Introduction to Line Graphs

Emphasizing their construction, clique decompositions, and regularity
A given graph $X$  

Its line graph $L(X)$
This vertex has **degree 1**....

...so it gives **no** new adjacencies.
This vertex has degree 1…. 

…so it gives no new adjacencies.
This vertex also has degree 1…. …so it gives no new adjacencies.
This vertex has degree 0….  

…so it gives no new adjacencies.
The given graph \( X \)

Its line graph \( L(X) \)
The given graph $X$  

Its line graph $L(X)$
Definition of Line Graph

**Def.** Given a graph $X=(V,E)$, its **line graph** is the graph $L(X)$ with vertex set $E$ and where two distinct vertices $e$ and $e'$ are adjacent iff the corresponding edges in graph $X$ share an endpoint.
Vertices of degree $\geq 2$

Cliques partitioning edge set
Note:
Each vertex is in $\leq 2$ of these cliques.
**Thm.** (Krausz 1943) Let $X$ be a nonempty simple graph. Then

$X$ is the line graph of some graph

if and only if

the edges of $X$ can be partitioned into cliques so that each vertex belongs to $\leq 2$.

For an exposition of the proof, see West.
A vertex in only 1 clique is a pendant. (A deg 0 vertex is an isolated edge.)
P is Δ-free, so maximal cliques have size 2.

So to cover the edges, each vertex must be in 3 cliques.

So P cannot be a line graph.
$P$ is $\Delta$-free, so maximal cliques have size 2.

So to cover the edges, each vertex must be in 3 cliques.

So $P$ cannot be a line graph.

Same problem. So $P$ cannot be a line graph.
Isolated vertices must correspond to isolated edges.
Each clique in the partition corresponds to a vertex.
Two vertices are adjacent iff their cliques share a vertex.
Vertices belonging to only a single clique correspond to pendants.
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**Theorem.** (Beineke, 1968) A simple graph is a line graph if and only if it has none of the following as an induced subgraph:
Alternatively, we could state the result as follows:

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This narrows the search; and in fact, there now exists a **linear** time algorithm to test whether a graph is a line graph!

(It also produces such a graph when the answer is affirmative.)
Non-uniqueness of the inverse problem....

This graph is the line graph of each of the following:
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This graph is the line graph of each of the following:
BUT… the problem is not any worse than these examples illustrate.

Theorem. (Whitney, 1932) $K_3$ and $K_{1,3}$ are the ONLY pair of non-isomorphic connected graphs whose line graphs are isomorphic.

“Star-Triangle” Issue
(or “claw-triangle”)

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2. If $L(X)$ is regular, then the components of $X$ are regular or bipartite semi-regular.
Finally, some results about regularity and line graphs….

1. If $X$ is regular (valency $k$), then $L(X)$ is regular (valency $2k-2$).

2. If $L(X)$ is regular, then the components of $X$ are regular or bipartite semi-regular.

\[ \text{deg}(x) + \text{deg}(y) = k + 2 \]

same for any $y \sim x$. 
All must have same degree as $y$. 

regular $L(X)$ 

$k$
$\deg(z) = \deg(y)$

regular $L(X)$

$\deg(z) = \deg(y)$
$deg(a) = deg(x)$
If $X$ is connected:

- there can be at most 2 degrees in $X$.
- if $X$ has an odd cycle, $X$ is regular.
- else $X$ is bipartite and semi-regular.

... Therefore ...

regular $L(X)$
Two results about regularity and line graphs

1. If $X$ is regular (valency $k$), then $L(X)$ is regular (valency $2k-2$).

2. If $L(X)$ is regular, then the components of $X$ are regular or bipartite semi-regular.
Review of topics

- Definition of line graphs
- Cliques partitioning edges of $L(X)$ using vertices in $X$ of degree $\geq 2$
- Beineke’s 9 forbidden subgraphs
- Non-uniqueness of inverse problem
- Whitney’s star-triangle result
- Regularity in $X$ and $L(X)$
Who knows?

Open problem: Start with a tree $T$ and form the sequence $T, L(T), L(L(T)), \text{etc}$...

Does the sequence $|V(T)|, |V(L(T))|, |V(L(L(T)))|, \text{etc}$ uniquely determine the initial tree $T$, up to isomorphism?
Unlocking more of Ch.1 HW

After working through Section 1.7, the following exercises in Ch.1 are likely to be accessible:

# 9-15, 17, 21, 24

(Recall the plan is to collect 8 problems from each chapter. This chapter has 8 sections and 26 exercises in total.)
Further thoughts:

– Of the 18 (isomorphism classes of) graphs on < 5 vertices (assuming you found them) or the 52 graphs on < 6 vertices, which are line graphs?

– Which of those are expressible as $L(L(X))$ for some $X$? How about $L(L(L(X)))$?
Further thoughts about $X$ when $L(X)$ is regular:

- Find an example of a connected, regular $L(X)$ for which $X$ is not regular
- Find infinitely many connected, regular line graphs whose input graphs are not regular
- For each positive integer $k$, find a regular $L(X)$ for which $X$ has exactly $k$ distinct vertex degrees
Extension #3

Beineke’s forbidden subgraph result:

– Check that each of the 9 forbidden graphs are not line graphs, using the Krausz criterion

– To see why the word “induced” is critical to the Beineke result, find a line graph that contains ALL of the forbidden graphs as subgraphs!
The End