Problem. "The Perplexing Puzzle Of The Proud Partygoers"
(From the Riddler at FiveThirtyEight)

Its Friday and that means its party time! A group of N people are in attendance at your shindig, some of whom are friends with each other. (Lets assume friendship is symmetric if person A is friends with person B, then B is friends with A.) Suppose that everyone has at least one friend at the party, and that a person is proud if her number of friends is strictly larger than the average number of friends that her own friends have. (A competitive lot, your guests.)

Importantly, more than one person can be proud. How large can the share of proud people at the party be?

Solution. (J. Caughman)

Suppose a graph G has minimum degree $\delta \geq 1$. A vertex v is *proud* iff its degree is strictly larger than the average degree of its neighbors. The complete graph K_n on n vertices has no proud vertices, but any edge-deleted subgraph $K_n - e$ has exactly $n-2$ proud vertices.

Claim. For any graph on $n \geq 2$ vertices, there are at most $n-2$ proud vertices.

Proof. Suppose G has n vertices and the distinct vertex degrees occurring in G are $d_1 < d_2 < \cdots < d_k$, for some integer k $(0 < k < n)$. For each i $(1 \leq i \leq k)$, let $V_i = \{v \mid \deg(v) = d_i\}$. If $x \in V_1$, then $\deg(x) = \delta$, so x is not proud. Therefore G has ≥ 1 vertex that is not proud, and G has $\leq n-1$ proud vertices.

By way of contradiction, suppose G has exactly 1 proud vertex. Let $V_1 = \{x\}$ and let $W = V - V_1 - V_2$.

For all $y \in V_2$, we must have $x \sim y$, or else y would not be proud. So $d_1 \geq |V_2|$. On the other hand, $d_1 < n-1$, or else $G = K_n$ which has no proud vertices, so the set W is not empty. By construction, every vertex in W has degree at least $d_2 + 1$.

Now, fix any $y \in V_2$ and let n_y be the number of neighbors of y in V_2. Notice that $n_y \leq |V_2| - 1$, so $n_y \leq d_1 - 1$. Since y is proud,

$$d_2 \quad > \quad \frac{1}{d_2}\left[1 \cdot d_1 + \sum_{\substack{w \in V_2 \\ w \sim y}} d_2 + \sum_{\substack{w \in W \\ w \sim y}} \deg(w)\right].$$

Therefore,

$$\begin{aligned} (d_2)^2 \quad &> \quad d_1 + n_y d_2 + \sum_{\substack{w \in W \\ w \sim y}} \deg(w) \\ &\geq \quad d_1 + n_y d_2 + \sum_{\substack{w \in W \\ w \sim y}} (d_2 + 1) \\ &= \quad d_1 + n_y d_2 + (d_2 - n_y - 1)(d_2 + 1). \end{aligned}$$

It follows that $n_y > d_1 - 1$, a contradiction.

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