbf{i}n\pi/2}\\ &= 2\cos\left(\frac{n\pi}{2}\right)\\ &= 0.\end{aligned}$$

On the other hand, simplifying (II) gives

$$q^{n(1-d)/2} + q^{-n(1-d)/2} = \left\langle \frac{n(1-d)}{2}\right\rangle = \left\langle \frac{-n(d-1)}{2}\right\rangle = \left\langle \frac{n(d-1)}{2}\right\rangle.$$

(The last equality is justified by (iii).)

In summary, if n is an odd integer, then

$$\left\langle \frac{n}{2} \right\rangle_q \left\langle \frac{nd}{2} \right\rangle_q = \left\langle \frac{n(d-1)}{2} \right\rangle_q.$$

Hence applying (i), (ii), and (iv) to each column of (3.4.4) (identities similar to (ii) above can be derived and applied to the remaining columns), we obtain

$$A^*\widetilde{Q_2} = \begin{pmatrix} \left\langle \frac{1}{2} \right\rangle_q & \langle 1 \rangle_q + 2 & \left\langle \frac{3}{2} \right\rangle_q + \left\langle \frac{1}{2} \right\rangle_q & \cdots & \left\langle \frac{d-1}{2} \right\rangle_q \\ \left\langle \frac{3}{2} \right\rangle_q & \langle 3 \rangle_q + 2 & \left\langle \frac{9}{2} \right\rangle_q + \left\langle \frac{3}{2} \right\rangle_q & \cdots & \left\langle \frac{3(d-1)}{2} \right\rangle_q \\ \left\langle \frac{5}{2} \right\rangle_q & \langle 5 \rangle_q + 2 & \left\langle \frac{15}{2} \right\rangle_q + \left\langle \frac{5}{2} \right\rangle_q & \cdots & \left\langle \frac{5(d-1)}{2} \right\rangle_q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left\langle \frac{2d+1}{2} \right\rangle_q & \langle 2d+1 \rangle_q + 2 & \left\langle \frac{3(2d+1)}{2} \right\rangle_q + \left\langle \frac{2d+1}{2} \right\rangle_q & \cdots & \left\langle \frac{(2d+1)(d-1)}{2} \right\rangle_q \end{pmatrix}.$$

On the other hand, the right-hand side of the matrix equation $A^*\widetilde{Q_2} = \widetilde{Q_2}T$ simplifies to

$$\widetilde{Q_2}T = \begin{pmatrix} 1 & \left\langle \frac{1}{2} \right\rangle_q & \langle 1 \rangle_q & \cdots & \left\langle \frac{d}{2} \right\rangle_q \\ 1 & \left\langle \frac{3}{2} \right\rangle_q & \langle 3 \rangle_q & \cdots & \left\langle \frac{3d}{2} \right\rangle_q \\ 1 & \left\langle \frac{5}{2} \right\rangle_q & \langle 5 \rangle_q & \cdots & \left\langle \frac{5d}{2} \right\rangle_q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \left\langle \frac{2d+1}{2} \right\rangle_q & \langle 2d+1 \rangle_q & \cdots & \left\langle \frac{(2d+1)d}{2} \right\rangle_q \end{pmatrix} \begin{pmatrix} 0 & 2 & & & 0 \\ 1 & \ddots & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \left\langle \frac{1}{2} \right\rangle_q & \langle 1 \rangle_q + 2 & \left\langle \frac{3}{2} \right\rangle_q + \left\langle \frac{1}{2} \right\rangle_q & \cdots & \left\langle \frac{d-1}{2} \right\rangle_q \\ \left\langle \frac{3}{2} \right\rangle_q & \langle 3 \rangle_q + 2 & \left\langle \frac{9}{2} \right\rangle_q + \left\langle \frac{3}{2} \right\rangle_q & \cdots & \left\langle \frac{3(d-1)}{2} \right\rangle_q \\ \left\langle \frac{5}{2} \right\rangle_q & \langle 5 \rangle_q + 2 & \left\langle \frac{15}{2} \right\rangle_q + \left\langle \frac{5}{2} \right\rangle_q & \cdots & \left\langle \frac{5(d-1)}{2} \right\rangle_q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left\langle \frac{2d+1}{2} \right\rangle_q & \langle 2d+1 \rangle_q + 2 & \left\langle \frac{3(2d+1)}{2} \right\rangle_q + \left\langle \frac{2d+1}{2} \right\rangle_q & \cdots & \left\langle \frac{(2d+1)(d-1)}{2} \right\rangle_q \end{pmatrix}.$$

This establishes the equality $A^*\widetilde{Q_2} = \widetilde{Q_2}T$. $\square$

3.5 Generalizing A^*

The following lemma will be used to prove the subsequent theorem which generalizes the companion matrix A^* for (A, A^*) to form an all ones DABLP.

Lemma 3.5.1. *Let (A, A^*) denote a Leonard pair on V. Let $\alpha, \beta, \alpha^*, \beta^*$ denote scalars in $\mathbb{K}$ with α, α^* nonzero. Then*

$$(\alpha A + \beta I, \alpha^* A^* + \beta^* I)$$

is also a Leonard pair on V.

We call the above pair the *affine transformation* of (A, A^*) associated with $\alpha, \beta, \alpha^*, \beta^*$.

Proof. Let (A, A^*) be a Leonard pair with conjugating matrices Q_1 and Q_2. By Lemma 2.6.1,

$$Q_1^{-1} A Q_1 = T_1 \qquad \text{and} \qquad Q_2^{-1} A^* Q_2 = T_2$$

for some irreducible tridiagonal matrices T_1 and T_2 and

$$Q_2^{-1} A Q_2 = D_1 \qquad \text{and} \qquad Q_1^{-1} A^* Q_1 = D_2$$

for some diagonal matrices D_1 and D_2. Conjugating both $\alpha A + \beta I$ and $\alpha^* A^* + \beta^* I$ by Q_1, we obtain

$$\begin{aligned} Q_1^{-1}(\alpha A + \beta I) Q_1 &= \alpha \overbrace{Q_1^{-1} A Q_1}^{=T_1} + \beta I \\ &= \alpha T_1 + \beta I, \end{aligned}$$

which is clearly irreducible tridiagonal and

$$\begin{aligned} Q_1^{-1}(\alpha^* A + \beta^* I) Q_1 &= \alpha^* \overbrace{Q_1^{-1} A^* Q_1}^{=D_2} + \beta^* I \\ &= \alpha^* D_2 + \beta^* I, \end{aligned}$$

which is clearly diagonal.

This shows that Lemma 2.6.1(i) is satisfied.

On the other hand, conjugating both $\alpha A + \beta I$ and $\alpha^* A^* + \beta^* I$ by Q_2 yields

$$Q_2^{-1}(\alpha A + \beta I)Q_2 = \alpha \underbrace{Q_2^{-1} A Q_2}_{=D_1} + \beta I$$

$$= \alpha D_1 + \beta I,$$

which is clearly diagonal and

$$Q_2^{-1}(\alpha^* A + \beta^* I)Q_2 = \alpha^* \underbrace{Q_2^{-1} A^* Q_2}_{=T_2} + \beta^* I$$

$$= \alpha^* T_2 + \beta^* I,$$

which is clearly irreducible tridiagonal. This shows that Lemma 2.6.1(ii) is also satisfied. $\square$

The next theorem generalizes Theorem 3.4.1.

Theorem 3.5.1. *Let*

$$A = \begin{pmatrix} 1 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ & & & 1 & 1 \end{pmatrix} \qquad \Delta^* = \begin{pmatrix} \delta_0^* & & & & \\ & \delta_1^* & & & \\ & & \delta_2^* & & \\ & & & \ddots & \\ & & & & \delta_d^* \end{pmatrix},$$

where δ_i's satisfy the following recursive relation

$$\frac{\delta_i^* - \delta_{i+1}^*}{\delta_{i+1}^* - \delta_{i+2}^*} = \frac{\theta_i^* - \theta_{i+1}^*}{\theta_{i+1}^* - \theta_{i+2}^*}, \tag{3.5.1}$$

where $0 \leq i \leq d-2$ (θ_i^* as in Theorem 3.4.1). Then (A, Δ^*) form an all ones DABLP on $\mathbb{K}^{d+1}$ via the identity matrix I and $\widetilde{Q_2}$.

Note. The A^* matrix in Theorem 3.4.1 is a special case of Δ^* where $\delta_i^* = \theta_i^*$ for all $i = 0, \ldots, d$.

Proof. We will show that Δ^* can be obtained by an affine transformation of A^* with suitable constants $\alpha^*, \beta^* \in \mathbb{K}$. First, notice that the left-hand side of (3.5.1) has 2 degrees of freedom since fixing the values of δ_0^* and δ_1^* will determine the rest of δ_i^* for $2 \leq i \leq d$. Thus, without loss of generality, fix the first two entries of Δ^* by letting $\delta_0^* \equiv \xi$ and $\delta_1^* \equiv \zeta$, where $\xi, \zeta \in \mathbb{K}$. Let

$$\alpha^* := \frac{\xi - \zeta}{\theta_0^* - \theta_1^*} \qquad \text{and} \qquad \beta^* := \frac{\zeta\theta_0^* - \xi\theta_1^*}{\theta_0^* - \theta_1^*}. \tag{3.5.2}$$

Observe that the ii-entry of an affine transformation $\alpha^* A^* + \beta^* I$ of A^* is given by

$$\left(\frac{\xi - \zeta}{\theta_0^* - \theta_1^*}\right)\theta_i^* + \frac{\zeta\theta_0^* - \xi\theta_1^*}{\theta_0^* - \theta_1^*}.$$

We will show that

$$\delta_i^* = \left(\frac{\xi - \zeta}{\theta_0^* - \theta_1^*}\right)\theta_i^* + \frac{\zeta\theta_0^* - \xi\theta_1^*}{\theta_0^* - \theta_1^*} = \zeta + (\zeta - \xi)\left(\frac{\theta_1^* - \theta_i^*}{\theta_0^* - \theta_1^*}\right) \tag{3.5.3}$$

for all $i = 0, \ldots, d$ by strong induction on i.

For the base cases, consider $i = 0$ and $i = 1$.

When $i = 0$, (3.5.3) simplifies to

$$\zeta + (\zeta - \xi)\left(\frac{\theta_1^* - \theta_0^*}{\theta_0^* - \theta_1^*}\right) = \xi = \delta_0^*.$$

When $i = 1$, (3.5.3) simplifies to

$$\zeta + (\zeta - \xi)\left(\frac{\theta_1^* - \theta_1^*}{\theta_0^* - \theta_1^*}\right) = \zeta = \delta_1^*.$$

Now, let $k \in \mathbb{N}$ with $k \geq 2$ be given and suppose (3.5.3) holds for all

$i = 0, 1, \ldots, k$. In particular, suppose (3.5.3) holds for $i = k-1$ and $i = k$:

$$\delta^*_{k-1} = \zeta + (\zeta - \xi)\left(\frac{\theta^*_1 - \theta^*_{k-1}}{\theta^*_0 - \theta^*_1}\right) \quad \text{and} \quad \delta^*_k = \zeta + (\zeta - \xi)\left(\frac{\theta^*_1 - \theta^*_k}{\theta^*_0 - \theta^*_1}\right).$$

By (3.5.1) with $j = k-1$, we have

$$\begin{aligned}
\frac{\delta^*_{k-1} - \delta^*_k}{\delta^*_k - \delta^*_{k+1}} &= \frac{\theta^*_{k-1} - \theta^*_k}{\theta^*_k - \theta^*_{k+1}} \\
\delta^*_{k+1} &= \delta^*_k + (\delta^*_k - \delta^*_{k-1})\left(\frac{\theta^*_k - \theta^*_{k+1}}{\theta^*_{k-1} - \theta^*_k}\right) \\
&= \left[\zeta + (\zeta - \xi)\left(\frac{\theta^*_1 - \theta^*_k}{\theta^*_0 - \theta^*_1}\right)\right] + \left[\left\{\zeta + (\zeta - \xi)\left(\frac{\theta^*_1 - \theta^*_k}{\theta^*_0 - \theta^*_1}\right)\right\}\right. \\
&\qquad \left. - \left\{\zeta + (\zeta - \xi)\left(\frac{\theta^*_1 - \theta^*_{k-1}}{\theta^*_0 - \theta^*_1}\right)\right\}\right]\left(\frac{\theta^*_k - \theta^*_{k+1}}{\theta^*_{k-1} - \theta^*_k}\right) \\
&= \zeta + (\zeta - \xi)\left(\frac{\theta^*_1 - \theta^*_k}{\theta^*_0 - \theta^*_1}\right) + (\zeta - \xi)\left(\frac{\theta^*_{k-1} - \theta^*_k}{\theta^*_0 - \theta^*_1}\right)\left(\frac{\theta^*_k - \theta^*_{k+1}}{\theta^*_{k-1} - \theta^*_k}\right) \\
&= \zeta + (\zeta - \xi)\left(\frac{\theta^*_1 - \theta^*_k}{\theta^*_0 - \theta^*_1}\right) + (\zeta - \xi)\left(\frac{\theta^*_k - \theta^*_{k+1}}{\theta^*_0 - \theta^*_1}\right) \\
&= \zeta + (\zeta - \xi)\left(\frac{\theta^*_1 - \theta^*_{k+1}}{\theta^*_0 - \theta^*_1}\right),
\end{aligned}$$

showing that (3.5.3) holds for $i = k+1$ and therefore, it is true for all $n = 0, \ldots, d$.

Applying Lemma 3.5.1 with $\alpha = 1, \beta = 0$ and α^*, β^* given in (3.5.2), the result follows. □

3.6 The *Modified* Chebyshev Polynomials of the First Kind

For $0 \leq i, j \leq d$, let T be the $(d+1) \times (d+1)$ matrix with ij entry

$$T_{ij} = T_j(\theta_i^*)$$

where $\theta_i^* = q^{(2i+1)/2} + q^{-(2i+1)/2}$ (see Theorem 3.4.1) and $T_j(x)$ is the j^{th} *modified Chebyshev polynomial of the first kind*. The first several modified Chebyshev polynomials are given by

$$\begin{aligned}
T_0(x) &= 2, \\
T_1(x) &= x, \\
T_2(x) &= 2x^2 - 1, \\
T_3(x) &= 4x^3 - 3x, \\
T_4(x) &= 8x^4 - 8x^2 + 1, \\
T_5(x) &= 16x^5 - 20x^3 + 5x.
\end{aligned}$$

The modified Chebyshev polynomials of the first kind can be obtained from the following recurrence relation[3]

$$\begin{aligned}
T_0(x) &= 2, \\
T_1(x) &= x, \\
T_{j+1}(x) &= xT_j(x) - T_{j-1}(x), \quad j \geq 2. \qquad (3.6.1)
\end{aligned}$$

By straight computation, we can see that the j^{th} column of the conjugating

[3]For the original Chebysev polynomials of the first kind, $T_0(x) = 1$ and $T_{j+1}(x) = 2xT_j(x) - T_{j-1}(x)$ for $j \geq 2$.

matrix $\widetilde{Q_2}$ in (3.4.1) is given by $T_j(\theta^*_i), i = 0, 1, \ldots, d$, which leads to the following proposition.

Proposition 3.6.1. *Let $i = 0, 1, \ldots d$ be fixed. For $0 \leq j \leq d$, $\widetilde{Q_2} = T_j(\theta^*_i)$. Consequently, the sequence $\{T_0, T_1, \ldots, T_d\}$ forms a basis for V.*

Proof. We will induct on j with i fixed.

For the base cases, consider $j = 0$ and $j = 1$. For each fixed i, we have $\left(\widetilde{Q_2}\right)_{i0} = 2 = T_0(\theta^*_i)$ and $\left(\widetilde{Q_2}\right)_{i1} = \theta^*_i = T_1(\theta^*_i)$.

Let $k \in \mathbb{N}$ with $k \geq 1$ be given and suppose the claim is true for all $j = 0, 1, \ldots, k$. In particular, assume the claim holds for $j = k - 1$ and $j = k$:

$$\begin{aligned} \left(\widetilde{Q_2}\right)_{i,k-1} &= q^{(2i+1)(k-1)/2} + q^{-(2i+1)(k-1)/2} = T_{k-1}(\theta^*_i), \\ \left(\widetilde{Q_2}\right)_{ik} &= q^{(2i+1)k/2} + q^{-(2i+1)k/2} = T_k(\theta^*_i) \end{aligned} \tag{3.6.2}$$

The $(i, k+1)$ entry of $\widetilde{Q_2}$ is given by

$$\left(\widetilde{Q_2}\right)_{i,k+1} = q^{(2i+1)(k+1)/2} + q^{-(2i+1)(k+1)/2}.$$

On the other hand, using (3.6.1) and the induction hypotheses (3.6.2),

$$\begin{aligned} T_{k+1}(\theta^*_i) &= \theta^*_i T_k(\theta^*_i) - T_{k-1}(\theta^*_i) \\ &= \left(q^{(2i+1)/2} + q^{-(2i+1)/2}\right)\left(q^{(2i+1)k/2} + q^{-(2i+1)k/2}\right) \\ &\qquad - \left(q^{(2i+1)(k-1)/2} + q^{-(2i+1)(k-1)/2}\right) \\ &= q^{(2i+1)(k+1)/2} + q^{-(2i+1)(k+1)/2}. \end{aligned}$$

Thus we have $\left(\widetilde{Q_2}\right)_{i,k+1} = T_{k+1}(\theta^*_i)$ and this completes the proof. □

3.7 Full-Characterization of All Ones DABLPs

Expressing the conjugating matrix $\widetilde{Q_2}$ in terms of the modified Chevyshev polynomials of the first kind allows us to prove the converse of Theorem 3.5.1.

Theorem 3.7.1. *Suppose (A, Δ^*) form an all ones DABLP (A and Δ^* given in the statement of Theorem 3.5.1). Then the δ_i^*'s satisfy the recursive relation given in (3.5.1):*

$$\frac{\delta_i^* - \delta_{i+1}^*}{\delta_{i+1}^* \delta_{i+2}^*} = \frac{\theta_i^* - \theta_{i+1}^*}{\theta_{i+1}^* - \theta_{i+2}^*},$$

where $0 \leq i \leq d-2$ (θ_i^* as in Theorem 3.4.1).

Proof. Assume (A, Δ^*) form all ones DABLP via the identity matrix I and the modified Chebyshev matrix of the first kind T. It suffices to show that the conjugation of Δ^* via T must yield an irreducible tridiagonal matrix, that is,

$$T^{-1}\Delta^* T = \begin{pmatrix} t_{00} & t_{01} & & & 0 \\ t_{10} & t_{11} & t_{12} & & \\ & t_{21} & \ddots & \ddots & \\ & & \ddots & t_{d-1,d-1} & t_{d-1,d} \\ 0 & & & t_{d,d-1} & t_{dd} \end{pmatrix}$$

or equivalently,

$$\Delta^* T = T \begin{pmatrix} t_{00} & t_{01} & & & 0 \\ t_{10} & t_{11} & t_{12} & & \\ & t_{21} & \ddots & \ddots & \\ & & \ddots & t_{d-1,d-1} & t_{d-1,d} \\ 0 & & & t_{d,d-1} & t_{dd} \end{pmatrix}. \qquad (3.7.1)$$

The first column of the left-hand side of (3.7.1) is given by

$$\begin{pmatrix} \delta_0^* T_0(\theta_0^*) \\ \delta_1^* T_0(\theta_1^*) \\ \delta_2^* T_0(\theta_2^*) \\ \delta_3^* T_0(\theta_3^*) \\ \vdots \\ \delta_{d-2}^* T_0(\theta_{d-2}^*) \\ \delta_{d-1}^* T_0(\theta_{d-1}^*) \\ \delta_d^* T_0(\theta_d^*) \end{pmatrix}. \tag{3.7.2}$$

On the other hand, the first column of the right-hand side of (3.7.1) is

$$\begin{pmatrix} t_{00}T_0(\theta_0^*) + t_{10}T_1(\theta_0^*) \\ t_{00}T_0(\theta_1^*) + t_{10}T_1(\theta_1^*) \\ t_{00}T_0(\theta_2^*) + t_{10}T_1(\theta_2^*) \\ t_{00}T_0(\theta_3^*) + t_{10}T_1(\theta_3^*) \\ \vdots \\ t_{00}T_0(\theta_{d-2}^*) + t_{10}T_1(\theta_{d-2}^*) \\ t_{00}T_0(\theta_{d-1}^*) + t_{10}T_1(\theta_{d-1}^*) \\ t_{00}T_0(\theta_d^*) + t_{10}T_1(\theta_d^*) \end{pmatrix}. \tag{3.7.3}$$

Equating (3.7.2) and (3.7.3) we obtain

$$\delta_0^* T_0(\theta_0^*) = t_{00}T_0(\theta_0^*) + t_{10}T_1(\theta_0^*), \tag{3.7.4a}$$

$$\delta_1^* T_0(\theta_1^*) = t_{00}T_0(\theta_1^*) + t_{10}T_1(\theta_1^*), \tag{3.7.4b}$$

$$\delta_2^* T_0(\theta_2^*) = t_{00}T_0(\theta_2^*) + t_{10}T_1(\theta_2^*), \tag{3.7.4c}$$

$$\delta_3^* T_0(\theta_3^*) = t_{00}T_0(\theta_3^*) + t_{10}T_1(\theta_3^*). \tag{3.7.4d}$$

$$\vdots$$

$$\delta_{d-2}^* T_0(\theta_{d-2}^*) = t_{00}T_0(\theta_{d-2}^*) + t_{10}T_1(\theta_{d-2}^*), \tag{3.7.4e}$$

$$\delta_{d-1}^* T_0(\theta_{d-1}^*) = t_{00}T_0(\theta_{d-1}^*) + t_{10}T_1(\theta_{d-2}^*), \tag{3.7.4f}$$

$$\delta_d^* T_0(\theta_d^*) = t_{00}T_0(\theta_d^*) + t_{10}T_1(\theta_d^*). \tag{3.7.4g}$$

Recall that $T_0(x) = 2$ and $T_1(x) = x$ for modified Chebyshev polynomials of the first kind so (3.7.4a)-(3.7.4g) simplify to

$$2\delta_0^* = t_{00} + t_{10}\theta_0^*, \tag{3.7.5a}$$

$$2\delta_1^* = t_{00} + t_{10}\theta_1^*, \tag{3.7.5b}$$

$$2\delta_2^* = t_{00} + t_{10}\theta_2^*, \tag{3.7.5c}$$

$$2\delta_3^* = t_{00} + t_{10}\theta_3^*, \tag{3.7.5d}$$

$$\vdots$$

$$2\delta_{d-2}^* = t_{00} + t_{10}\theta_{d-2}^*, \tag{3.7.5e}$$

$$2\delta_{d-1}^* = t_{00} + t_{10}\theta_{d-1}^*, \tag{3.7.5f}$$

$$2\delta_d^* = t_{00} + t_{10}\theta_d^*. \tag{3.7.5g}$$

We may subtract (3.7.5b) from (3.7.5a) to eliminate t_{00}:

$$2(\delta_0^* - \delta_1^*) = t_{10}(\theta_0^* - \theta_1^*). \tag{3.7.6}$$

Similarly, we may eliminate t_{00} by subtracting (3.7.5c) from (3.7.5b):

$$2(\delta_1^* - \delta_2^*) = t_{10}(\theta_1^* - \theta_2^*). \tag{3.7.7}$$

Dividing (3.7.6) by (3.7.7), we get

$$\frac{\delta_0^* - \delta_1^*}{\delta_1^* - \delta_2^*} = \frac{\theta_0^* - \theta_1^*}{\theta_1^* - \theta_2^*}$$

Similar calculations (using (3.7.5b)-(3.7.5d)) will show that

$$\frac{\delta_1^* - \delta_2^*}{\delta_2^* - \delta_3^*} = \frac{\theta_1^* - \theta_2^*}{\theta_2^* - \theta_3^*}$$

and continuing in this manner, we obtained the desired result. $\square$

Now we have a full characterization of all ones DABLPs and state the result as a corollary below.

Corollary 3.7.1. *The pair* (A, Δ^*) *given in the statement of Theorem 3.5.1 form an all ones DABLP if and only if the diagonal entries* δ_i^**'s satisfy the recursive relation given in (3.5.1):*

$$\frac{\delta_i^* - \delta_{i+1}^*}{\delta_{i+1}^* - \delta_{i+2}^*} = \frac{\theta_i^* - \theta_{i+1}^*}{\theta_{i+1}^* - \theta_{i+2}^*},$$

where $0 \leq i \leq d-2$ *(*θ_i^* *as in Theorem 3.4.1).*

Proof. Apply Theorems 3.5.1 and 3.7.1. $\square$

4 Classification of DABLPs Using Leonard's Theorem

In this chapter we formulate some conditions on the dual eigenvalues that allow us to use *Leonard's Theorem*. With this result, we are able to classify the DABLPs. The key ingredients for this classification are the *Askey-Wilson Relations*. Throughout this chapter we will assume that the field $\mathbb{K}$ has the characteristic of 0.

4.1 Askey-Wilson Relations

Theorem 4.1.1. [46, Theorem 1.5] *Let* (A, A^*) *denote a Leonard pair on* V*. There exists a sequence of scalars* $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^* \in \mathbb{K}$ *such that*

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* = \gamma^*A^2 + \omega A + \eta I, \tag{4.1.1a}$$

$$(A^*)^2A - \beta A^*AA^* + A(A^*)^2 - \gamma^*(A^*A + AA^*) - \varrho^*A = \gamma(A^*)^2 + \omega A^* + \eta^*I. \tag{4.1.1b}$$

The sequence is uniquely determined by the pair (A, A^*) provided $\dim(V) \geq 4$. The relations (4.1.1a) and (4.1.1b) are called the *Askey-Wilson relations* (*AWRs*) and the sequence of 8 scalars are called the *Askey-Wilson coefficients* (*AWCs*).

We denote the pair of equations (4.1.1a) and (4.1.1b) by

$$AW(\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*)$$

and they first appeared in [52].

In that article it is shown that the Askey-Wilson polynomials give a pair of infinite matrices which satisfy (4.1.1a) and (4.1.1b). For related work and the proof of this theorem, see [19, 20, 21, 46, 53].

The following theorem displays some formulae which can be used to compute the *AWCs* using Theorem 4.1.1.

Theorem 4.1.2. [46, Theorem 4.5 and 5.3] *Given a Leonard pair* (A, A^*) *on* V*, expressions for the 8 AWCs in terms of parameter arrays* $\mathcal{P}$ *are given by the following formulas:*

$$\beta = \frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} - 1 = \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} - 1, \tag{4.1.2a}$$

$$\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1}, \tag{4.1.2b}$$

$$\gamma^* = \theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1}, \tag{4.1.2c}$$

$$\varrho = \theta_i^2 - \beta\theta_i\theta_{i-1} + \theta_{i-1}^2 - \gamma(\theta_i + \theta_{i-1}), \tag{4.1.2d}$$

$$\varrho^* = \theta_i^{*2} - \beta\theta^*_i\theta^*_{i-1} + {\theta^*_{i-1}}^2 - \gamma^*(\theta^*_i + \theta^*_{i-1}), \tag{4.1.2e}$$

$$\omega = a_i(\theta^*_i - \theta^*_{i+1}) + a_{i-1}(\theta^*_{i-1} - \theta^*_{i-2}) - \gamma(\theta^*_i + \theta^*_{i-1}), \tag{4.1.2f}$$

$$= a^*_i(\theta_i - \theta_{i+1}) + a^*_{i-1}(\theta_{i-1} - \theta_{i-2}) - \gamma^*(\theta_i + \theta_{i-1}), \tag{4.1.2g}$$

$$\eta = a^*_i(\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1}) - \gamma^*\theta_i^2 - \omega\theta_i, \tag{4.1.2h}$$

$$\eta^* = a_i(\theta^*_i - \theta^*_{i-1})(\theta^*_i - \theta^*_{i+1}) - \gamma\theta_i^{*2} - \omega\theta^*_i. \tag{4.1.2i}$$

Note that β *in (4.1.2a) is the fundamental constant defined in (2.9.2) and valid for* $2 \le i \le d-1$*. The expressions for* γ, γ^* *are valid for* $1 \le i \le d-1$*, the expressions for* $\varrho, \varrho^*, \omega$ *are valid for* $1 \le i \le d$*, and the expressions for* η, η^* *are valid for* $0 \le i \le d$*.*

Corollary 4.1.1. *For all ones DABLP given in (3.4.1), the 8 AWCs are as follows:*

$$\beta = q + q^{-1},$$

$$\gamma = \gamma^* = \omega = \eta = \eta^* = 0,$$

$$\varrho = 4 - (q + q^{-1})2 = 4 - \beta^2,$$

$$\varrho^* = 1 - \frac{q + q^{-1}}{4} = \varrho/4,$$

where $q = e^{\mathbf{i}\pi/(d+1)}$.

Proof. Simple calculations using the expressions for θ_i's, θ_i^*'s, and q. □

4.2 Extended Dual Eigenvalues

As stated in Lemma 2.9.2(v), together with Theorem 4.1.2, the *AWRs* imply certain ratios are independent of i. In particular, by (4.1.2a),

$$\beta + 1 = \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}. \tag{4.2.1}$$

By (4.1.2c), the dual eigenvalue sequence $\{\theta_i^*\}$ satisfy a 3-term recurrence and if we assume $d \geq 3$, this allows us to *extend* the sequence. To this end, letting $i = 1, 2$ in (4.2.1) and setting the two expressions equal to each other, we have

$$\frac{\theta^*_{-1} - \theta^*_2}{\theta^*_0 - \theta^*_1} = \frac{\theta^*_0 - \theta^*_3}{\theta^*_1 - \theta^*_2}.$$

Solving the above equation for θ^*_{-1} yields

$$\theta^*_{-1} = \theta^*_2 + (\theta^*_0 - \theta^*_1)\frac{\theta^*_0 - \theta^*_3}{\theta^*_1 - \theta^*_2}.$$

Similarly, letting $i = d-1, d$ in (4.2.1) and setting the two expressions equal to each other, we get

$$\frac{\theta^*_{d-3} - \theta^*_d}{\theta^*_{d-2} - \theta^*_{d-1}} = \frac{\theta^*_{d-2} - \theta^*_{d+1}}{\theta^*_{d-1} - \theta^*_d}.$$

Solving the above equation for θ^*_{d+1} gives

$$\theta^*_{d+1} = \theta^*_{d-2} + (\theta^*_{d-1} - \theta^*_d)\frac{\theta^*_{d-3} - \theta^*_d}{\theta^*_{d-2} - \theta^*_{d-1}}.$$

This leads to the following definition.

Definition 4.2.1. (Extended Dual Eigenvalues)

$$\theta^*_{-1} = \theta^*_2 + (\theta^*_0 - \theta^*_1)\frac{\theta^*_0 - \theta^*_3}{\theta^*_1 - \theta^*_2}, \tag{4.2.2a}$$

$$\theta^*_{d+1} = \theta^*_{d-2} + (\theta^*_{d-1} - \theta^*_d)\frac{\theta^*_{d-3} - \theta^*_d}{\theta^*_{d-2} - \theta^*_{d-1}}. \tag{4.2.2b}$$

4.3 Classification of DABLPs

The following two lemmas will be used to prove the main theorem in this section (Theorem 4.3.1).

Lemma 4.3.1. *Suppose (A, A^*) is a LP with $d \geq 4$. Assume $a_1 = a_2 = a_3 = 0$. Then the AWCs $\gamma, \omega,$ and η^* satisfy $\gamma = \omega = \eta^* = 0$.*

Proof. Set $i = 1, 2, 3$ in (4.1.2i), which relates the a_i to the θ^*_i:

$$\eta^* = a_1(\theta^*_1 - \theta^*_0)(\theta^*_1 - \theta^*_2) - \gamma\theta^{*2}_1 - \omega\theta^*_1, \tag{4.3.1a}$$

$$\eta^* = a_2(\theta^*_2 - \theta^*_1)(\theta^*_2 - \theta^*_3) - \gamma\theta^{*2}_2 - \omega\theta^*_2, \tag{4.3.1b}$$

$$\eta^* = a_3(\theta^*_3 - \theta^*_2)(\theta^*_3 - \theta^*_4) - \gamma\theta^{*2}_3 - \omega\theta^*_3. \tag{4.3.1c}$$

By assumption $a_i = 0$ for $i = 1, 2, 3$ and hence (4.3.1a)-(4.3.1c) simplify to

$$\gamma\theta_1^{*2}\omega\theta_1^* + \eta^* = 0, \tag{4.3.2a}$$

$$\gamma\theta_2^{*2} - \omega\theta_2^* + \eta^* = 0, \tag{4.3.2b}$$

$$\gamma\theta_3^{*2}\omega\theta_3^* + \eta^* = 0. \tag{4.3.2c}$$

Since θ_i^* are assumed to be distinct, the above linear system in γ, ω, and η^* only has a trivial solution, so the result follows. □

Remark. These results are consistent with Corollary 4.1.1.

Lemma 4.3.2. *Suppose (A, A^*) is a LP with $d \geq 4$. Assume $a_1 = a_2 = a_3 = 0$.*

(i) *If $a_0 \neq 0$, then $\theta_{-1}^* = \theta_0^*$.*

(ii) *If $a_d \neq 0$, then $\theta_{d+1}^* = \theta_d^*$.*

Proof. Applying the result in Lemma 4.3.1 to (4.1.2i), we obtain

$$a_i(\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) = 0. \tag{4.3.3}$$

Setting $i = 0$ in (4.3.3) yields

$$a_0(\theta_0^* - \theta_{-1}^*)(\theta_0^* - \theta_1^*) = 0.$$

Since $\theta_0^* \neq \theta_1^*$ and $a_0 \neq 0$, this forces $\theta_{-1}^* = \theta_0^*$.

The second claim (ii) follows similarly by setting $i = d$ in (4.3.3). □

Theorem 4.3.1. *Let (A, A^*) be a DABLP with $d \geq 4$. Then*

(i) *(A, A^*) must be of type q-Racah, q-Hahn, or q-Krawtchouk.*

(ii) *In each of these cases, $s^* = 1$ and $q^{d+1} = -1$.*

Proof. (i) Since $d \geq 4$, it is clear that (A, A^*) cannot be of Orphan type. (See Appendix C: 13. Orphan - page 96.)

Let us first show that (A, A^*) cannot be of either Dual Hahn or Krawtchouk type. By (C.10b) and (C.11b) in Appendix C, the dual eigenvalue sequences of these types of LPs are given by $\theta^*_i = \theta^*_0 + s^* i$ for some nonzero constant $s^* \in \mathbb{K}$. By Lemma 4.3.2(i),

$$0 = \theta^*_0 - \theta^*_{-1} = \theta^*_0 - (\theta^*_0 - s^*) = s^*,$$

contradiction.

Next, let us show that (A, A^*) cannot be of either Racah or Hahn type. By (C.8b) and (C.9b) in Appendix C, the dual eigenvalue sequences of these types of LPs are given by $\theta^*_i = \theta^*_0 + h^* i(i + 1 + s^*)$ for some nonzero constant $h^* \in \mathbb{K}$. By Lemma 4.3.2(i),

$$0 = \theta^*_0 - \theta^*_{-1} = \theta^*_0 - \big(\theta^*_0 + h^*(-1)(-1 + 1 + s^*)\big) = h^* s^*.$$

Since $h^* \neq 0$, s^* must vanish identically. Now using Lemma 4.3.2(ii),

$$\begin{aligned} 0 &= \theta^*_{d+1} - \theta^*_d \\ &= [\theta^*_0 + h^*(d+1)(d+1+1+s^*)] - [\theta^*_0 + h^* d(d+1+s^*)] \\ &= h^*(2d + 2 + s^*). \end{aligned}$$

Once again, $h^* \neq 0$ by assumption so this forces $2d + 2 + s^* = 0$ or $s^* = -2d - 2 \neq 0$, impossibility.

We now claim that (A, A^*) cannot be of Dual q-Hahn, Quantum q-

Krawtchouk, Affine q-Krawtchouk, or Dual q-Krawtchouk type. By (C.3b), (C.4b), (C.6b), (C.7b) in Appendix C, the dual eigenvalue sequences of these types of LPs are given by $\theta_i^* = \theta_0^* + h^*(1-q^i)q^{-i}$ for some nonzero constant $h^* \in \mathbb{K}$ and $q^i \neq 1$ for $1 \leq i \leq d$. Using Lemma 4.3.2(i) again, we see that

$$0 = \theta_0^* - \theta_{-1}^* = \theta_0^* - [\theta_0^* + h^*(1-q^{-1})q^{-(-1)}] = h^*(1-q),$$

contradiction since neither $h^* \neq 0$ nor $q = 1$.

Lastly, we claim that (A, A^*) cannot be of Bannai/Ito type. By (C.12b) in Appendix C, the dual eigenvalue sequence of a LP of Bannai/Ito type is given by $\theta_i^* = \theta_0^* + h^*[s^* - 1 + (1 - s^* + 2i)(-1)^i]$ for some nonzero constant $h^* \in \mathbb{K}$. By Lemma 4.3.2(i),

$$\begin{aligned} 0 &= \theta_0^* - \theta_{-1}^* \\ &= \theta_0^* - (\theta_0^* + h^*[s^* - 1 + (1 - s^* + 2(-1))(-1)^{-1}]) \\ &= -2s^*h^*. \end{aligned}$$

Since $h^* \neq 0$, we must have $s^* = 0$. This simplifies the dual eigenvalue sequence of the LP of Bannai/Ito type as follows

$$\theta_i^* = \theta_0^* + h^*[-1 + (1+2i)(-1)^i]. \tag{4.3.4}$$

Now using Lemma 4.3.2(ii),

$$
\begin{aligned}
0 &= \theta^*_{d+1} - \theta^*_d \\
&= \big(\theta^*_0 + h^*[-1 + (1 + 2(d+1))(-1)^{d+1}]\big) - \big(\theta^*_0 + h^*[-1 + (1+2d)(-1)^d]\big) \\
&= h^*\big((1 + 2(d+1))(-1)^{d+1} - (1+2d)(-1)^d\big) \\
&= h^*\big((1 + 2(d+1))(-1)^{d+1} + (1+2d)(-1)^{d+1}\big) \\
&= 4h^*(-1)^{d+1}(d+1),
\end{aligned}
$$

which holds if and only if $h^* = 0$, contradiction.

This proves (i).

(ii) By (C.1b), (C.2b), (C.5b) in Appendix C, the dual eigenvalue sequences of LPs of q-Racah, q-Hahn, and q-Krawtchouk types are given by $\theta^*_i = \theta^*_0 + h^*(1-q^i)(1-s^*q^{i+1})q^{-i}$ for some nonzero constant $h^*, q, s^* \in \mathbb{K}$. By Lemma 4.3.2(i),

$$
\begin{aligned}
0 &= \theta^*_0 - \theta^*_{-1} \\
&= \theta^*_0 - (\theta^*_0 + h^*(1 - q^{-1})(1 - s^*)q) \\
&= h^*(1-q)(1-s^*),
\end{aligned}
$$

provided that $s^* = 1$ since $h^* \neq 0$ nor $q \neq 1$. This proves the first part of the claim made in (ii) and simplifies the dual eigenvalue sequence as follows

$$\theta^*_i = \theta^*_0 + h^*(1-q^i)(1-q^{i+1})q^{-i}. \tag{4.3.5}$$

Using Lemma 4.3.2(ii) for one last time, we have

$$
\begin{aligned}
0 &= \theta^*_{d+1} - \theta^*_d \\
&= \big(\theta^*_0 + h^*(1-q^{d+1})(1-q^{d+2})q^{-(d+1)}\big) - \big(\theta^*_0 + h^*(1-q^d)(1-q^{d+1})q^{-d}\big) \\
&= h^* q^{-(d+1)}(1-q^{d+1})\big[(1-q^{d+2}) - (1-q^d)q\big] \\
&= h^* q^{-(d+1)}(1-q^{d+1})\big[1-q^{d+2} - q + q^{d+1}\big] \\
&= h^* q^{-(d+1)}(1-q^{d+1})\big[q^{d+1}(1-q) + (1-q)\big] \\
&= h^* q^{-(d+1)}(1-q)(1-q^{d+1})(1+q^{d+1}) \\
&= h^* q^{-(d+1)}(1-q)(1-q^{2(d+1)}).
\end{aligned}
$$

Since none of the first three factors above equal to 0, we must have $1 - q^{2(d+1)} = 0$ or $q^{d+1} = \pm 1$. The assumption $s^* q^i = q^i \neq 1$ for $2 \leq i \leq 2d$ implies $q^{d+1} = -1$, verifying the second claim made in (ii).

□

Corollary 4.3.1. *The all-ones DABLP is of q-Racah type with*

$$h = s^* = 1, \quad h^* = q^{-1/2}, \quad s = q^{-1}, \quad r_1 r_2 = q^d,$$

where $q = e^{\mathbf{i}\pi/(d+1)}$.

In future work, we intend to explore the DABLPs of q-Hahn and q-Krawtchouk type.

5 Future Directions

In this final chapter, we collect and discuss some of the potential future directions by introducing doubly almost bipartite analogues of several related objects, including *Leonard triples* and *Modular Leonard triples* (Section 5.1), *Spin Leonard pairs* (Section 5.2), and a connection to *Near-bipartite Leonard pairs* (Section 5.3).

5.1 Leonard Triples (LTs) and Modular Leonard Triples (MLTs)

The notion of *Leonard triples* was introduced as a natural extension of Leonard pairs by Curtin in [11]. See the following definition.

Definition 5.1.1. [11, Definition 1.2] A *Leonard triple* (LT) on V is an ordered triple (A, A^*, A^ϵ) of linear transformations $A : V \to V, A^* : V \to V, A^\epsilon : V \to V$ in $\mathrm{End}(V)$ that satisfy conditions (i)-(iii) below.

(i) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrices representing A^* and A^ϵ are each irreducible tridiagonal.

(ii) There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrices representing A and A^ϵ are each irreducible tridiagonal.

(iii) There exists a basis for V with respect to which the matrix representing A^ϵ is diagonal and the matrices representing A and A^* are each irreducible tridiagonal.

As in LPs, the *diameter* of the LT (A, A^*, A^ϵ) is defined to be one less than the dimension of V.

There is a a LTs-analogue of 2.6.1.

Lemma 5.1.1. [37, Lemma 1.8] *An ordered triple* (A, A^*, A^ϵ) *of matrices* $A, A^*, A^\epsilon \in \mathrm{Mat}_{d+1}(\mathbb{K})$ *is a LT on* V *if and only if the following hold.*

(i) *There exists a non-singular matrix* Q_1 *such that* $Q_1^{-1}AQ_1$ *is diagonal and* $Q_1^{-1}A^*Q_1$ *and* $Q_1^{-1}A^\epsilon Q_1$ *are irreducible tridiagonal.*

(ii) *There exists a non-singular matrix* Q_2 *such that* $Q_2^{-1}A^*Q_2$ *is diagonal and* $Q_2^{-1}AQ_2$ *and* $Q_2^{-1}A^\epsilon Q_2$ *are irreducible tridiagonal.*

(iii) *There exists a non-singular matrix* Q_3 *such that* $Q_3^{-1}A^\epsilon Q_3$ *is diagonal and* $Q_3^{-1}AQ_3$ *and* $Q_3^{-1}A^*Q_3$ *are irreducible tridiagonal.*

(When (i)-(iii) hold we say that (A, A^*, A^ϵ) form a Leonard triple via *conjugating matrices* Q_1, Q_2, and Q_3.)

The notion of a LT and the corresponding notion of TB, TAB, and TDAB are similarly defined below.

Definition 5.1.2. In the definition of a LT in Definition 5.1.1, we mentioned six irreducible tridiagonal matrices (i.e., A^* and A^ϵ in (i), A and A^ϵ in (ii), and A and A^* in (iii)). The LT (A, A^*, A^ϵ) is said to be *totally bipartite* (resp., *totally almost bipartite*, *totally doubly almost bipartite*) whenever each of the six irreducible tridiagonal matrices is bipartite (resp., almost bipartite, doubly almost bipartite).

For any LTs, any two of the three form a LP. We say that these LPs are *associated* with the LT. So the LP is TB if and only if all the associated LPs are TB. Similarly, the LT is TAB (respectively TDAB) if and only if all of the associated LPs are TAB (TDAB).

Known results on LTs:

- Given a TBLP (A, A^*) on V of q-Racah type, Gao, Hou, Zhang determined all matrices A^ϵ such that (A, A^*, A^ϵ) forms a LT on V and classified up to isomorphism the TBLTs of q-Racah type [37].

- Given a TBLP (A, A^*) on V of Bannai-Ito type, Brown determined all matrices A^ϵ such that (A, A^*, A^ϵ) forms a LT on V and classified up to isomorphism the TBLTs of Bannai-Ito type [7].

- Given a TBLP (A, A^*) on V of Krawtchouk type, Balmaceda and Maralit determined all matrices A^ϵ such that (A, A^*, A^ϵ) forms a LT on V and classified up to isomorphism the TBLTs of Krawtchouk type [2].

As stated on page 16, Terwilliger classified all LPs and the isomorphism classes of LPs fall naturally into 13 families listed. It remains an open problem to fully classy the LTs (up to isomorphism). However, Curtin classifed a family of LTs said to be *modular* [11]. This leads to the following two definitions.

Definition 5.1.3 (Antiautomorphism)**.** By an *antiautomorphism* of $\mathrm{End}(V)$, we mean a $\mathbb{K}$-linear bijection $\tau : \mathrm{End}(V) \to \mathrm{End}(V)$ such that $\tau(AB) = \tau(B)\tau(A)$ for all $A, B \in \mathrm{End}(V)$.

Definition 5.1.4. Let (A, A^*, A^ϵ) be a LT on V. It is said to be *modular* whenever for each $B \in \{A, A^*, A^\epsilon\}$ there exists an antiautomorphism of $\mathrm{End}(V)$ which fixes B and swaps the other two members of the triple.

We pose the following two questions.

Future Research Problems

Problem 5.1.1. Classify up to isomorphism of doubly almost bipartite Leonard triples.

Problem 5.1.2. Find an appropriate autiautomorphism and classify up to isomorphism of doubly almost bipartite modular Leonard triples.

5.2 Spin Leonard Pairs (SLPs)

Let us define the following new class of Leonard pairs.

Definition 5.2.1. [12, Definition 1.2] A Leonard pair (A, A^*) on V is said to be a *spin Leonard pair* (SLP) whenever there exist invertible linear transformations B, B^* in $\mathrm{End}(V)$ such that

(i) $BA = AB$,

(ii) $B^*A^* = A^*B^*$,

(iii) $BA^*B^{-1} = (B^*)^{-1}AB^*$.

In this case, we refer to (B, B^*) as a *Boltzmann pair* for (A, A^*).

The notion of a SLP was first introduced by V.F.R. Jones for a statistical mechanical construction of link invariants [25]. Jaeger [24] and Nomura [34] then showed that spin models are contained in Bose-Mesner algebra arising from distance-regular graphs [3, 6]. In many instances, the irreducible representations of the Terwilliger algebra are LPs and thus if the Bose-Mesner algebra of a distance-regular graph supports a spin model, then every irreducible representation of the associated Terwilliger algebra is not only a LP, but a SLP [8].

The SLPs are classified up to isomorphism involving explicit formulas for the entries of the matrices representing A and A^* with respect to a particular basis and the corresponding Boltzmann pair for (A, A^*) are also described. Furthermore, Curtin showed that there is an intimate connection between SLPs and MLTs. See the following two theorems.

Theorem 5.2.1. [12, Theorem 1.5] *Let (A, A^*, A^ϵ) be a MLT on V. Then A, A^* is a SLP.*

Theorem 5.2.2. [12, Theorem 1.6] *Let (S, S^*) be a SLT on V and let (B, B^*) denote a Boltzmann pair for (S, S^*). Set $T := BS^*B^{-1}(= B^{*-1}SB^*)$ and $T^* := B^{-1}S^*B(= B^*SB^{*-1})$. Then (S, S^*, T) and (S, S^*, T^*) are both MLTs.*

We pose the following problem regarding SLPs.

Future Research Problem

Problem 5.2.1. Classify up to isomorphism of spin DABLPs (S, S^*) and describe the corresponding Boltzmann pair (B, B^*) for (S, S^*).

5.3 Near-Bipartite Leonard Pairs (near-BLPs)

In [35], Nomura and Terwilliger introduced a notion of *near-bipartite Leonard pairs.*

Start with a LS $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ with a Φ-standard basis $\{v_i\}_{i=0}^d$ for V so that the matrices representing A and A^* are irreducible tridiagonal and diagonal, respectively. For $0 \leq i \leq d$ define an $\mathbb{K}$-linear map $E_i^* : V \to V$ such that $E_i^* v_i = \delta_i v_i$ (i.e., the dual primitive idempotent given on page 14). Define a linear map

$$F := \sum_{i=0}^{d} E_i^* A E_i^*. \tag{5.3.1}$$

Recall F is the *flat part of* A defined in (3.1.13b). (Fact: (A, A^*) is a BLP if and only if $F = 0$.)

Definition 5.3.1. The LP (A, A^*) is said to be *near-bipartite* whenever the pair $(A - F, A^*)$ is a LP on V and in this case, the pair $(A - F, A^*)$ is a BLP and called the *bipartite contraction of* (A, A^*). Let (B, B^*) be a LP on V. By a *near-bipartite expansion of* (B, B^*) we mean a near-bipartite LP (N, N^*) on V with bipartite contraction (B, B^*).

Nomura and Terwilliger showed several important results regarding near-bipartite LP: A LP (A, A^*) over $\mathbb{K}$ with $d \geq 3$ is near-bipartite if and only if at least one of the following holds:

(i) (A, A^*) is *essentially bipartite*[4];

(ii) (A, A^*) has *reinforced*[5] dual q-Krawtchouk type;

(iii) (A, A^*) has Krawtchouk type.

We pose three more problems regarding near-BLPs.

Future Research Problem

Problem 5.3.1. Classify up to isomorphism of near-DABLPs over $\mathbb{K}$.

Problem 5.3.2. For each near-DABLP, describe its bipartite contraction.

Problem 5.3.3. For each DABLP, describe its near-DAB expansions.

[4] A LP (A, A^*) is said to be *essentially bipartite* whenever the flat part F of A is a scalar multiple of the identity I.

[5] The notion of *reinforced* LP applies to the dual q-Krawtchouk LP and it means that $q^{2i} \neq -1$.

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Appendix A Eigenvalues/vectors of All Ones DABITM

Proof of Theorem 3.3.1. Typically one first determines the eigenvalues and then the eigenvectors of a square matrix. For A given in (3.2.1), it ends up being simpler first to find the eigenvectors due to the three-term recurrence nature. To this end, let θ be an eigenvalue (not necessarily real) and $\vec{x} = \begin{pmatrix} x_0 & x_1 & \cdots & x_d \end{pmatrix}^T$ be a corresponding eigenvector of A. (Let us relabel the indices from $0, \ldots, d$ to $1, \ldots, n$ instead where $n = d+1$.) With hindsight it will be convenient to write $\theta = 2\lambda$. Then

$$
\begin{aligned}
\vec{0} &= (\theta I - A)\vec{x} \\
&= (2\lambda I - A)\vec{x} \\
&= \begin{pmatrix}
2\lambda - 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\
-1 & 2\lambda & -1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 2\lambda & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & & \ddots & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2\lambda & -1 & 0 \\
0 & 0 & 0 & \cdots & -1 & 2\lambda & -1 \\
0 & 0 & 0 & \cdots & 0 & -1 & 2\lambda - 1
\end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{pmatrix} \\
&= \begin{pmatrix}
(2\lambda - 1)x_1 - x_2 \\
-x_1 + 2\lambda x_2 - x_3 \\
-x_2 + 2\lambda x_3 - x_4 \\
\vdots \\
-x_{k-1} + 2\lambda x_k - x_{k+1} \\
\vdots \\
-x_{n-2} + 2\lambda x_{n-1} - x_n \\
-x_{n-1} + (2\lambda - 1)x_n
\end{pmatrix}
\end{aligned}
$$

$$
= \begin{pmatrix} (2\lambda-1)x_1 - x_2 \\ x_2 - x_1 + (2\lambda-1)x_2 - x_3 \\ x_3 - x_2 + (2\lambda-1)x_3 - x_4 \\ \vdots \\ x_k - x_{k-1} + (2\lambda-1)x_k - x_{k+1} \\ \vdots \\ x_{n-1} - x_{n-2} + (2\lambda-1)x_{n-1} - x_n \\ -x_{n-1} + (2\lambda-1)x_n \end{pmatrix}. \tag{A.1}
$$

Introducing two new auxiliary variables x_0 and x_{n+1}, the first and the last entries of (A.1) can be written as

$$
x_1 - x_0 + (2\lambda-1)x_1 - x_2 \quad \text{and} \quad x_n - x_{n-1} + (2\lambda-1)x_n - x_{n+1},
$$

respectively. Note that we must have $x_1 - x_0 = 0$ and $x_n - x_{n+1} = 0$. Hence

$$
\begin{pmatrix} x_1 - x_0 + (2\lambda-1)x_1 - x_2 \\ x_2 - x_1 + (2\lambda-1)x_2 - x_3 \\ x_3 - x_2 + (2\lambda-1)x_3 - x_4 \\ \vdots \\ x_k - x_{k-1} + (2\lambda-1)x_k - x_{k+1} \\ \vdots \\ x_{n-1} - x_{n-2} + (2\lambda-1)x_{n-1} - x_n \\ x_n - x_{n-1} + (2\lambda-1)x_n - x_{n+1} \end{pmatrix} = \vec{0}. \tag{A.2}
$$

Observe that for $k = 1, \ldots, n$, each entry of (A.2) has the form

$$
x_k - x_{k-1} + (2\lambda-1)x_k - x_{k+1} = 0, \tag{A.3}
$$

which is a *second-order homogeneous linear difference equation* with constant coefficients along with two conditions (i) $x_1 - x_0 = 0$ and (ii) $x_n - x_{n+1} = 0$. Assuming (A.3) has a solution of the form $x_k = r^k$ ($r \neq 0$), the characteristic equation of this difference equation is

$$r^k - r^{k-1} + (2\lambda - 1)r^k - r^{k+1} = 0$$

or simply

$$r^2 - 2\lambda r + 1 = 0 \tag{A.4}$$

whose roots are $r_\pm = \lambda \pm \sqrt{\lambda^2 - 1}$. Rearrange (A.4) in the following way to obtain

$$2\lambda = r + r^{-1}. \tag{A.5}$$

Also, the product of these two roots is found to be

$$r_+ r_- = 1. \tag{A.6}$$

Let us consider the following three cases.

Case 1. $\lambda \neq \pm 1$. In this case the two roots r_+ and r_- are distinct. For notational convenience, let $r := r_+ = \lambda + \sqrt{\lambda^2 - 1}$. Then we can express the other root r_- in terms of r as follows:

$$r_- = \lambda - \sqrt{\lambda^2 - 1} = \frac{1}{\lambda + \sqrt{\lambda^2 - 1}} = \frac{1}{r} = r^{-1}.$$

Therefore, the general solution of (A.3) is

$$x_k = c_1 r_+^k + c_2 r_-^k = c_1 r^k + c_2 r^{-k}, \qquad k = 1, \ldots, n$$

for some constants c_1 and c_2.

Using the first condition (i) $x_1 - x_0 = 0$, we see that $c_2 = c_1 r$. Thus

$$x_k = c_1 r^k + (c_1 r) r^{-k} = c_1 (r^k + r^{1-k}). \tag{A.7}$$

Notice that we require $c_1 \neq 0$ for a non-trivial solution of (A.3).

Next, using the second condition (ii) $x_n - x_{n+1} = 0$, we get

$$r^n (1 - r) = r^{-n} (1 - r).$$

If $r = 1$, then (A.5) implies $\lambda = 1$, which is a contradiction and hence we conclude that $r \neq 1$. Dividing each side of the above equation by $1 - r$ and further simplifying, we obtain

$$r^{2n} = 1, \tag{A.8}$$

which implies $|r| = 1$. Taking the absolute value of each side of (A.5) and using the fact that $|r| = 1$,

$$2|\lambda| = |2\lambda| = |r + r^{-1}| \leq |r| + |r^{-1}| = 2$$

so $|\lambda| \leq 1$ and since $\lambda \neq \pm 1$, we have $|\lambda| < 1$.

Case 2. $\lambda = 1$. In this case, $r = \lambda$ so the general solution of (A.3) is give by

$$x_k = (c_1 + c_2 k)\lambda^k = c_1 + c_2 k \tag{A.9}$$

for some constants c_1 and c_2. The first condition (i) $x_1 - x_0 = 0$ implies $c_2 = 0$ and therefore, $x_k = c_1$ ($c_1 \neq 0$). Notice that the second condition (ii) $x_n - x_{n+1} = 0$ is automatically satisfied. So in this case, $x_k = c_1$ for all $1 \leq k \leq n$, where c_1 is some nonzero constant.

Case 3. $\lambda = -1$. Once again, in this case, $r = \lambda$ so the general solution of (A.3) is give by

$$x_k = (c_1 + c_2 k)\lambda^k = (c_1 + c_2 k)(-1)^k \tag{A.10}$$

for some constants c_1 and c_2. The first condition (i) $x_1 - x_0 = 0$ implies $c_2 = -2c_1$ and therefore,

$$x_k = (c_1 - 2c_1 k)(-1)^k = c_1(1 - 2k)(-1)^k. \tag{A.11}$$

Notice that we require $c_1 \neq 0$ for a non-trivial solution of (A.3).

Next, using the second condition (ii) $x_n - x_{n+1} = 0$, we get $n = 0$ which is clearly absurd. Consequently, the original eigenvalue equation has no non-trivial solution for $\lambda = -1$.

Let us go back to Case 1 above. Since $|r| = 1$, write r as $r = e^{\mathbf{i}\alpha}$ for some real variable α and $\mathbf{i} = \sqrt{-1}$ is the imaginary unit. Equation (A.8) implies $1 = r^{2n} = e^{2\mathbf{i}n\alpha}$. So $2n\alpha = 2i\pi$ or simply $\alpha = i\pi/n$ for $1 \leq i \leq n-1$. (We exclude $i = n$ since then $\alpha = \pi$ and so $r = e^{\mathbf{i}\pi} = -1$. This implies (by A.5) $\lambda = -1$ which is not allowed based on Case 3.) On the other hand, if we allowed $i = 0$, then $\alpha = 0$ and $r = 1$ and thus $\lambda = 1$, which is simply the second case we considered. Therefore,

$$r = e^{\mathbf{i}(i\pi/n)} = \left(e^{\mathbf{i}\pi/n}\right)^i = \left(e^{\mathbf{i}\pi/(d+1)}\right)^i. \tag{A.12}$$

(Recall that the indices were relabeled such that $n = d+1$ and so (A.12) is valid for $0 \leq i \leq d$.) Define $q \equiv e^{\mathbf{i}\pi/(d+1)}$. By (A.12), $r = q^i$ and substituting this result in (A.5), together with (A.7) yields the desired result. $\square$

Appendix B Generalization of the A Matrix

See [1] - Lemma 8.1, 8.2, Corollary 8.3, as well as Equation (1.1) for the following results.)

Define the following two $(d+1) \times (d+1)$ tridiagonal matrices

$$\widetilde{A} := \begin{pmatrix} a_0 & 1-c_1 & & & \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ & & & 1-b_{d-1} & a_d \end{pmatrix}, \tag{B.1a}$$

$$B := k \begin{pmatrix} a+a_0 & \gamma_1^{-1}(1-c_1) & & & \\ \gamma_1 & a & \gamma_2^{-1} & & \\ & \gamma_2 & \ddots & \ddots & \\ & & \ddots & a & \gamma_d^{-1} \\ & & & \gamma_d(1-b_{d-1}) & a+a_d \end{pmatrix}, \tag{B.1b}$$

where $a, k,$ and $\{\gamma_j\}_{j=1}^{d}$ are arbitrary constants in $\mathbb{K}$ such that k and γ_j (for all j) are nonzero and $c_1, b_{d-1} \neq 1$ to ensure that both $\widetilde{A}$ and B are irreducible. Observe that $\widetilde{A}$ becomes an all ones DABITM given in (3.2.1) when $a_0 = a_d = 1$ and $c_1 = b_{d-1} = 0$.

The next three lemmas will be helpful and can be proven by simple computations.

Lemma B.1. [1, Lemma 8.1] *Let $D \in Mat_{d+1}(\mathbb{K})$ be the diagonal matrix with $\epsilon_i \neq 0$ as its i^{th} diagonal element. Let $M \in Mat_{d+1}(\mathbb{K})$ be arbitrary. Then*

$$(D^{-1}MD)_{ij} = \epsilon_i^{-1}\epsilon_j M_{ij}. \tag{B.2}$$

Proof. Since $\epsilon_i \neq 0$ for each i, D is invertible and

$$D^{-1} = \mathrm{diag}(\epsilon_0^{-1}, \epsilon_1^{-1}, \ldots, \epsilon_d^{-1}).$$

Pre-multiplying M by D^{-1} scales the i^{th} row of M by ϵ_i^{-1} and post-multiplying M by D scales the j^{th} column of M by ϵ_j. Therefore, the ij-entry of $D^{-1}MD$ is given by $\epsilon_i^{-1}\epsilon_j M_{ij}$ where M_{ij} is the ij-entry of M. □

Lemma B.2. [1, Lemma 8.2] *Let* $D = \mathrm{diag}(\epsilon_0, \epsilon_1, \ldots, \epsilon_\mathrm{d})$ *where* $\epsilon_0 \equiv 1$ *and* $\epsilon_i = \prod_{j=1}^{i} \gamma_j$ *for* $j = 1, \ldots, d$*. Furthermore, let* $\widetilde{A}$ *be the matrix given in (B.1a). Then the matrix* B *given in (B.1b) is given by*

$$B = k(D\widetilde{A}D^{-1} + aI) \quad \textit{or equivalently} \quad \widetilde{A} = D^{-1}(k^{-1}B - aI)D. \tag{B.3}$$

Proof. Simply apply Lemma B.1 to $D^{-1}(k^{-1}B - aI)D$. □

Lemma B.3. [1, Corollary 8.3] *Let* $a, \theta \in \mathbb{K}$ *and* $\vec{x} \in \mathbb{K}^{d+1}$*. Then* $\{k(\theta + a), D\vec{x}\}$ *is an eigenpair of* B *if and only if* $(\theta, \vec{x})$ *is an eigenpair of* $\widetilde{A}$*.*

Proof.

$$\begin{aligned}
\{k(\theta + a), D\vec{x}\} \text{ is an eigenpair of } B &\iff B(D\vec{x}) = k(\theta + a)D\vec{x} \\
&\iff k(D\widetilde{A}D^{-1} + aI)(D\vec{x}) = k(\theta + a)D\vec{x} \\
&\iff A\vec{x} = \theta\vec{x} \\
&\iff \{\theta, \vec{x}\} \text{ is an eigenpair of } A.
\end{aligned}$$

(The second '$\iff$' is justified by B.3.) □

In order to ensure that B is doubly almost bipartite irreducible tridiagonal, choose $a = 0$. By (B.3) in Lemma B.2, we have $B = kDAD^{-1}$ (or $D^{-1}BD = kA$). Then by Lemma B.3, $\{k\theta, D\vec{x}\}$ is an eigenpair of B if and only if $(\theta, \vec{x})$ is an eigenpair of A.

The following is the generalization of Corollary 3.7.1.

Theorem B.1. *Let B be the tridiagonal matrix in (B.3) with $a = c_1 = b_{d-1} = 0$ and $a_0 = a_d = 1$. The pair (B, Δ) form an all ones DABLP on $\mathbb{K}^{d+1}$ via the identity matrix I and $D\widetilde{Q_2}$ if and only if the diagonal entries δ_i's satisfy the recursive relation give in (3.5.1):*

$$\frac{\delta_i - \delta_{i+1}}{\delta_{i+1} - \delta_{i+2}} = \frac{\theta^*_i - \theta^*_{i+1}}{\theta^*_{i+1} - \theta^*_{i+2}},$$

where $0 \leq i \leq d-2$ (θ^*_i as in Theorem 3.4.1).

Proof. It suffices to show that (i) $\big(D\widetilde{Q_2}\big)^{-1}B\big(D\widetilde{Q_2}\big)$ is diagonal and (ii) $\big(D\widetilde{Q_2}\big)^{-1}\Delta\big(D\widetilde{Q_2}\big)$ is irreducible tridiagonal.

To prove (i), we see that

$$\begin{aligned}
\big(D\widetilde{Q_2}\big)^{-1}B\big(D\widetilde{Q_2}\big) &= \widetilde{Q_2}^{-1}(D^{-1}BD)\widetilde{Q_2} \\
&= \widetilde{Q_2}^{-1}(kA)\widetilde{Q_2} \\
&= k\widetilde{Q_2}^{-1}A\widetilde{Q_2} \\
&= k\Lambda,
\end{aligned}$$

where Λ is the diagonal matrix consisting of the eigenvalues of A. This shows that $\big(D\widetilde{Q_2}\big)^{-1}B\big(D\widetilde{Q_2}\big)$ is indeed diagonal.

On the other hand,

$$\begin{aligned}\big(D\widetilde{Q_2}\big)^{-1}\Delta\big(D\widetilde{Q_2}\big) &= \widetilde{Q_2}^{-1}(D^{-1}\Delta D)\widetilde{Q_2}\\ &= \widetilde{Q_2}^{-1}\Delta\widetilde{Q_2}.\end{aligned}$$

(Note that the product of diagonal matrices commute.) By Theorem 3.5.1, we know that (A, Δ) form an all ones DABLP via the identity matrix I and $\widetilde{Q_2}$ and hence the conjugation of Δ by $\widetilde{Q_2}$ is guaranteed to be irreducible tridiagonal, showing that $\big(D\widetilde{Q_2}\big)^{-1}\Delta\big(D\widetilde{Q_2}\big)$ is irreducible tridiagonal, as claimed. $\square$

Appendix C Parameter/Intersection Arrays

In this section we display the parameter and intersection arrays

$$\mathcal{P} = (\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d) \quad \text{and} \quad \mathcal{I} = (\{b_i\}_{i=0}^{d-1}, \{c_i\}_{i=1}^d),$$

respectively, of all 13 types of LPs over $\mathbb{K}$ (see page 16). For more detailed information, see [45, 47].

1. ***q*-Racah** Assume $h, h^*, q, s, s^*, r_1, r_2$ are nonzero and $r_1 r_2 = ss^* q^{d+1}$. Furthermore, assume none of $q^i, r_1 q^i, r_2 q^i, s^* q^i / r_1, s^* q^i / r_2$ is equal to 1 for $1 \leq i \leq d$ and neither of $sq^i, s^* q^i$ is equal to 1 for $2 \leq i \leq 2d$.

$$\theta_i = \theta_0 + h(1 - q^i)(1 - sq^{i+1})q^{-i}, \tag{C.1a}$$

$$\theta_i^* = \theta_0^* + h^*(1 - q^i)(1 - s^* q^{i+1})q^{-i}, \tag{C.1b}$$

$$\varphi_i = hh^* q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(1 - r_1 q^i)(1 - r_2 q^i), \tag{C.1c}$$

$$\phi_i = hh^* q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(r_1 - s^* q^i)(r_2 - s^* q^i)/s^*, \tag{C.1d}$$

$$b_i = \frac{h(1 - q^{i-d})(1 - s^* q^{i+1})(1 - r_1 q^{i+1})(1 - r_2 q^{i+1})}{(1 - s^* q^{2i+1})(1 - s^* q^{2i+2})}, \tag{C.1e}$$

$$c_i = \frac{h(1 - q^i)(1 - s^* q^{i+d+1})(r_1 - s^* q^i)(r_2 - s^*(q^i)}{s^* q^d (1 - s^* q^{2i})(1 - s^* q^{2i+1})}. \tag{C.1f}$$

To obtain $\{b_i^*\}_{i=0}^{d-1}$ and $\{c_i^*\}_{i=1}^d$, exchange $h \leftrightarrow h^*, s \leftrightarrow s^*$ in (C.1e) and (C.1f) and preserve r_1, r_2, q.

2. ***q*-Hahn** Assume h, h^*, q, s^*, r are nonzero. Furthermore, assume none of $q^i, rq^i, s^*q^i/r$ is equal to 1 for $1 \leq i \leq d$ and $s^*q^i \neq 1$ for $2 \leq i \leq 2d$.

$$\theta_i = \theta_0 + h(1-q^i)q^{-i}, \tag{C.2a}$$

$$\theta^*_i = \theta^*_0 + h^*(1-q^i)(1-s^*q^{i+1})q^{-i}, \tag{C.2b}$$

$$\varphi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-rq^i), \tag{C.2c}$$

$$\phi_i = -hh^*q^{1-i}(1-q^i)(1-q^{i-d-1})(r-s^*q^i), \tag{C.2d}$$

$$b_i = \frac{h(1-q^{i-d})(1-s^*q^{i+1})(1-rq^{i+1})}{(1-s^*q^{2i+1})(1-s^*q^{2i+2})}, \tag{C.2e}$$

$$c_i = \frac{-hq^{i-d}(1-q^i)(1-s^*q^{i+d+1})(r-s^*q^i)}{(1-s^*q^{2i})(1-s^*q^{2i+1})}, \tag{C.2f}$$

$$b^*_i = h^*(1-q^{i-d})(1-rq^{i+1}) \quad (0 \leq i \leq d-1), \tag{C.2g}$$

$$c^*_i = h^*(1-q^i)(qs^*-rq^{i-d}) \quad (1 \leq i \leq d). \tag{C.2h}$$

3. **Dual *q*-Hahn** Assume h, h^*, q, s, r are nonzero. Furthermore, assume none of $q^i, rq^i, sq^i/r$ is equal to 1 for $1 \leq i \leq d$ and $sq^i \neq 1$ for $2 \leq i \leq 2d$.

$$\theta_i = \theta_0 + h(1-q^i)(1-sq^{i+1})q^{-i}, \tag{C.3a}$$

$$\theta^*_i = \theta^*_0 + h^*(1-q^i)q^{-i}, \tag{C.3b}$$

$$\varphi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-rq^i), \tag{C.3c}$$

$$\phi_i = hh^*q^{d+2-2i}(1-q^i)(1-q^{i-d-1})(s-rq^{i-d-1}), \tag{C.3d}$$

$$b_i = h(1-q^{i-d})(1-rq^{i+1}), \tag{C.3e}$$

$$c_i = h(1-q^i)(qs-rq^{i-d}), \tag{C.3f}$$

$$b^*_i = \frac{h^*(1-q^{i-d})(1-sq^{i+1})(1-rq^{i+1})}{(1-sq^{2i+1})(1-sq^{2i+2})} \quad (0 \leq i \leq d-1), \tag{C.3g}$$

$$c^*_i = \frac{-h^*q^{i-d}(1-q^i)(1-sq^{i+d+1})(r-sq^i)}{(1-sq^{2i})(1-sq^{2i+1})} \quad (1 \leq i \leq d). \tag{C.3h}$$

4. **Quantum q-Krawtchouk** Assume h^*, q, s, r are nonzero. Furthermore, assume neither of $q^i, sq^i/r$ is equal to 1 for $1 \leq i \leq d$.

$$\theta_i = \theta_0 - sq(1-q^i), \tag{C.4a}$$

$$\theta^*_i = \theta^*_0 + h^*(1-q^i)q^{-i}, \tag{C.4b}$$

$$\varphi_i = -rh^*q^{1-i}(1-q^i)(1-q^{i-d-1}), \tag{C.4c}$$

$$\phi_i = h^*q^{d+2-2i}(1-q^i)(1-q^{i-d-1})(s-rq^{i-d-1}), \tag{C.4d}$$

$$b_i = -rq^{i+1}(1-q^{i-d}), \tag{C.4e}$$

$$c_i = (1-q^i)(qs-rq^{i-d}), \tag{C.4f}$$

$$b^*_i = \frac{h^*r(1-q^{i-d})}{sq^{2i+1}} \qquad (0 \leq i \leq d-1), \tag{C.4g}$$

$$c^*_i = \frac{h^*(1-q^i)(r-sq^i)}{sq^{2i}} \qquad (1 \leq i \leq d). \tag{C.4h}$$

5. **q-Krawtchouk** Assume h, h^*, q, s^* are nonzero. Furthermore, assume $q^i \neq 1$ for $1 \leq i \leq d$ and $s^*q^i \neq 1$ for $2 \leq i \leq 2d$.

$$\theta_i = \theta_0 + h(1-q^i)q^{-i}, \tag{C.5a}$$

$$\theta^*_i = \theta^*_0 + h^*(1-q^i)(1-s^*q^{i+1})q^{-i}, \tag{C.5b}$$

$$\varphi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1}), \tag{C.5c}$$

$$\phi_i = hh^*s^*q(1-q^i)(1-q^{i-d-1}), \tag{C.5d}$$

$$b_i = \frac{h(1-q^{i-d})(1-s^*q^{i+1})}{(1-s^*q^{2i+1})(1-s^*q^{2i+2})}, \tag{C.5e}$$

$$c_i = \frac{hs^*q^{2i-d}(1-q^i)(1-s^*q^{i+d+1})}{(1-s^*q^{2i})(1-s^*q^{2i+1})}, \tag{C.5f}$$

$$b^*_i = h^*(1-q^{i-d}) \qquad (0 \leq i \leq d-1), \tag{C.5g}$$

$$c^*_i = h^*s^*q(1-q^i) \qquad (1 \leq i \leq d). \tag{C.5h}$$

6. **Affine *q*-Krawtchouk** Assume h, h^*, q, r are nonzero. Furthermore, assume neither q^i, rq^i is equal to 1 for $1 \le i \le d$.

$$\theta_i = \theta_0 + h(1-q^i)q^{-i}, \tag{C.6a}$$

$$\theta^*_i = \theta^*_0 + h^*(1-q^i)q^{-i}, \tag{C.6b}$$

$$\varphi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-rq^i), \tag{C.6c}$$

$$\phi_i = -hh^*rq^{1-i}(1-q^i)(1-q^{i-d-1}), \tag{C.6d}$$

$$b_i = h(1-q^{i-d})(1-rq^{i+1}), \tag{C.6e}$$

$$c_i = -hrq^{i-d}(1-q^i). \tag{C.6f}$$

To obtain $\{b^*_i\}_{i=0}^{d-1}$ and $\{c^*_i\}_{i=1}^{d}$, exchange $h \leftrightarrow h^*, s \leftrightarrow s^*$ in (C.6e) and (C.6f) and preserve r and q.

7. **Dual *q*-Krawtchouk** Assume h, h^*, q, s are nonzero. Furthermore, assume $q^i \neq 1$ for $1 \le i \le d$ and $sq^i \neq 1$ for $2 \le i \le 2d$.

$$\theta_i = \theta_0 + h(1-q^i)(1-sq^{i+1})q^{-i}, \tag{C.7a}$$

$$\theta^*_i = \theta^*_0 + h^*(1-q^i)q^{-i}, \tag{C.7b}$$

$$\varphi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1}), \tag{C.7c}$$

$$\phi_i = hh^*sq^{d+2-2i}(1-q^i)(1-q^{i-d-1}), \tag{C.7d}$$

$$b_i = h(1-q^{i-d}), \tag{C.7e}$$

$$c_i = hsq(1-q^i), \tag{C.7f}$$

$$b^*_i = \frac{h^*(1-q^{i-d})(1-sq^{i+1})}{(1-sq^{2i+1})(1-sq^{2i+2})} \qquad (0 \le i \le d-1), \tag{C.7g}$$

$$c^*_i = \frac{h^*sq^{2i-d}(1-q^i)(1-sq^{i+d+1})}{(1-sq^{2i})(1-sq^{2i+1})} \qquad (1 \le i \le d). \tag{C.7h}$$

8. **Racah** Assume h, h^* are nonzero and $r_1+r_2 = s+s^*+d+1$. Furthermore, $\text{char}(\mathbb{K}) = 0$ or a prime greater than d and none of $r_1, r_2, s^* - r_1, s^* - r_2$ is equal to $-i$ for $1 \le i \le d$ and neither s, s^* is equal to $-i$ for $2 \le i \le 2d$.

$$\theta_i = \theta_0 + hi(i+1+s), \tag{C.8a}$$

$$\theta^*_i = \theta^*_0 + h^*i(i+1+s^*), \tag{C.8b}$$

$$\varphi_i = hh^*i(i-d-1)(i+r_1)(i+r_2), \tag{C.8c}$$

$$\phi_i = hh^*i(i-d-1)(i+s^*-r_1)(i+s^*-r_2), \tag{C.8d}$$

$$b_i = \frac{h(i-d)(i+1+s^*)(i+1+r_1)(i+1+r_2)}{(2i+1+s^*)(2i+2+s^*)}, \tag{C.8e}$$

$$c_i = \frac{hi(i+d+1+s^*)(i+s^*-r_1)(i+s^*-r_2)}{(2i+s^*)(2i+1+s^*)}. \tag{C.8f}$$

To obtain $\{b^*_i\}_{i=0}^{d-1}$ and $\{c^*_i\}_{i=1}^{d}$, exchange $h \leftrightarrow h^*, s \leftrightarrow s^*$ in (C.8e) and (C.8f) and preserve r_1 and r_2.

9. **Hahn** Assume h^*, s are nonzero. Furthermore, $\text{char}(\mathbb{K}) = 0$ or a prime greater than d and neither of $r, s^* - r$ is equal to $-i$ for $1 \le i \le d$ and that $s^* \ne -i$ for $2 \le i \le 2d$.

$$\theta_i = \theta_0 + si, \tag{C.9a}$$

$$\theta^*_i = \theta^*_0 + h^*i(i+1+s^*), \tag{C.9b}$$

$$\varphi_i = h^*si(i-d-1)(i+r), \tag{C.9c}$$

$$\phi_i = -h^*si(i-d-1)(i+s^*-r), \tag{C.9d}$$

$$b_i = \frac{s(i-d)(i+1+s^*)(i+1+r)}{(2i+1+s^*)(2i+2+s^*)}, \tag{C.9e}$$

$$c_i = \frac{-si(i+d+1+s^*)(i+s^*-r)}{(2i+s^*)(2i+1+s^*)}, \tag{C.9f}$$

$$b^*_i = h^*(i-d)(i+1+r) \qquad (0 \le i \le d-1), \tag{C.9g}$$

$$c^*_i = h^*i(i-d-1-s^*+r) \qquad (1 \le i \le d). \tag{C.9h}$$

10. **Dual Hahn** Assume h, s^* are nonzero. Furthermore, $\mathrm{char}(\mathbb{K}) = 0$ or a prime greater than d and neither of $r, s - r$ is equal to $-i$ for $1 \leq i \leq d$ and $s \neq -i$ for $2 \leq i \leq 2d$.

$$\theta_i = \theta_0 + hi(i + 1 + s), \tag{C.10a}$$

$$\theta^*_i = \theta^*_0 + s^* i, \tag{C.10b}$$

$$\varphi_i = hs^* i(i - d - 1)(i + r), \tag{C.10c}$$

$$\phi_i = hs^* i(i - d - 1)(i + r - s - d - 1), \tag{C.10d}$$

$$b_i = h(i - d)(i + 1 + r), \tag{C.10e}$$

$$c_i = hi(i - d - 1 - s + r), \tag{C.10f}$$

$$b^*_i = \frac{s^*(i - d)(i + 1 + s)(i + 1 + r)}{(2i + 1 + s)(2i + 2 + s)} \qquad (0 \leq i \leq d - 1) \tag{C.10g}$$

$$c^*_i = \frac{-s^* i(i + d + 1 + s)(i + s - r)}{(2i + s)(2i + 1 + s)} \qquad (1 \leq i \leq d). \tag{C.10h}$$

11. **Krawtchouk** Assume r, s, s^* are nonzero. Furthermore, $\mathrm{char}(\mathbb{K}) = 0$ or a prime greater than d and $r \neq ss^*$.

$$\theta_i = \theta_0 + si, \tag{C.11a}$$

$$\theta^*_i = \theta^*_0 + s^* i, \tag{C.11b}$$

$$\varphi_i = ri(i - d - 1), \tag{C.11c}$$

$$\phi_i = i(r - ss^*)(i - d - 1), \tag{C.11d}$$

$$b_i = r(i - d)/s^*, \tag{C.11e}$$

$$c_i = i(r - ss^*)/s^*. \tag{C.11f}$$

To obtain $\{b^*_i\}_{i=0}^{d-1}$ and $\{c^*_i\}_{i=1}^{d}$, exchange $s \leftrightarrow s^*$ in (C.11e) and (C.11f) and preserve r.

12. **Bannai/Ito** Assume h, h^* are nonzero and that $r_1 + r_2 = -s - s^* + d + 1$. Furthermore, $\mathrm{char}(\mathbb{K}) = 0$ or a prime greater than $d/2$, neither of $r_1, -s^* - r_1$ is equal to $-i$ for $1 \leq i \leq d, d - i$ even. Assume further that neither of $r_2, -s^* - r_2$ is equal to $-i$ for $1 \leq i \leq d, i$ odd and neither of s, s^* is equal to $2i$ for $1 \leq i \leq d$.

$$\theta_i = \theta_0 + h[s - 1 + (1 - s + 2i)(-1)^i], \tag{C.12a}$$

$$\theta^*_i = \theta^*_0 + h^*[s^* - 1 + (1 - s^* + 2i)(-1)^i], \tag{C.12b}$$

$$\varphi_i = \begin{cases} -4hh^*i(i + r_1), & i \text{ even, } d \text{ even;} \\ -4hh^*i(i - d - 1)(i + r_2), & i \text{ odd, } d \text{ even;} \\ -4hh^*i(i - d - 1), & i \text{ even, } d \text{ odd;} \\ -4hh^*(i + r_1)(i + r_2), & i \text{ odd, } d \text{ odd.} \end{cases} \tag{C.12c}$$

$$\phi_i = \begin{cases} 4hh^*i(i - s^* - r_1), & i \text{ even, } d \text{ even;} \\ 4hh^*(i - d - 1)(i - s^* - r_2), & i \text{ odd, } d \text{ even;} \\ -4hh^*i(i - d - 1), & i \text{ even, } d \text{ odd;} \\ -4hh^*(i - s^* - r_1)(i - s^* - r_2), & i \text{ odd, } d \text{ odd.} \end{cases} \tag{C.12d}$$

$$b_i = \begin{cases} \dfrac{2h(i - d)(i + 1 + r_2)}{2i + 2 - s^*}, & i \text{ even, } d \text{ even;} \\ \dfrac{2h(i + 1 - s^*)(i + 1 + r_1)}{2i + 2 - s^*}, & i \text{ odd, } d \text{ even;} \\ \dfrac{2h(i + 1 + r_1)(i + 1 + r_2)}{2i + 2 - s^*}, & i \text{ even, } d \text{ odd;} \\ \dfrac{2h(i - d)(i + 1 - s^*)}{2i + 2 - s^*}, & i \text{ odd, } d \text{ odd.} \end{cases} \tag{C.12e}$$

$$c_i = \begin{cases} \dfrac{-2hi(i - s^* - r_1)}{2i - s^*}, & i \text{ even, } d \text{ even;} \\ \dfrac{-2h(i + d + 1 - s^*)(i - s^* - r_2)}{2i - s^*}, & i \text{ odd, } d \text{ even;} \\ \dfrac{-2hi(i + d + 1 - s^*)}{2i - s^*}, & i \text{ even, } d \text{ odd;} \\ \dfrac{-2h(i - s^* - r_1)(i - s^* - r_2)}{2i - s^*}, & i \text{ odd, } d \text{ odd.} \end{cases} \tag{C.12f}$$

To obtain $\{b^*_i\}_{i=0}^{d-1}$ and $\{c^*_i\}_{i=1}^{d}$, exchange $h \leftrightarrow h^*, s \leftrightarrow s^*$ in (C.12e) and (C.12f) and preserve r_1, r_2, q.

13. **Orphan** Assume $h, h^*, s, s^{*}*$ are nonzero. Furthermore, the char($\mathbb{K}$) = 2, $d = 3$, and neither of s, s^* is equal to 1 and that r is equal to none of $s + s^*, s(1 + s^*), s^*(1 + s)$.

$$\theta_1 = \theta_0 + h(1+s), \qquad \theta_2 = \theta_0 + h, \qquad \theta_3 = \theta_0 + hs, \tag{C.13a}$$

$$\theta^*_1 = \theta^*_0 + h^*(1+s^*), \qquad \theta^*_2 = \theta^*_0 + h^*, \qquad \theta^*_3 = \theta^*_0 + h^*s^*, \tag{C.13b}$$

$$\varphi_1 = hh^*r, \qquad \varphi_2 = hh^*, \qquad \varphi_3 = hh^*(r + s + s^*), \tag{C.13c}$$

$$\phi_1 = hh^*(r + s + ss^*), \quad \phi_2 = hh^*, \qquad \phi_3 = hh^*(r + s^* + ss^*), \tag{C.13d}$$

$$b_0 = \frac{hr}{1+s^*}, \qquad b_1 = \frac{h(1+s^*)}{s^*}, \quad b_0 = \frac{h(r+s+s^*)}{1+s^*}, \tag{C.13e}$$

$$c_1 = \frac{h(r+s+ss^*)}{1+s^*}, \qquad c_2 = \frac{h(1+s^*)}{s^*}, \quad c_3 = \frac{h(r+s^*+ss^*)}{1+s^*}. \tag{C.13f}$$

To obtain $\{b^*_i\}_{i=0}^2$ and $\{c^*_i\}_{i=1}^3$, exchange $h \leftrightarrow h^*, s \leftrightarrow s^*$ in (C.13e) and (C.13f) and preserve r.