Counting and Coloring Sudoku Graphs

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Abstract

A sudoku puzzle is most commonly a 9×9 grid of 3×3 boxes wherein the puzzle player writes the numbers $1 - 9$ with no repetition in any row, column, or box. We generalize the notion of the $n^2 \times n^2$ sudoku grid for all $n \in \mathbb{Z}^{\geq 2}$ and codify the empty sudoku board as a graph. In the main section of this paper we prove that sudoku boards and sudoku graphs exist for all such n; we prove the equivalence of [3]'s construction using unions and products of graphs to the definition of the sudoku graph; we show that sudoku graphs are Cayley graphs for the direct product group $\mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n$; and we find the automorphism group of the sudoku graph. In the subsequent section, we find and prove several graph theoretic properties for this class of graphs, and we offer some conjectures on these and other properties.

Contents

1 Introduction

The mathematics of sudoku has been a subject of inquiry and interest to many combinatorialists (we recommend *Taking Sudoku Seriously* by Jason Rosenhouse and Laura Taalman for a wonderful primer). Many mathematicians have also applied the notions of graph theory to sudoku. In this paper we primarily aim to generalize and expand the knowledge of the class of sudoku graphs. For the purposes of this paper as a literature project, our primary reference is the article [3] by Cooper and Kirkpatrick.

We begin our main results with a seemingly obvious result, that sudoku grids (and graphs) of all appropriate sizes do exist. The rest of the main results are generally concerned with the structure and construction of sudoku graphs. Included here is a proof that these graphs are the union of graph products, a claim we credit to a draft version of a paper by Cooper and Kirkpatrick; the proof here is original, having been extrapolated from their claim. The wonderful symmetry and repetition found within sudoku graphs informs our remaining main results, the relationship between sudoku graphs and particular direct product groups, as well as the automorphism group of these graphs.

The remainder of this paper is an attempt to give an overview, from a graph theoretic perspective, of sudoku graphs—this includes some of their perhaps more mundane properties, as well as some perhaps more surprising (or at least interesting).

In the interest of brevity, we are compelled to assume basic knowledge of graph theory and of group theory. For references on graph theory definitions and concepts, we recommend West [12]; and for group theory, we recommend Fraleigh [5].

During this research, occasional automation was used to confirm or reject hypotheses. The code implemented can be found through [8].

2 Definitions

Definition 2.1. Sudoku board
For each $n \in \mathbb{Z}^{\geq 2}$, define a *sudoku board* Bn as a grid consisting of $n^2 \times n^2$ cells; the grid is subdivided into n^2 disjoint *boxes*, sub-grids of $n \times n$ cells. Observe that there is one such arrangement of boxes for any $n^2 \times n^2$ grid. Hence the 4×4 sudoku board is $B2$, and the standard 9×9 sudoku board is $B3$. Note that while $B1$ does satisfy the definition of a sudoku board, it is a trivial example and will not be considered in our study.

A *properly filled* sudoku board contains the numbers 1 through n^2 in each row, each column, and each box (i.e. a properly filled Bn is a fully solved sudoku puzzle).

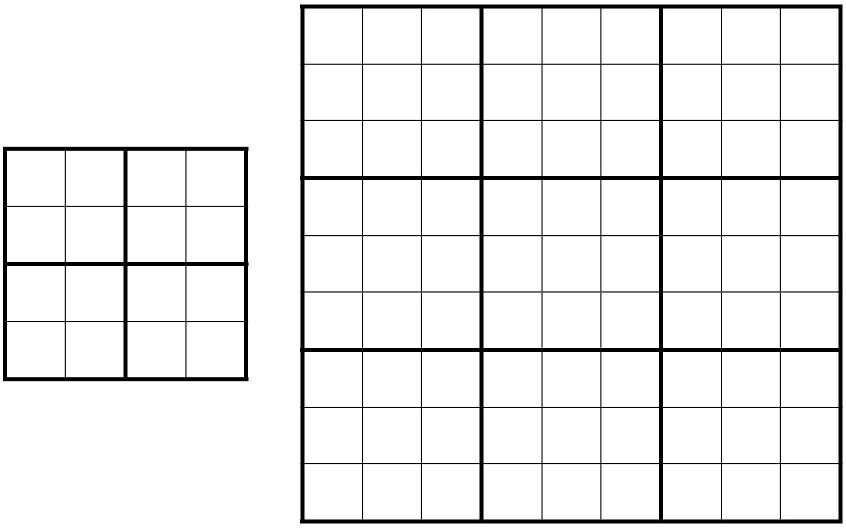

Figure 1: Left, $B2$; Right, $B3$

Definition 2.2. Band, Stack
A *band* of Bn is a maximal set of horizontally consecutive boxes. A *stack* of Bn is a maximal set of vertically consecutive boxes.

Definition 2.3. Sudoku graph
Define the simple graph Sn as the *sudoku graph* on n^4 vertices where each cell of the $n^2 \times n^2$ sudoku board is a vertex, and two vertices are adjacent iff their corresponding cells in the sudoku board Bn lie in the same row, column, or $n \times n$ box.

As in Definition 2.1, we forestall any consideration of $S1$. While many of our results do hold for the graph consisting of one vertex and no edges, the triviality of this graph obviates any desire to include it in our studies. Henceforth we consider only those n in $\mathbb{Z}^{\geq 2}$.

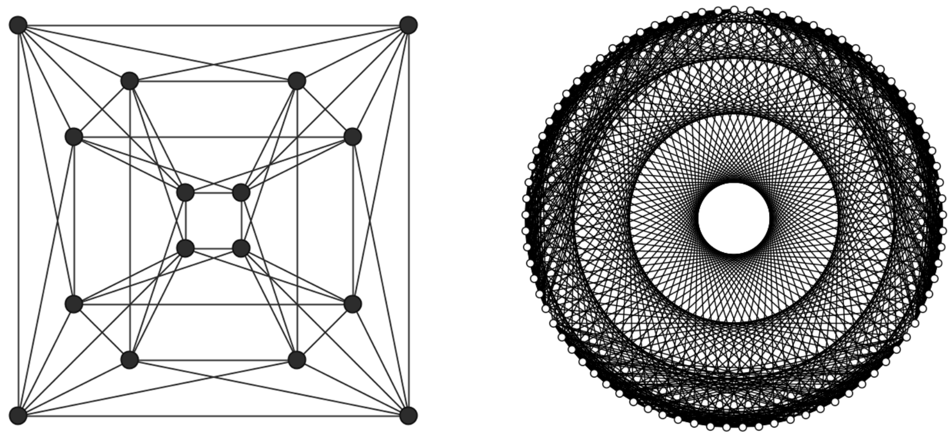

Figure 2: Embeddings of: Left, $S2$, created with GraphTea [10]; Right, $S3$, created with Sage [11]

Definition 2.4. Canonical Labeling
Let the top row of Bn be row 1, ..., the bottom row of Bn be row n^2, the leftmost column of Bn be column 1, ..., the rightmost column of Bn be column n^2. Then the *canonical labeling* of the cells of Bn and the corresponding vertices of Sn is that labeling by which the cell in the xth row, yth column of Bn has coordinate label (x, y), $x, y \in \{1, \ldots, n^2\}$.

$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$
$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$
$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$
$(4,1)$	$(4,2)$	$(4,3)$	$(4,4)$

$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$	$(1,7)$	$(1,8)$	$(1,9)$
$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$	$(2,7)$	$(2,8)$	$(2,9)$
$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$	$(3,5)$	$(3,6)$	$(3,7)$	$(3,8)$	$(3,9)$
$(4,1)$	$(4,2)$	$(4,3)$	$(4,4)$	$(4,5)$	$(4,6)$	$(4,7)$	$(4,8)$	$(4,9)$
$(5,1)$	$(5,2)$	$(5,3)$	$(5,4)$	$(5,5)$	$(5,6)$	$(5,7)$	$(5,8)$	$(5,9)$
$(6,1)$	$(6,2)$	$(6,3)$	$(6,4)$	$(6,5)$	$(6,6)$	$(6,7)$	$(6,8)$	$(6,9)$
$(7,1)$	$(7,2)$	$(7,3)$	$(7,4)$	$(7,5)$	$(7,6)$	$(7,7)$	$(7,8)$	$(7,9)$
$(8,1)$	$(8,2)$	$(8,3)$	$(8,4)$	$(8,5)$	$(8,6)$	$(8,7)$	$(8,8)$	$(8,9)$
$(9,1)$	$(9,2)$	$(9,3)$	$(9,4)$	$(9,5)$	$(9,6)$	$(9,7)$	$(9,8)$	$(9,9)$

Figure 3: Canonical labeling of: Left, $B2$; Right, $B3$

Definition 2.5. Box Identification

By the canonical labeling, the cells of any box of Bn have the following labels:

$$\begin{array}{c|cccc|c}
 & & & & & \\ \hline
 & (i,j) & (i,j+1) & \cdots & (i,j+n-1) & \\
 & (i+1,j) & & & & \\
 & \vdots & & \ddots & \vdots & \\
 & (i+n-1,j) & & \cdots & (i+n-1,j+n-1) & \\ \hline
 & & & & &
\end{array}$$

for some $i, j \in \{1, n+1, 2n+1, \ldots, (n-1)n+1\}$. That is, for any $(x_1, y_1), (x_2, y_2)$ in the same box, $x_1, x_2 \in \{i, \ldots, i+n-1\}$, $y_1, y_2 \in \{j, \ldots, j+n-1\}$. Note that $i = an + 1$ for some $a \in \{0, \ldots, n-1\}$. Then

$$\left\lceil \frac{i}{n} \right\rceil = \left\lceil \frac{an+1}{n} \right\rceil = \left\lceil a + \frac{1}{n} \right\rceil = a + 1 \text{ (recall that } n \in \mathbb{Z}^{\geq 2}\text{). And}$$

$$\left\lceil \frac{i+n-1}{n} \right\rceil = \left\lceil \frac{an+1+n-1}{n} \right\rceil = \left\lceil \frac{an+n}{n} \right\rceil = \lceil a+1 \rceil = a+1.$$

That is, for any $x_1, x_2 \in \{i, \ldots, i+n-1\}$, $\left\lceil \frac{x_1}{n} \right\rceil = \left\lceil \frac{x_2}{n} \right\rceil$.

Similarly, where $j = bn + 1$ for some $b \in \{0, \ldots, n-1\}$,

$$\left\lceil \frac{j}{n} \right\rceil = b + 1 = \left\lceil \frac{j+n-1}{n} \right\rceil,$$

and so for any $y_1, y_2 \in \{j, \ldots, j+n-1\}$, $\left\lceil \frac{y_1}{n} \right\rceil = \left\lceil \frac{y_2}{n} \right\rceil$.

Suppose (x_3, y_3) is not in the same box as (x_1, y_1)—then $x_3 \leq i-1$ or $x_3 \geq i+n$. In the case of the former,

$$\left\lceil \frac{x_3}{n} \right\rceil \leq \left\lceil \frac{i-1}{n} \right\rceil = \left\lceil \frac{an+1-1}{n} \right\rceil = \lceil a \rceil = a < \left\lceil \frac{x_1}{n} \right\rceil.$$

In the case of the latter,

$$\left\lceil \frac{x_3}{n} \right\rceil \geq \left\lceil \frac{i+n}{n} \right\rceil = \left\lceil \frac{an+1+n}{n} \right\rceil = \left\lceil a+1+\frac{1}{n} \right\rceil = a+2 > \left\lceil \frac{x_1}{n} \right\rceil.$$

In either case, if $(x_1, y_1), (x_3, y_3)$ are not in the same box, then $\left\lceil \frac{x_3}{n} \right\rceil \neq \left\lceil \frac{x_1}{n} \right\rceil$. By a similar argument, $\left\lceil \frac{y_3}{n} \right\rceil \neq \left\lceil \frac{y_1}{n} \right\rceil$.

This gives a labeling of the boxes analogous to the canonical labeling—the box containing cell (x, y) is *box* $\left(\left\lceil \frac{x}{n} \right\rceil, \left\lceil \frac{y}{n} \right\rceil\right)$.

Definition 2.6. Adjacencies of Sn

Cooper and Kirkpatrick [3] use these properties to generate a formal adjacency definition for Sn: Let Sn have the canonical labeling. Then distinct vertices $v_1 = (x_1, y_1), v_2 = (x_2, y_2) \in V(Sn)$ are adjacent iff:

1. $x_1 = x_2$,
2. $y_1 = y_2$, or
3. $\lceil \frac{x_1}{n} \rceil = \lceil \frac{x_2}{n} \rceil$ and $\lceil \frac{y_1}{n} \rceil = \lceil \frac{y_2}{n} \rceil$.

Unless otherwise stated, we will henceforth use this definition for the vertices and adjacencies of Sn. Note that this definition is equivalent to the definition given by [7], which uses the floor function and $V(Sn) = \{0, \ldots, n^2 - 1\} \times \{0, \ldots, n^2 - 1\}$.

Definition 2.7. Classification

A simple graph G is a sudoku graph Sn iff there exists a bijective relabeling φ,

$$\varphi : V(G) \to \{1, \ldots, n^2\} \times \{1, \ldots, n^2\},$$

such that for all $u, v \in V(G)$, edge $uv \in E(G)$ iff $\varphi(u), \varphi(v)$ satisfy Definition 2.6 (i.e., if G is isomorphic to some Sn).

We will often refer to neighboring vertices u, v as being in the same row, column, or box—this is meant to indicate that, for the sudoku board Bn corresponding to the sudoku graph Sn, the cells in Bn corresponding to $u, v \in V(Sn)$ are in the same row, column, or box. In this way we use *sudoku board* and *sudoku graph* somewhat interchangeably.

If we let vertices $(x_1, y_1), (x_2, y_2)$ of Sn respectively correspond to cells u, v of Bn, then we see that the conditions for adjacency given in Definition 2.6 respectively correspond to u, v being in the same row, column, or box of Bn.

3 Main Results

Theorem 3.1. The chromatic number of the sudoku graph is $\chi(Sn) = n^2$.

Proof. Each row, column, and box of Bn is an induced K_{n^2} in Sn, so $\chi(Sn) \geq n^2$. In order to prove that $\chi(Sn) \leq n^2$, we will show that there is a proper n^2-coloring of Sn by demonstrating that a properly filled Bn exists for all n.

Note: All counting of rows, columns, and boxes, will be from left to right or from top to bottom. For illustration, we will include figures of this construction for $B2$ and $B3$.

Into row 1, place the numbers 1 through n^2 in order. This sequence of numbers contains n disjoint subsequences of n distinct numbers. Call these sub-sequences $s_1, s_2, \ldots, s_n$.

1	2	3	4

1	2	3	4	5	6	7	8	9

Figure 4: Left, $B2$: $s_1 = 12$, $s_2 = 34$; Right, $B3$: $s_1 = 123$, $s_2 = 456$, $s_3 = 789$

For $j = 1, \ldots, n$, fill out row j with the sequence $s_j, s_{j+1}, \ldots, s_n, s_1, \ldots, s_{j-1}$, (where the subscripts are mod n). It is clear that the top n rows of Bn, being the top band, will each contain the numbers 1 through n^2. And since we completed the 2nd through nth rows of the boxes by shifting the initial permutation by one position for each subsequent row, no row will have any repeated numbers, and no box will have any repeated numbers.

1	2	3	4
3	4	1	2

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6

Figure 5: The first n rows filled out of: Left, $B2$; Right, $B3$

Now, each box in the top band contains the number 1 through n^2 arranged in a square, and each of the n boxes has a distinct such arrangement. From left to right, label the columns of the stacks by ${s^1}_1, {s^1}_2, \ldots, {s^1}_n; {s^2}_1, \ldots, {s^2}_n; {s^n}_1, \ldots, {s^n}_n$, where, for $i, j = 1, \ldots, n$, ${s^i}_j$ is the jth column of the ith stack, both counting from left to right.

1	2	3	4
3	4	1	2
$s^1{}_1 = 13$		$s^2{}_1 = 31$	
$s^1{}_2 = 24$		$s^2{}_2 = 42$	

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
$s^1{}_1 = 147$			$s^2{}_1 = 471$			$s^3{}_1 = 714$		
$s^1{}_2 = 258$			$s^2{}_2 = 582$			$s^3{}_2 = 825$		
$s^1{}_3 = 369$			$s^2{}_3 = 693$			$s^3{}_3 = 936$		

Figure 6: The $s^i{}_j$ as described for: Left, $B2$; Right, $B3$

We will complete each column by permuting the $s^i{}_j$: for the jth column of stack i, the completed top-to-bottom sequence shall be $s^i{}_j, s^i{}_{j+1}, \ldots, s^i{}_n, s^i{}_1, \ldots, s^i{}_{j-1}$ (subscripts again mod n).

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
2	3	1	5	6	4	8	9	7
5	6	4	8	9	7	2	3	1
8	9	7	2	3	1	5	6	4
a	b	c	d	e	f	g	h	i

$$\begin{aligned} a &= s^1{}_1, s^1{}_2 &= 147, 258 \\ &\vdots \\ e &= s^2{}_2, s^2{}_3 &= 582, 643 \\ &\vdots \\ i &= s^3{}_3, s^3{}_1 &= 936, 714 \end{aligned}$$

Figure 7: The partial column permutations for $B3$, given partial columns $a, \ldots, i$

Clearly this construction will fill out Bn using the numbers $1, \ldots, n^2$. We claim that the rules of sudoku are obeyed.

Each $s^i{}_j$ is a column of a square arrangement of the numbers $1, \ldots, n^2$—so for any given $i \in \{1, \ldots, n\}$, the collection $s^i{}_1, \ldots, s^i{}_n$ partitions $1, \ldots, n^2$. Since each column is built exactly out of one such collection, we see that no column can contain any number twice.

For any set of columns in the same box, each of those columns is a permutation of $s^i{}_1, \ldots, s^i{}_n$ for some $i \in \{1, \ldots, n\}$; moreover, from left to right, each of those columns after the first is formed by the permutation of the previous column shifted once in its order (but otherwise maintaining its order). The box contains n columns, each a permutation of n sub-sequences of length n, and so there are n shifts possible without any sub-sequence returning to its original position—that is, we have n shifts available to us before the same number will return to the same box (in a different column); as we only require n shifts, we see that no number can repeat within a box. Thus, should our completed Bn disobey the rules of sudoku, it must be that some row contains a number repeated–say, the ith row of the jth box, $i, j \in \{1, \ldots, n\}$, contains the number k, $k \in \{1, \ldots, n^2\}$, in columns m, ℓ.

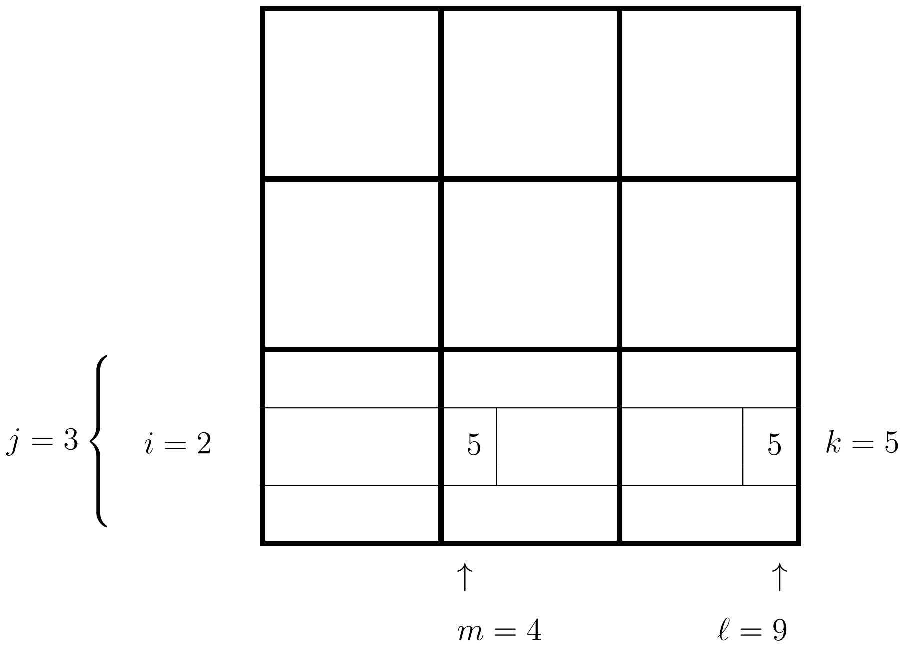

Figure 8: $B3$ with a repeated $k = 5$, in columns $m = 4, \ell = 9$ of the $i = 2$ row of box $j = 3$.

When we form each column by permuting the sub-columns of the top box of the column, we only permute these sub-columns horizontally—we do not permute any entries vertically. Then for k being in the ith row in columns m and ℓ, k was originally in the ith row of the topmost boxes in both columns m and ℓ, contradicting our completion of the top row of boxes.

We thus see that our construction indeed yields a properly filled sudoku board. Now consider the numbers $1, \ldots, n^2$ to be color classes—this particular completion of Bn is thus an n^2-coloring of Sn, and so $\chi(Sn) \leq n^2$. Hence equality is achieved, and $\chi(Sn) = n^2$. $\square$

Remark 3.2. This particular construction is formulated by the following: fill in the top row with any permutation, $f(1, j)$, of $1, \ldots, n^2$. Then for $i, j \in \{1, \ldots, n^2\}$, the remaining vertices are properly colored by

$$f(i,j) = \begin{cases} f(i-1, j+n) \mod n^2 & \text{if } 2 \leq i \leq n \\ f(i-n, j+1-n \cdot \delta_{n|j}) & \text{if } i > n, \end{cases}$$

where $\delta_{n|j} = 1$ if n divides j and 0 otherwise.

Remark 3.3. From Sage, the chromatic polynomial for $S2$ (denoted $\chi(S2, x)$) is found to be

$$\begin{aligned} \chi(S2, x) = x^{16} &- 56x^{15} + 1492x^{14} - 25072x^{13} + 296918x^{12} - 2621552x^{11} + 17795572x^{10} \\ &- 94352168x^9 + 392779169x^8 - 1279118840x^7 + 3217758336x^6 - 6107865464x^5 \\ &+ 8413745644x^4 - 7877463064x^3 + 4436831332x^2 - 1117762248x \end{aligned}$$

Thus far, all of our computational attempts to find $\chi(S3, x)$ have been unsuccessful.

Remark 3.4. For graphs G, H, any graph product of G and H has as its vertex set $V(G) \times V(H)$.

Definition 3.5. In the Cartesian graph product $G \square H$, $(u_G, u_H) \sim (v_G, v_H)$ iff

1. $u_G = v_G$ and $u_H \sim v_H$, or
2. $u_G \sim v_G$ and $u_H = v_H$.

Definition 3.6. In the Strong product $G \boxtimes H$, $(u_G, u_H) \sim (v_G, v_H)$ iff

1. $u_G = v_G$ and $u_H \sim v_H$, or
2. $u_G \sim v_G$ and $u_H = v_H$, or
3. $u_G \sim v_G$ and $u_H \sim v_H$.

Remark 3.7. $|E(G \square H)| = |V(G)| \cdot |E(H)| + |V(H)| \cdot |E(G)|$
Suppose $u_G v_G \in E(G)$. Then $(u_G, x_H)(v_G, x_H) \in E(G \square H)$ for each $x \in V(H)$: for each edge in $E(G)$, $G \square H$ has $|V(H)|$ adjacencies. Similarly, for each $u_H v_H \in E(H), (x_G, u_H)(x_G, v_H) \in E(G \square H)$ for each $x \in V(G)$—for each edge in $E(H)$, $G \square H$ has $|V(G)|$ adjacencies. We further see from this that $\deg_{G \square H}(u_G, v_H) = \deg_G(u) + \deg_H(v)$.

Theorem 3.8. Cooper and Kirkpatrick [3] observe that Sn is a union of graph products:

$$Sn = (K_{n^2} \square K_{n^2}) \cup (nK_n \boxtimes nK_n).$$

Proof. We take the same vertex set for these products, $V(K_{n^2} \square K_{n^2}) = V(nK_n \boxtimes nK_n)$, and define their union to be the union of their edge sets. Note that nK_n is the disjoint union of n copies of K_n, that $G \square H$ is the Cartesian product of graphs G, H, and that $G \boxtimes H$ is the Strong (or Normal) product of graphs G, H.

Let G, H be disjoint copies of K_{n^2} with $V(G) = \{u_1, ..., u_{n^2}\}$ and $V(H) = \{v_1, ..., v_{n^2}\}$. Then $|V(G \square H)| = n^4 = |V(Sn)|$. Fix $i, j \in \{1, ..., n^2\}$ and consider $(u_i, v_j) \in V(G \square H)$. From Remark 3.7, $\deg_{G \square H}(u_i, v_j) = \deg_G(u_i) + \deg_H(v) = n^2 - 1 + n^2 - 1$ (since G, H are each K_{n^2}). From clause (1) of the definition of $G \square H$, (u_i, v_j) has neighbors (u_i, v_x), where $x \in \{1, ..., n^2\}, x \neq j$; and from clause (2), (u_i, v_j) has neighbors (u_y, v_j), where $y \in \{1, ..., n^2\}, y \neq i$. In relation to Bn, we consider row i to consist of vertices $(u_i, v_1), (u_i, v_2), ..., (u_i, v_{n^2})$; and column j to consist of vertices $(u_1, v_j), (u_2, v_j), ..., (u_{n^2}, v_j)$. In this respect, we see that $K_{n^2} \square K_{n^2}$ gives the necessary relations between the rows and columns of Bn and Sn. However, this product neglects those cells in the $n \times n$ boxes that are not in the same row/column.

Now let S, T be disjoint copies of nK_n with $V(S) = \{u_1, ..., u_{n^2}\}$ and $V(T) = \{v_1, ..., v_{n^2}\}$ such that the distinct K_n's of nK_n are on successively numbered groups of n vertices and observe the following:

1. $V(S) = V(G)$ and $V(T) = V(H)$, and so $V(nK_n \boxtimes nK_n) = V(K_{n^2} \,\square\, K_{n^2})$;

2. $G \cong H$, $S \cong T$, and $S \subseteq G$ (and so $T \subseteq H$);

3. The adjacencies of both the Cartesian product and the Strong product are based on adjacencies of their factors (as opposed to being based on non-adjacencies as, for example, the Modular product partly is); and

4. Clauses (1) and (2) of the definition of the Strong product are identical to the definition of the Cartesian product.

Together, these observations tell us that clauses (1) and (2) of the Strong product will not result in any adjacencies in $S \boxtimes T$ that are not already present in $G \,\square\, H$. We may then focus only on clause (3) of the Strong product: For $(u_i, v_j), (u_k, v_\ell) \in V(S \boxtimes T)$, $(u_i, v_j)(u_k, v_\ell) \in E(S \boxtimes T)$ iff $u_i u_k \in E(S)$ and $v_j v_\ell \in E(T)$.

Let $u_a, u_{a+1}, ..., u_{a+n-1}$ be an arbitrarily chosen group of n consecutively labeled vertices in $V(S)$ and let $v_m, v_{m+1}, ..., v_{m+n-1}$ be an arbitrarily chosen group of n consecutively labeled vertices in $V(T)$ such that the subgraph induced on each is a copy of K_n. Let $i, j \in \{a, ..., a+n-1\}, i \neq j$, and $k, \ell \in \{m, ..., m+n-1\}, k \neq \ell$. Then $u_i u_j \in E(S)$ and $v_k v_\ell \in E(T)$. By clause (3) of the Strong product, $(u_i, v_k)(u_j, v_\ell) \in E(S \boxtimes T)$. For each u_i, there are $n-1$ such u_j, and for each v_k, there are $n-1$ such v_ℓ; so each such (u_i, v_k) has, by clause (3), $(n-1) \cdot (n-1)$ adjacencies in $S \boxtimes T$. Further, since $i \neq j$ and $k \neq \ell$, none of these adjacencies are present in $G \,\square\, H$ (by Definition 3.5). For any $u_x \in V(S)$ such that $x \notin \{a, ..., a+n-1\}$, by our labeling of $V(S)$, for any $i \in \{a, ..., a+n-1\}$ we have that $u_i u_x \notin E(S)$, and so (by Definition 3.6) $(u_i, v_k)(u_x, v_\ell) \notin E(S \boxtimes T)$ for any $v_k, v_\ell \in V(T)$.

Then each vertex $(u_i, v_k) \in V(S \boxtimes T)$ has, by clause (3) alone, degree $(n-1)^2$. Since, by the above remarks, any adjacencies in $S \boxtimes T$ given by clauses (1) and (2) of the Strong product are already present in $G \,\square\, H$, and since $\deg_{G \square H}(u_i, v_k) = 2(n^2 - 1)$ for each $(u_i, v_k) \in V(G \,\square\, H)$, we have that for any $(u_i, v_k) \in V((G \square H) \cup (S \boxtimes T))$, $\deg(u_i, v_k) = 2(n^2-1) + (n-1)^2 = 3n^2 - 2n - 1 = \deg_{Sn}(x)$ for all $x \in V(Sn)$.

We continue our convention of corresponding row i of Bn to $\{(u_i, v_j) : j = 1, ..., n^2\}$ and column j to $\{(u_i, v_j) : i = 1, ..., n^2\}$. By our labeling of $V(S)$ and $V(T)$, each copy of K_n in S (and thus each consecutive group of n vertices of S) corresponds to the rows of an $n \times n$ box of Bn; likewise, each copy of K_n in T (and thus each consecutive group of n vertices of T) corresponds to the columns of an $n \times n$ box of Bn—then each set of n^2 vertices

$$(u_a, v_m),(u_{a+1}, v_m), ..., (u_{a+n-1}, v_m); (u_a, v_{m+1}), (u_{a+1}, v_{m+1}),$$
$$..., (u_{a+n-1}, v_{m+1}); ...; (u_{a+n-1}, v_m), ..., (u_{a+n-1}, v_{m+n-1})$$

such that $a, m \in \{1, n+1, 2n+1, ..., n^2-n+1\}$ corresponds to an $n \times n$ box of Bn. As discussed above, any such set of vertices receives from clause (3) of Strong product all possible adjacencies $(u_i, v_k)(u_j, v_\ell)$ where $i \neq j, k \neq \ell, i, j \in \{a, ..., a+n-1\}, k, \ell \in \{m, ..., m+n-1\}$.

Hence $S \boxtimes T$ provides the adjacencies of Bn (and Sn) for all cells in the same $n \times n$ box but not in the same row/column. Together with the row/column adjacencies of $G \square H$, we therefore have that $Sn = (K_{n^2} \square K_{n^2}) \cup (nK_n \boxtimes nK_n)$. $\square$

Theorem 3.9. *Sn is a Cayley graph for $\mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n$.*

Proof. We will use $(\mathbb{Z}_n)^4$ to denote $\mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n$. We will first relabel the vertices of Sn from the canonical labeling to the elements of $(\mathbb{Z}_n)^4$, where the set of elements of $(\mathbb{Z}_n)^4$ is $\{(a, b, c, d) : a, b, c, d = 0, \ldots, n-1\}$. Recall that under the canonical labeling, $V(Sn) = \{(x, y) : x, y = 1, \ldots, n^2\}$ and define $\varphi : V(Sn) \to (\mathbb{Z}_n)^4$ by $\varphi(x, y) = (a, b, c, d)$ where $\overline{ab}$ is the two-digit base-n representation of $x - 1$ and $\overline{cd}$ is the two-digit base-n representation of $y - 1$. Clearly φ is everywhere defined and well defined. Suppose $(x_1, y_1), (x_2, y_2) \in V(Sn)$ such that $\varphi(x_1, y_1) = \varphi(x_2, y_2)$. Then $(a_1, b_1, c_1, d_1) = (a_2, b_2, c_2, d_2)$ coordinate-wise; so $\overline{a_1b_1} = \overline{a_2b_2}$ and $\overline{c_1d_1} = \overline{c_2d_2}$ base n; then $x_1 - 1 = x_2 - 1$ and $y_1 - 1 = y_2 - 1$ base n. Ergo $x_1 = x_2$ and $y_1 = y_2$, so $(x_1, y_1) = (x_2, y_2)$, and we have that φ is 1–1. Further, $|V(Sn)| = n^4 = |(\mathbb{Z}_n)^4|$—then since the size of the domain equals the size of the codomain and φ is everywhere defined, well defined, and 1–1, then φ is also onto, and consequently bijective. We claim that that φ is a homomorphism by the following: for $(x_1, y_1), (x_2, y_2) \in V(Sn)$ where $\varphi(x_1, y_1) = (a, b, c, d)$ and $\varphi(x_2, y_2) = (e, f, g, h)$, $(x_1, y_1) \sim (x_2, y_2)$ iff

1. $a = e$ and $b = f$,

2. $c = g$ and $d = h$, or

3. $a = e$ and $c = g$.

The definition of φ clearly indicates that $x_1 = x_2$ iff $\overline{ab} = \overline{ef}$ iff $a = e$ and $b = f$; and $y_1 = y_2$ iff $\overline{cd} = \overline{gh}$ iff $c = g$ and $d = h$.

Now suppose that $a = e$ and $c = g$. We will show that this is equivalent to

$$\left\lceil \frac{x_1}{n} \right\rceil = \left\lceil \frac{x_2}{n} \right\rceil \text{ and } \left\lceil \frac{y_1}{n} \right\rceil = \left\lceil \frac{y_2}{n} \right\rceil.$$

Let $a = e = p$ and $c = g = q$ for some $p, q \in \{0, \ldots, n-1\}$. Then $(a, b, c, d) = (p, b, q, d)$ and $(e, f, g, h) = (p, f, q, h)$. By reversing the base conversion and accounting for φ's subtraction of 1 (that is, by applying φ^{-1}), $(x_1, y_1) = (n \cdot p + b + 1, n \cdot q + d + 1)$ and $(x_2, y_2) = (n \cdot p + f + 1, d \cdot q + h + 1)$.

Observe the following:

$$\left\lceil \frac{x_1}{n} \right\rceil = \left\lceil \frac{np+b+1}{n} \right\rceil = p + \left\lceil \frac{b+1}{n} \right\rceil = p+1;$$

$$\left\lceil \frac{x_2}{n} \right\rceil = \left\lceil \frac{np+f+1}{n} \right\rceil = p + \left\lceil \frac{f+1}{n} \right\rceil = p+1;$$

$$\left\lceil \frac{y_1}{n} \right\rceil = \left\lceil \frac{nq+d+1}{n} \right\rceil = q + \left\lceil \frac{d+1}{n} \right\rceil = q+1; \text{ and}$$

$$\left\lceil \frac{y_2}{n} \right\rceil = \left\lceil \frac{nq+h+1}{n} \right\rceil = q + \left\lceil \frac{h+1}{n} \right\rceil = q+1$$

(since $p, q \in \mathbb{Z}$ and $b, d, f, h \in \{0, \ldots, n-1\}$). Hence the adjacencies rules of Definition 2.6 are preserved by φ, and so $\varphi : V(Sn) \to (\mathbb{Z}_n)^4$ is a graph isomorphism.

(0,0,0,0)	(0,0,0,1)	(0,0,0,2)	(0,0,1,0)	(0,0,1,1)	(0,0,1,2)	(0,0,2,0)	(0,0,2,1)	(0,0,2,2)
(0,1,0,0)	(0,1,0,1)	(0,1,0,2)	(0,1,1,0)	(0,1,1,1)	(0,1,1,2)	(0,1,2,0)	(0,1,2,1)	(0,1,2,2)
(0,2,0,0)	(0,2,0,1)	(0,2,0,2)	(0,2,1,0)	(0,2,1,1)	(0,2,1,2)	(0,2,2,0)	(0,2,2,1)	(0,2,2,2)
(1,0,0,0)	(1,0,0,1)	(1,0,0,2)	(1,0,1,0)	(1,0,1,1)	(1,0,1,2)	(1,0,2,0)	(1,0,2,1)	(1,0,2,2)
(1,1,0,0)	(1,1,0,1)	(1,1,0,2)	(1,1,1,0)	(1,1,1,1)	(1,1,1,2)	(1,1,2,0)	(1,1,2,1)	(1,1,2,2)
(1,2,0,0)	(1,2,0,1)	(1,2,0,2)	(1,2,1,0)	(1,2,1,1)	(1,2,1,2)	(1,2,2,0)	(1,2,2,1)	(1,2,2,2)
(2,0,0,0)	(2,0,0,1)	(2,0,0,2)	(2,0,1,0)	(2,0,1,1)	(2,0,1,2)	(2,0,2,0)	(2,0,2,1)	(2,0,2,2)
(2,1,0,0)	(2,1,0,1)	(2,1,0,2)	(2,1,1,0)	(2,1,1,1)	(2,1,1,2)	(2,1,2,0)	(2,1,2,1)	(2,1,2,2)
(2,2,0,0)	(2,2,0,1)	(2,2,0,2)	(2,2,1,0)	(2,2,1,1)	(2,2,1,2)	(2,2,2,0)	(2,2,2,1)	(2,2,2,2)

Figure 9: Canonical labeling of $B3$ under φ

We will now show that by judicious selection of the graph generating set, Sn fits the criteria for a Cayley graph. Recall that in a Cayley graph C (see [6]), the vertices are labeled by the group elements, and for $x, y \in V(C)$, directed edge (x, y) is in $E(C)$ iff $xy^{-1} \in T$, where T is the *generating set* of C (so T is a subset of the group elements). Observe that $xy^{-1} \in T$ iff $xy^{-1} = t$ for some $t \in T$, iff $x = ty$ for some $t \in T$. That is, to find the "from" neighbor set of any vertex

$y \in V(C)$, we fix y and multiply on the left with each member of T (in the case of $(\mathbb{Z}_n)^4$, the group is abelian and the operation is component-wise addition mod n).

In order for a Cayley graph Cn to be isomorphic to Sudoku graph Sn, Cn must be a simple, undirected graph: i.e., no directed edges, no loops (edges whose endpoints are not distinct), and no multiple edges (distinct edges with the same endpoints).

Observe that for any t and any y in a group, ty has a unique result, so no edge (x, y) is generated twice, and Cn has no multiple edges.

Remark 3.10. *Cn has loops iff the group identity element is in the generating set.*

Proof. Cn has a loop iff there is some $x \in V(Cn)$ such that $(x, x) \in E(Cn)$; by the criteria for adjacency in a Cayley graph, $(x, x) \in E(Cn)$ iff $xx^{-1} = e \in T$. □

Remark 3.11. *A Cayley graph C is undirected iff the generating set is closed under inverses (i.e. for any $t \in T$, $t^{-1} \in T$).*

Proof. A Cayley graph is considered to be undirected iff it is actually completely *bi*-directed—that is, C is "undirected" iff for every directed edge $(u, v) \in E(C)$, directed edge (v, u) is also in $E(C)$. In this case, the two directed edges (u, v) and (v, u) are collectively considered to be the single undirected edge uv.

Suppose that C is undirected. Then for $xy^{-1} \in T$, we also have $yx^{-1} \in T$. Let $xy^{-1} = t_1$ and $yx^{-1} = t_2$ for some $t_1, t_2 \in T$. Then $x = t_1 y$ and $y = t_2 x$. By substitution, $y = t_2 t_1 y$, and so $t_2 t_1 = e$. Thus $t_2 = {t_1}^{-1} \in T$ and $t_1 = {t_2}^{-1} \in T$. Hence if C is undirected, then T is closed under inverses.

Now suppose T is closed under inverses and let $(x, y) \in E(C)$. Then $xy^{-1} = t$ for some $t \in T$, iff $(xy^{-1})^{-1} = t^{-1}$, iff $yx^{-1} = t^{-1}$. Since T is closed under inverses, $t^{-1} \in T$, and so $yx^{-1} = t^{-1}$ iff $yx^{-1} \in T$, iff $(y, x) \in E(C)$ (by the adjacency criteria for a Cayley graph). Hence if T is closed under inverses, then C is undirected. □

Now let T be the image under φ (as defined above) of the neighbor set of $(1, 1)$. That is, for $(x, y) \in V(Sn)$, $\varphi(x, y)$ is in T iff $(1, 1)(x, y)$ is an edge in Sn. Note that $\varphi(1, 1) = (0, 0, 0, 0)$, the identity element of $(\mathbb{Z}_n)^4$. Since $(1, 1)$ is not adjacent to itself, $\varphi(1, 1)$ is not in T; ergo, by Remark 3.10, Cn has no loops. Additionally, by the above, Cn has no multiple edges. We will now show that T is closed under inverses.

Observe that the neighbor set in Sn of $(1, 1)$ is precisely the first row, first column, and top-left box of Bn, each without $(1, 1)$.

The first row of Bn less $(1, 1)$ is $\{(1, 2), \ldots, (1, n^2)\}$; under φ, the first row of Cn less $(0, 0, 0, 0)$ is

$$\{(0, 0, 0, 1), \ldots, (0, 0, n-1, n-1)\} = \{(0, 0, i, j) : i = 0, \ldots, n-1; j = 1, \ldots, n-1\}.$$

We denote this set by R'.

The first column of Bn less $(1,1)$ is $\{(2,1),\ldots,(n^2,1)\}$; under φ, the first column of Cn less $(0,0,0,0)$ is

$$\{(0,1,0,0),\ldots,(n-1,n-1,0,0)\}=\{(i,j,0,0): i=0,\ldots,n-1; j=1,\ldots,n-1\}.$$

We denote this set by C'.

The top-left box of Bn excepting those cells in the first row or first column is, row by row, $\{(2,2),\ldots,(2,n),(3,2),\ldots,(3,n),\ldots,(n,2),\ldots,(n,n)\}$. Under φ, the top-left box of Cn less R' and C' is

$$\begin{aligned}&\{(0,1,0,1),\ldots,(0,1,0,n-1);\ldots;(0,n-1,0,1),\ldots,(0,n-1,0,n-1)\}\\&=\{(0,i,0,j): i,j=1,\ldots,n-1\}.\end{aligned}$$

We denote this set by B'; and so $T=R'\cup C'\cup B'$.

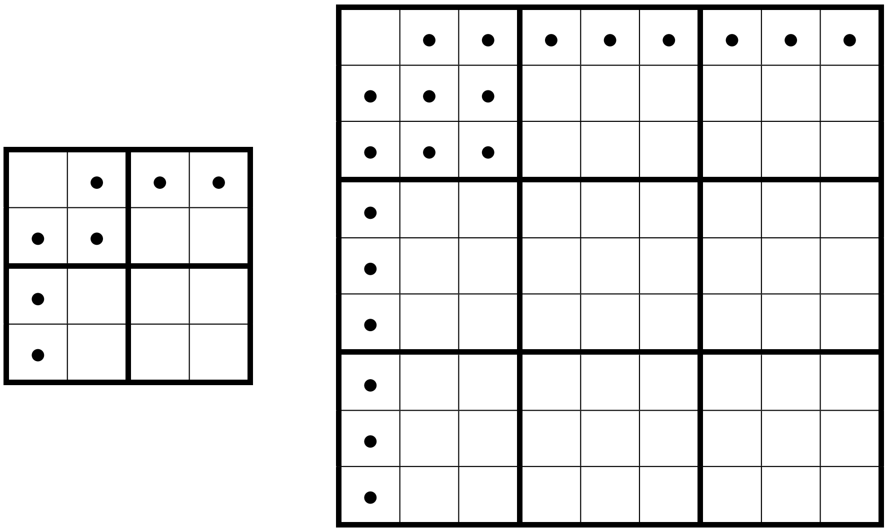

Figure 10: The described generating set T for: Left, $S2$; Right, $S3$

Note that for any $(a,b,c,d)\in(\mathbb{Z}_n)^4$, by the group operation we have $(a,b,c,d)^{-1}=(p,q,r,s)$ where

$$a+p=b+q=c+r=d+s=0 \mod n$$

for $p,q,r,s\in\{0,\ldots,n-1\}$.

Let $(0,0,x,y)\in R'$. Then $1\le y\le n-1$, so $1\le n-y\le n-1$. Suppose $x=0$. Then $(0,0,0,n-y)=(0,0,0,y)^{-1}\in R'$. Now suppose $x\ge 1$; then $1\le n-x\le n-1$, and so $(0,0,n-x,n-y)=(0,0,x,y)^{-1}\in R'$.

Let $(x,y,0,0)\in C'$. As in our argument for R', we have that $1\le n-y\le n-1$; then if $x=0$, then $(x,y,0,0)^{-1}=(0,n-y,0,0)\in C'$; and if $x\ge 1$, then $(x,y,0,0)^{-1}=(n-x,n-y,0,0)\in C'$.

Now let $(0,x,0,y) \in B'$. Then we have that $1 \leq n-x \leq n-1$ and $1 \leq n-y \leq n-1$, and so $(0,x,0,y)^{-1} = (0,n-x,0,n-y) \in B' \subset T$.

Since R', C', and B' are all closed under inverses and $T = R' \cup C' \cup B'$, we see that T is closed under inverses. We will now show that for any $(a,b,c,d),(e,f,g,h) \in (\mathbb{Z}_n)^4$, edge $(a,b,c,d)(e,f,g,h)$ is in $E(Cn)$ (by the adjacency definition for Sn or its equivalence under φ) iff $(a,b,c,d)(e,f,g,h)^{-1}$ is in T.

Let $(a,b,c,d),(e,f,g,h) \in V(Cn)$ such that $(a,b,c,d)(e,f,g,h) \in E(Cn)$. Note that for any $(e,f,g,h) \in (\mathbb{Z}_n)^4$, $(e,f,g,h)^{-1}$ uniquely exists.

Case 1. $a = e$ and $b = f$.
Pick $t = (0,0,x,y) \in R'$ such that $x = c - g \mod n$ and $y = d - h \mod n$. By modular arithmetic and $c,d,g,h \in \{0,\ldots,n-1\}$, we are guaranteed that such x,y exist; and by the definition of R', that such a t exists in R'. Then

$$\begin{aligned}(0,0,x,y)(e,f,g,h) &= (0,0,c-g,d-h)(e,f,g,h) \mod n\\ &= (e,f,c,d)\\ &= (a,b,c,d) \text{ (by assumption).}\end{aligned}$$

By the group's operation, there exists a $t \in T$ such that

$$(a,b,c,d) = t(e,f,g,h) \text{ iff } (a,b,c,d)(e,f,g,h)^{-1} = t.$$

So if $a = e$ and $b = f$, then $(a,b,c,d)(e,f,g,h)^{-1} \in T$.

Case 2. $c = g$ and $d = h$.
Pick $t = (x,y,0,0) \in C'$ such that $x = a - e \mod n$ and $y = b - f \mod n$. As in Case 1, we are guaranteed the existence of such a $t \in C'$. Then

$$\begin{aligned}(x,y,0,0)(e,f,g,h) &= (a-e,b-f,0,0)(e,f,g,h) \mod n\\ &= (a,b,g,h)\\ &= (a,b,c,d) \text{ (by assumption).}\end{aligned}$$

A $t \in T$ exists such that $(a,b,c,d) = t(e,f,g,h)$ iff $(a,b,c,d)(e,f,g,h)^{-1} = t$. So if $c = g$ and $d = h$, then $(a,b,c,d)(e,f,g,h)^{-1} \in T$.

Case 3. $a = e$ and $c = g$.
Pick $t = (0,x,0,y) \in B'$ such that $x = b - f \mod n$ and $y = d - h \mod n$. Again, existence of this t is guaranteed in B'. Then

$$\begin{aligned}(0,x,0,y)(e,f,g,h) &= (0,b-f,0,d-h)(e,f,g,h) \mod n\\ &= (e,b,g,d)\\ &= (a,b,c,d) \text{ (by assumption).}\end{aligned}$$

A $t \in T$ exists such that $(a,b,c,d) = t(e,f,g,h)$ iff $(a,b,c,d)(e,f,g,h)^{-1} = t$. So if $a = e$ and $c = g$, then $(a,b,c,d)(e,f,g,h)^{-1} \in T$.

Then for any edge $(a,b,c,d)(e,f,g,h) \in E(Cn)$, the criteria for adjacency in a Cayley graph is upheld.

Now let $(a,b,c,d),(e,f,g,h) \in (\mathbb{Z}_n)^4$ and suppose that $(a,b,c,d)(e,f,g,h)^{-1} \in T$. Recall that $T = R' \cup C' \cup B'$, and that by their definitions, R', C', and B' are pairwise disjoint. Then if $(a,b,c,d)(e,f,g,h)^{-1}$ is in T, $(a,b,c,d)(e,f,g,h)^{-1}$ is in exactly one of R', C', or B'. By the group operation, $(a,b,c,d)(e,f,g,h)^{-1} \in T$ iff there exists a $t \in T$ such that $(a,b,c,d) = t(e,f,g,h)$.

Case 1. $(a,b,c,d)(e,f,g,h)^{-1} \in R'$.
Then there exists $(0,0,x,y) \in R'$ such that

$$\begin{aligned}(a,b,c,d) &= (0,0,x,y)(e,f,g,h)\\ &= (e,f,x+g,y+h).\end{aligned}$$

Then $a = e$ and $b = f$.

Case 2. $(a,b,c,d)(e,f,g,h)^{-1} \in C'$.
Then there exists $(x,y,0,0) \in C'$ such that

$$\begin{aligned}(a,b,c,d) &= (x,y,0,0)(e,f,g,h)\\ &= (x+e,y+f,g,h).\end{aligned}$$

Then $c = g$ and $d = h$.

Case 3. $(a,b,c,d)(e,f,g,h)^{-1} \in B'$.
Then there exists $(0,x,0,y) \in B'$ such that

$$\begin{aligned}(a,b,c,d) &= (0,x,0,y)(e,f,g,h)\\ &= (e,x+f,g,y+h).\end{aligned}$$

Then $a = e$ and $c = g$.

Then for any $(a,b,c,d),(e,f,g,h) \in (\mathbb{Z}_n)^4$ such that $(a,b,c,d)(e,f,g,h)^{-1} \in T$, the definition of adjacency for Sn (rather, the equivalence of the definition under φ) is upheld.

We have now shown there is an bijection between the vertex sets of Sn and a Cayley graph for the group $\mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n$; that under this isomorphism, the rules for adjacency in the Sudoku graph have a homomorphic equivalent; and that two vertices are adjacent in Sn if and only if their images under the bijection respect the criteria for adjacency in a Cayley graph. We thus conclude that Sn is isomorphic to a Cayley graph for the direct product group $(\mathbb{Z}_n)^4$. $\square$

Corollary 3.12. *Sn is vertex transitive.*

By [6], all Cayley graphs are vertex transitive.

Definition 3.13. Symmetry group of Bn

The *symmetry group* of Bn is the set of all transformations of Bn (with the operation of composition) under which any proper sudoku board is mapped to a proper sudoku board. We will refer to the symmetry group of Bn as $\mathrm{Sym}(Bn)$.

Definition 3.14. Automorphism group of Sn

The *automorphism group* of Sn is the set of all graph isomorphisms (with the operation of composition) $\varphi : V(Sn) \to V(Sn)$ under which, for $x, y \in V(Sn)$, $xy \in E(Sn)$ iff $\varphi(x)\varphi(y) \in E(Sn')$, where Sn' is the image of $V(Sn)$ under φ. We will refer to the automorphism group of Sn as $\mathrm{Aut}(Sn)$.

Lemma 3.15. The symmetry group of Bn is precisely the automorphism group of Sn.

Proof. Define $\theta : \{1, \ldots, n^2\} \times \{1, \ldots, n^2\} \to V(Sn)$, where θ is the correspondence between cells of Bn and vertices of Sn, as described in Definition 2.3. By that definition, θ is a bijection—so it here suffices to show that θ composed with any symmetry of Bn is a homomorphism (the other direction, θ^{-1} composed with any automorphism of $\mathrm{Aut}(Sn)$, will immediately follow).
Let $(x_1, y_1), (x_2, y_2) \in Bn$ such that (x_1, y_1) and (x_2, y_2) share a row, column, or box in Bn (see Definition 2.6). By θ, there exist $u, v \in V(Sn)$ such that $\theta(x_1, y_1) = u$, $\theta(x_2, y_2) = v$, and $uv \in E(Sn)$. Let $\sigma \in \mathrm{Sym}(Bn)$, and let $\sigma(x_1, y_1) = ({x_1}', {y_1}'), \sigma(x_2, y_2) = ({x_2}', {y_2}')$. Then $({x_1}', {y_1}')$ and $({x_2}', {y_2}')$ share a row, column, or box by Definition 2.3, there exist $u', v' \in V(Sn)$ with $u'v' \in E(Sn)$ where $u' = \theta({x_1}', {y_1}') = \theta \circ \sigma(x_1, y_1)$ and $v' = \theta({x_2}', {y_2}') = \theta \circ \sigma(x_2, y_2)$.
Hence $\theta \circ \sigma$ preserves the defined relation between cells of Bn which share a row, column, or box with adjacent vertices of Bn; and since θ, σ are both bijections, the result is achieved. $\square$

Theorem 3.16. *The automorphism group of Sn is found as*

$$\mathrm{Aut}(Sn) \cong [((\underbrace{S_n \times \cdots \times S_n}_{n \text{ times}}) \rtimes S_n) \times ((\underbrace{S_n \times \cdots \times S_n}_{n \text{ times}}) \rtimes S_n)] \rtimes \mathbb{Z}_2$$

Proof. Note that we use $G \times H$ to refer to the direct product of groups G, H and $G \rtimes H$ to refer to a semidirect product of G and H (where G is normal in the product). For more on group structure, see Dummit and Foote [4].

Since $\mathrm{Aut}(Sn) \cong \mathrm{Sym}(Bn)$, we will instead prove that the above group is the symmetry group for the sudoku board Bn. Note that we make no claim as to the particular semi-direct products; merely that the subgroups in question do interact as semi-direct products. (However, the two

$(\underbrace{S_n \times \cdots \times S_n}_{n \text{ times}}) \rtimes S_n$ subgroups are isomorphic to each other.) We will first show how to generate each of the subgroups, then show that the particular subgroup isomorphic to $\mathbb{Z}_2$ cannot be generated by the other subgroups of $\mathrm{Sym}(Bn)$, and finally show that this group captures all symmetries of Bn.

We will first demonstrate permutations of the rows. Label the rows of Bn, top to bottom, by $r_{1,1}, r_{1,2}, \ldots, r_{1,n}; r_{2,1}, \ldots, r_{2,n}; \ldots; r_{n,1}, \ldots, r_{n,n}$, where row $r_{i,j}$ is the jth row down of band i. Fix i and swap any two rows $r_{i,j}, r_{i,k}$, $j, k \in \{1, \ldots, n\}$. Any cells in row j or k remain together in that row; any cells in the same box remain so; and all cells together in a column remain together in that column, though permuted. Then transposing any two rows within a band is a symmetry of Bn. Since any permutation of rows $r_{i,1}, r_{i,2}, \ldots, r_{i,n}$ is equivalent to a permuting the row labels $1, \ldots, n$, and any permutation can be written as a product of transpositions, each group of permutations of the n rows within a band is isomorphic to S_n. Now, since permuting the cells of a column keeps those cells in the same column and also keeps all cells of a box together in the same box, permutations of rows in distinct bands are wholly independent acts; i.e., given permutations ρ_i, ρ_j on the rows of bands i and j, $\rho_i\rho_j = \rho_j\rho_i$. And since, as before, the cells together in in any given row, column, or box remain in the same row, column, or box (though their row numbers may be changed), any composition of these permutations is not only a symmetry of Bn but also commutative. Hence the subgroup of all row permutations is a direct product of the subgroups of permutations of the rows in a given band. As there are n bands in Bn, we have the subgroup $\underbrace{S_n \times \cdots \times S_n}_{n \text{ times}}$.

We will now demonstrate permutations of the bands. Label the bands top to bottom by $R_1, \ldots, R_n$. Swap any two bands R_i, R_j, leaving the rows within each band in their original top to bottom order. Any cells that were in the same box together remain so; any cells in the same row remain so; and the columns, though permuted, contain the same cells. Thus any transposition of bands is a symmetry of Bn. As any permutation of the n bands $R_1, \ldots, R_n$ is a sequence of transpositions and is equivalent to permuting the band numbers $1, \ldots, n$, the group of permutations of the bands is isomorphic to S_n.

We now show that permutations of rows and permutations of bands are together non-abelian. Fix the row labeling of Bn as $r_{1,1}, r_{1,2}, \ldots, r_{1,n}; r_{2,1}, \ldots, r_{2,n}; \ldots; r_{n,1}, \ldots, r_{n,n}$, top to bottom. Let ρ denote the permutation of swapping the top two rows of Bn and consider $\rho(Bn)$. Now Bn has its rows in the order $r_{1,2}, r_{1,1}, r_{1,3}, \ldots, r_{1,n}; \ldots, r_{n,n}$. Now perform the band permutation β of swapping the top two bands of Bn. Now Bn has rows in order $r_{2,1}, r_{2,2}, \ldots, r_{2,n}; r_{1,2}, r_{1,1}, r_{1,3}, \ldots, r_{n,n}$, and we have performed the composite symmetry $\beta \circ \rho$. We will now consider $\rho \circ \beta$. Performing β first, Bn has row order $r_{2,1}, r_{2,2}, \ldots, r_{2,n}; r_{1,1}, \ldots, r_{1,n}; \ldots, r_{n,n}$. Now performing ρ, Bn has row order $r_{2,2}, r_{2,1}, \ldots, r_{2,n}; r_{1,1}, \ldots, r_{n,n}$. Since $\rho \circ \beta(Bn)$ has top row

$r_{2,2}$ and $\beta \circ \rho(Bn)$ has top row $r_{2,1}$, clearly these two permutations do not commute—hence the subgroup of all row swaps does not, in general, commute with the subgroup of band swaps.

Though these two subgroups are not together abelian, they do interact with normality. Let β be any permutation of the bands and let ρ be any permutation of the rows in some band. If β leaves fixed the band that ρ acts on, then $\rho \circ \beta = \beta \circ \rho$. So suppose that β does act on the same band as ρ. Let this be band i, and suppose that β maps band i to band position j. Let ρ be the row permutation $(r_{i,a}\, r_{i,b}\, \cdots\, r_{i,k})$. Under $\beta \circ \rho$, the row originally in position $r_{i,a}$ first mapped under ρ to position $r_{i,k}$, and then under β to position $r_{j,k}$. That is, $\beta \circ \rho(r_{i,a}) = r_{j,k}$. Let ρ' be the row permutation acting on band j by $\rho' = (r_{j,a}\, r_{j,b}\, \cdots\, r_{j,k})$, and define β as before. Then $\rho' \circ \beta(r_{i,a}) = \rho'(r_{j,a}) = r_{j,k}$. Noting the arbitrariness of the subscripts, we note that for any row permutation ρ and any band permutation β, there exists a row permutation ρ' such that $\beta \circ \rho = \rho' \circ \beta$. Then $\rho = \beta^{-1}\rho'\beta$, and hence ρ acts normally on β. Therefore the (direct product) subgroup of row permutations acts normally on the subgroup of band permutations, and we have the semi-direct product subgroup $(S_n \times \cdots \times S_n) \rtimes S_n$.

Next, we demonstrate the same for columns and stacks. In the above arguments, replace every "r" with "c", every "R" with "C", the word "row" with the word "column", and the word "band" with "stack" (alternatively, the reader may rotate the page a quarter turn and re-read the above argument), and we see that the argument holds for permutations of columns, permutations of stacks, and the non-commutativity and the normality of their combination. Thus we have another subgroup isomorphic to $(S_n \times \cdots \times S_n) \rtimes S_n$.

Observe that any row/band permutation leaves the column number of every cell fixed, and every column/stack permutation leaves the row number of every cell fixed. Let Bn have the canonical labeling and consider a row/band permutation $\rho \circ \beta$ and a column/stack permutation $\gamma \circ \sigma$. Since ρ and β each only affect the row number of any cell (x, y), suppose $\rho \circ \beta(x_1, y_1) = (x_2, y_1)$; and since γ and σ each only affect the column number of any cell (x, y), suppose $\gamma \circ \sigma(x_1, y_1) = (x_1, y_2)$. Then

$$(\rho \circ \beta) \circ (\gamma \circ \sigma)(x_1, y_1) = (\rho \circ \beta)(x_1, y_2) = (x_2, y_2) = (\gamma \circ \sigma)(x_2, y_1) = (\gamma \circ \sigma) \circ (\rho \circ \beta)(x_1, y_1).$$

Hence the subgroup of row/band permutations commutes with the subgroup of column/stack permutations. And, as mentioned, no row/band permutation changes the column number of any cell and and no column/stack permutation changes the row number of cell; so the only permutation in both subgroups is the identity permutation, and we thus have the direct product subgroup $((S_n \times \cdots \times S_n) \rtimes S_n) \times ((S_n \times \cdots \times S_n) \rtimes S_n)$.

We will now show that a diagonal flip of of Bn cannot be generated by this subgroup. First observe that flipping Bn along the main diagonal keeps any two cells in the same row in the same column, any two cells in the same column, and any two cells in the same box in the same box—so this flip is a valid symmetry of Bn; per the canonical labeling, this has the effect that $\varphi(x, y) =$

(y, x) for all cells (x, y) (so $\varphi(Bn) = Bn'$ is the transpose of Bn). Suppose for contradiction that there is some sequence of row, band, column, and stack permutations equivalent to the diagonal flip. Since the entire collection of these permutations is a group, there is some single permutation equivalent to φ. As shown above, no row or band permutation changes the column number of a cell; and no column or stack permutation changes the row number of a cell. That is, there is no nontrivial row/band permutation that is also a nontrivial column/stack permutation; and there is no nontrivial column/stack permutation that is also a nontrivial row/band permutation. Since the elements of the subgroup of row and band permutations commute with the elements of the subgroup of column and stack permutations, this sequence of permutations can be re-ordered so that all row/band actions take place before all column/stack actions. Under the flip (and this supposed equivalent sequence), we have that $\varphi(1, 2) = (2, 1)$—so the end result (local to the top-left box) of the sequence of row/band actions is that rows 1 and 2 swap; and the (local) end result of the sequence of column/stack actions is that columns 1 and 2 swap. Under these actions, $(1, 1)$ first maps to $(2, 1)$ and then to $(2, 2)$—but under the flip, all cells on the main diagonal are fixed points. Hence the diagonal flip cannot be achieved only by permuting the rows, bands, columns, or stacks, and hence is not an element of that direct product subgroup. As the diagonal flip is an order-2 element, we have a subgroup isomorphic to $\mathbb{Z}_2$ which has a trivial intersection with our already-found subgroup. From this we see that $((S_n \times \cdots \times S_n) \rtimes S_n) \times ((S_n \times \cdots \times S_n) \rtimes S_n)$ is exactly half of the symmetry group found so far (as any symmetry can be paired with the diagonal flip to achieve a new symmetry), and is therefore normal in the group found so far. This gives us the group $[((S_n \times \cdots \times S_n) \rtimes S_n) \times ((S_n \times \cdots \times S_n) \rtimes S_n)] \rtimes \mathbb{Z}_2$.

Finally, we will show that this group captures all of the possible symmetries of the sudoku board. In service of this claim, we will first prove the following:

- if cells u, v share a box in Bn, then $\varphi(u), \varphi(v)$ share a box in $\varphi(Bn) = Bn'$;
- if u, v share a row or column in Bn, then $\varphi(u), \varphi(v)$ share a row or column in Bn';
- if some pair u, v share a row in Bn and $\varphi(u), \varphi(v)$ share a column in Bn', then for every pair x, y sharing a row in Bn, $\varphi(x), \varphi(y)$ share a column in Bn';
- and if some pair u, v share a column in Bn and $\varphi(u), \varphi(v)$ share a row in Bn', then for every pair x, y sharing a column in Bn, $\varphi(x), \varphi(y)$ share a row in Bn'.

That is to say, boxes map to boxes (though likely permuted), rows and columns map to rows and columns (and not to boxes), and if one row maps to a column that all rows map to columns (and if one column maps to a row then all columns map to rows).

All of the cells sharing a box in Bn are adjacent in Sn; under any automorphism, their images must be adjacent—so in Bn', their images must share a row, column, or box. Suppose for contradiction that a box of Bn maps to a row in Bn'. Pick any cell u in this box in Bn.

Let x be one of the cells in u's box not in u's row or column. Since u and x share a box, they have $n^2 - 2$ common neighbor cells in the box (the entirety of the $n \times n$ box minus u and x themselves). Since u and x do not share a row or column, they have no common neighbors outside of their box. Then $\varphi(u)$ and $\varphi(x)$ must also have exactly $n^2 - 2$ common neighbors. Since the box, by supposition, maps to a row (of n^2 cells), $\varphi(u)$ and $\varphi(x)$ have $n^2 - 2$ common neighbors just in their row of Bn'. If $\varphi(u)$ and $\varphi(x)$ also share a box in Bn', then they will have an additional $n^2 - n$ common neighbors (the rest of the box minus the cells of their row that are in the box)—so $\varphi(u)$ and $\varphi(x)$ cannot be in the box. Now, in $\varphi(u)$'s row outside of its box, there are $n^2 - n$ cells. In u's box in Bn, there are $(n-1)^2$ cells not also in u's row or column (i.e., there are $(n-1)^2$ candidates for x). Then there are exactly $(n-1)^2 = n^2 - 2n + 1$ cells from u's box in Bn that must be mapped to exactly $n^2 - n$ spots in $\varphi(u)$'s row in Bn'. If $n^2 - 2n + 1 = n^2 - n$, then $n = 1$. Thus the cells of a box cannot all be mapped to the cells of a single row under any symmetry. (The argument that a box cannot map to a column is similar.) But since the cells of a box are all adjacent in Sn and must remain adjacent in $\varphi(Sn)$, they must map from the K_{n^2} that is their induced subgraph in Sn to an induced K_{n^2} in $\varphi(Sn)$—in Bn', this is exactly a row, column, or box (see Proposition 4.4). By process of elimination, boxes must map to boxes.

As a corollary, the cells sharing a row of Bn map to a row or column of Bn' (and the cells sharing a column of Bn map to a row or column of Bn'): since a row of Bn is an induced K_{n^2} of Sn, it maps to an induced K_{n^2} of $\varphi(Sn)$, which is a row, column, or box of Bn'. Since every box of Bn maps to a box of Bn' and there are the same number of boxes in Bn and Bn', a row of Bn is pigeonholed to a row or column of Bn'.

We now show that if one row maps to a column, then all rows must map to columns. Suppose for contradiction that some row of Bn maps to a column of Bn' but another row of Bn maps to a row Bn'. Let Bn have the canonical labeling and suppose that cells $(x_1, y_1), (x_1, y_2), \ldots, (x_1, y_{n^2})$ comprise the row of Bn that is mapped to a column of Bn' under symmetry φ; and suppose that cells $(u_1, y_1), (u_1, y_2), \ldots, (u_1, y_{n^2})$ comprise the row of Bn that is mapped to a row of Bn'. Now, for $i = 1, \ldots, n^2$, cells (x_1, y_i) and (u_1, y_i) are adjacent. If the two rows are in different bands of Bn, then (x_1, y_i) is not adjacent to any (u_1, y_j) if $j \neq i$. If the two rows are in the same band, then (x_1, y_i) is adjacent with exactly n of the (u_1, y_j), those in row u_1 and the same box as (x_1, y_i). So each (x_1, y_i) is adjacent with at most n of the cells in row u_1. We observe that in any Bn, any row intersects with every column in exactly one cell. This means that there is a cell (x_1, y_i) such that $\varphi(x_1, y_i) = \varphi(u_1, y_j)$ for some $i, j \in \{1, \ldots, n^2\}$, which means that $\varphi(x_1, y_i)$ is adjacent with all n^2 of $\varphi(u_1, y_1), \ldots, \varphi(u_1, y_{n^2})$, which means that φ does not preserve the adjacencies of Bn, contradicting that such a symmetry exists. Hence if one row is mapped to a column, there cannot be any row mapped to a row; i.e., if one row maps to a column then all rows map to columns. Similarly, if one column maps to a row, then all columns map to rows.

Corollary to this, if boxes share a band or a stack in Bn, then they will share a band or a stack in Bn'; and if one band maps to a stack then all bands map to stacks.

Let Bn be a sudoku board and $\varphi(Bn) = Bn'$ be any symmetry of Bn. The above arguments will provide an algorithm for determining a sequence of permutations $\varphi_1, \ldots, \varphi_m$ such that $\varphi = \varphi_m \circ \cdots \circ \varphi_1$ (note that since Bn has a finite number of cells, Sym(Bn) is a finite group, so such a sequence exists and is finite). Let Bn have the canonical labeling. Since $(1,1)$ and $(1,2)$ are in the same row, $\varphi(1,1)$ and $\varphi(1,2)$ must be in the same row or column. As the first step, determine whether $\varphi(1,1)$ and $\varphi(1,2)$ share a row or a column. Since mapping a row to a column changes both the row number and the cell number of every cell in the row, and this is not possible using only row, band, column, or stack permutations, this necessitates the diagonal flip. Also, a diagonal flip swaps the row and column numbers of every cell of Bn. Then every row has been matched to a column and every column to a row. Hence if any row is mapped to a column, then a diagonal flip has been used (specifically, an odd number of diagonal flips, since an even number of flips will return rows $\to$ columns $\to$ rows). So a flip is necessary iff $\varphi(1,1)$ and $\varphi(1,2)$ are in the same column.

Case 1. $\varphi(1,1)$ and $\varphi(1,2)$ share a row. Then all boxes of any given band of Bn are in the same band of Bn', all boxes of any given stack of Bn are in the same stack of Bn', all cells of a given row of Bn are in the same row in Bn', and all cells of a given column of Bn are in the same column in Bn'. Observe that the cells $(1,1),(1+n,1),(1+2n,1),\ldots,(1+(n-1)n,1)$ are those cells of the top corner of each box in stack 1; respectively, these cells are bands $1,2,\ldots,n$. Since in this case bands map to bands, each of $\varphi(1,1),\varphi(1+n,1),\ldots,\varphi(1+(n-1)n,1)$ are in distinct bands. Suppose $\varphi(1,1) = (x_1,y_1)$, $\varphi(1+n,1) = (x_2,y_2)$, $\ldots$, $\varphi(1+(n-1)n,1) = (x_n,y_n)$. Note that the x_i are all in different bands. Then φ_1 is the product of band transpositions $\varphi_1 = (1\ x_1)(2\ x_2)\cdots(n\ x_n)$.

Now perform the same with the top left cells of the stacks of Bn, these cells being $(1,1),(1,1+n),(1,1+2n),\ldots,(1,1+(n-1)n)$. These cells are in stacks $1,\ldots,n$, respectively. Since stacks map to stacks in this case, each of $\varphi(1,1),\varphi(1+n,1),\ldots,\varphi(1,1+(n-1)n)$ are in distinct stacks. Suppose $\varphi(1,1) = (x_1,y_1)$, $\varphi(1,1+n) = (x_2,y_2)$, $\ldots$, $\varphi(1,1+(n-1)n) = (x_n,y_n)$. Note that the y_i are all in different stacks. Then φ_2 is the product of stack transpositions $\varphi_2 = (1\ y_1)(2\ y_2)\cdots(n\ y_n)$.

Now look at each band of Bn and note the row order of the first column of stack one. Label these rows by $r_{i,j}$, $i,j \in \{1,\ldots,n\}$, where row $r_{i,j}$ is the jth row of band i. Suppose $\varphi(r_{1,1}) = r_{x_1,a_1}$, $\ldots$, $\varphi(r_{1,n}) = r_{x_1,a_n}$. Note that the a_i are all distinct rows. Then $\varphi_{3^{(1)}}$ is the product of row transpositions $\varphi_{3^{(1)}} = (1\ a_1)(2\ a_2)\cdots(n\ a_n)$ applied to band x_1 of Bn'. Suppose $\varphi(r_{i,1}) = r_{x_i,b_1}$, $\ldots$, $\varphi_(r_{i,n}) = r_{x_i,b_n})$. Then the b_i are all distinct rows, and $\varphi_{3^{(i)}}$ is the product of row transpositions $\varphi_{3^{(i)}} = (1\ b_1)(1\ b_2)\cdots(n\ b_n)$ applied to band x_i of Bn'. Recognizing that the row permutations of band j commute with the row permutations of band k for all $j \neq k$, apply this process to band x_i of Bn' for $i = 1,\ldots,n$. Then, collectively, φ_3 is the product of row transpositions $\varphi_3 = \underbrace{(1\ a_1)\cdots(n\ a_n)}_{Bn' \text{ band } 1}\cdots\underbrace{(1\ k_1)\cdots(n\ k_n)}_{Bn' \text{ band } n}$.

Finally we permute the columns within each stack of Bn' as necessary. Look at each band of Bn and note the column order of the first row of stack one. Label these columns by $c_{i,j}$, $i, j \in \{1, \ldots, n\}$, where column $r_{i,j}$ is the jth column of stack i. Suppose $\varphi(c_{1,1}) = c_{y_1,a_1}, \ldots, \varphi(c_{1,n}) = c_{y_1,a_n}$. Note that the a_i are all distinct columns. Then $\varphi_{4^{(1)}}$ is the product of column transpositions $\varphi_{4^{(1)}} = (1\ a_1)(2\ a_2)\cdots(n\ a_n)$ applied to stack y_1 of Bn'. Suppose $\varphi(c_{i,1}) = c_{y_i,b_1}, \ldots, \varphi(c_{i,n}) = r_{y_i,b_n})$. Then the b_i are all distinct columns, and $\varphi_{4^{(i)}}$ is the product of column transpositions $\varphi_{4^{(i)}} = (1\ b_1)(1\ b_2)\cdots(n\ b_n)$ applied to stack y_i of Bn'. Recognizing that the column permutations of stack j commute with the column permutations of stack k for all $j \neq k$, apply this process to stack y_i of Bn' for $i = 1, \ldots, n$. Then, collectively, φ_4 is the product of column transpositions $\varphi_4 = \underbrace{(1\ a_1)\cdots(n\ a_n)}_{Bn' \text{ stack } 1}\cdots\underbrace{(1\ k_1)\cdots(n\ k_n)}_{Bn' \text{ stack } n}$.

In determining φ_1, we are justified in considering only one cell from one box of each band since the collection of cells comprising this band must be mapped to a collection of cells also comprising a band. This is specifically what it is meant by "bands map to bands", and it enables us to determine the mapping of a band by representative. Indeed, since stacks map to stacks, rows map to rows, and columns map to columns, we may determine the specific necessary permutations φ_i whose composition is φ by considering only representatives from each type of configuration (bands, stacks, rows, columns). In this way we see that, by φ_1, what is good for any cell in a band is good for all of the cells in the band; by φ_2, what works for any cell in a stack works for all of the cells in the stack; by φ_3, cells that row together stick together; and by φ_4, that which is columnal shall remain columnar.

Case 2. $\varphi(1, 1)$ and $\varphi(1, 2)$ share a column. Then all boxes comprising a band of Bn are in the same stack of Bn', all boxes comprising a stack of Bn lie in the same band of Bn', all cells comprising a row of Bn share a column of Bn', and all cells in the same column of Bn are in the same row of Bn'. For the sake of the reader, we will outline the argument of the algorithm but with far less notation than used in Case 1 (though the argument is similar).

Since every band of Bn has become a stack in Bn', first perform the diagonal flip on Bn. Now, whereas step 1 of Case 1 was to determine where the bands of Bn sat amid the bands of Bn', now the question is where the bands of Bn sit amid the stacks of Bn'. Choose a representative cell from each band of Bn. The Bn-band number of each cell is its "original" stack number in Bn'. Apply the stack permutation to Bn' that sends the "original" stack numbers to the stack numbers where the representative cells are now found. This product of transpositions of stack numbers is φ_2.

Now pick a representative cell from every stack of Bn. After the diagonal flip, the Bn-stack number of each cell is its starting band number in Bn'. Note the band number in Bn' where each representative has ended up, and apply φ_3, the product of band transpositions that sends every starting number to the final band number in Bn'.

Note the cells (in order) of column 1 of Bn. This labeling designates representatives from each row of Bn—which becomes the "start" labeling for the columns of Bn'. Apply the appropriate product of transpositions of columns within stacks to send these "start" column labels to the column labels of the columns where the representative cells chosen are in Bn'. This sequence of column transpositions within stacks is φ_4.

Finally, note the cells in order of row 1 of Bn. This labeling designates representatives from each column of Bn—which becomes the "start" labeling for the rows of Bn'. Apply the appropriate product of transpositions of rows with bands to send these "start" row labels to the row labels of the rows where the representative cells chosen are in Bn'. This sequence of row transpositions within bands is φ_5.

Similar to Case 1, since we know that any collection of cells in a band of Bn must together comprise a stack of Bn', picking any cell from this band tells us the original stack number of each of these cells, and informs the proper transposition to apply to this stack based on the stack number of a single chosen cell. This equally applies to determining the transpositions of bands, rows, and columns.

A sharp-eyed reader may be concerned that between these two cases the order of choosing permutations was altered—the algorithm presented in Case 1 permuted, in order, bands, stacks, rows, and columns; whereas Case 2 permuted, in order, stacks, bands, columns, and rows. Recall that the vertically-applying permutations and the horizontally-applying permutations of the symmetry group are relatively abelian (any row/band permutation will commute with any column/stack permutation). Also note that, while row permutations do not commute with band permutations (nor column permutations with stack permutations), as long as careful and appropriate selection is made, the algorithm could be made to work in the opposite order (rows first, then bands). Our selections of permutations was based not on row/column/band/stack number, but rather on the starting and ending locations of representative cells and, by representation, those cells sharing a row/column/band/stack. Certainly the transpositions (row 1 $\leftrightarrow$ row 2) and (band 1 $\leftrightarrow$ band 2) do not commute—but the transpositions (the row containing cell 1 $\leftrightarrow$ the row containing cell 2) and (the band containing cell 1 $\leftrightarrow$ the band containing cell 2) do commute—but this is a possibly different set of transpositions than the first.

Through this constructive algorithm, we see that any Bn' can be reached in a well-defined finite number of permutations in the group we have thus far defined—hence this is the entire group of symmetries of Bn; so

$$\mathrm{Sym}(Bn) \cong [((\underbrace{S_n \times \cdots \times S_n}_{n \text{ times}}) \rtimes S_n) \times ((\underbrace{S_n \times \cdots \times S_n}_{n \text{ times}}) \rtimes S_n)] \rtimes \mathbb{Z}_2.$$

And since $\mathrm{Aut}(Sn) \cong \mathrm{Sym}(Bn)$ by Lemma 3.15, we conclude that this is also the automorphism group of the sudoku graph Sn. $\square$

Corollary 3.17. The order of the automorphism group of Sn is $2 \cdot (n!)^{2n+2}$.
From the argument of Theorem 3.16, the intersection of any two of the generating subgroups of $\mathrm{Aut}(Sn)$ is trivial, and $\mathrm{Aut}(Sn)$ is a product of these subgroups. Hence the order of the group is equal to the product of the orders of the generating subgroups.

4 Compendium of Properties

Proposition 4.1. For all $v \in V(Sn), \deg(v) = 3n^2 - 2n - 1$.

Proof. Let $v \in V(Sn)$. The box containing v has n^2 vertices, one of which is v. The row containing v has n^2 vertices, n of which are in the already-counted box containing v; same for the column containing v. Then $|N_{box}(v)| = n^2-1$; $|N_{row\backslash box}(v)| = n^2-n$; and $|N_{col\backslash box}(v)| = n^2-n$, so $|N(v)| = n^2 - 1 + 2(n^2 - n) = 3n^2 - 2n - 1$. $\square$

Corollary 4.2. $|E(Sn)| = n^4(3n^2 - 2n - 1)/2 = (3n^6 - 2n^5 - n^4)/2$.
This follows the above and $2|E(G)| = \sum_{v \in V(G)} \deg(v)$.

Proposition 4.3. The clique number of Sn is $\omega(Sn) = n^2$.

Proof. Since every row, column, and box is an induced K_{n^2}, $\omega(Sn) \geq n^2$. Since $\chi(Sn) = n^2$ (see 3.1), there is no clique of size $n^2 + 1$, and equality is achieved. $\square$

Proposition 4.4. Sn contains exactly $3n^2$ copies of K_{n^2} as subgraphs.

Proof. Suppose u, v share a row. Note that if they also share a column, then $u = v$. Suppose u, v are distinct. By Definition 2.3, they are adjacent. Moreover, they are both adjacent to each of the other $n^2 - 2$ vertices in this row, and those $n^2 - 2$ vertices are mutually adjacent. Then this row is a K_{n^2} subgraph. As $\omega(Sn) = n^2$ (see Proposition 4.3), this is a maximum (and hence maximal) clique; i.e., there is no other vertex adjacent to all of these n^2 vertices. That is, if two vertices share a row, regardless of whether they share a box, then they belong to a maximum K_{n^2} subgraph that comprises vertices sharing a row and no vertices not in that row.

A symmetric argument can be made for u, v sharing a column (see proof of Theorem 3.16).

Now suppose that u, v share a box. By Definition 2.3, they are also adjacent to each of the other $n^2 - 2$ vertices in this box, and those $n^2 - 2$ vertices are mutually adjacent. Then this box is a K_{n^2} subgraph, which is again a maximum (and hence maximal) clique. That is, if two vertices share a box, regardless of whether they share a row or column, then they belong to a maximum K_{n^2} subgraph that comprises vertices sharing a box and no vertices not in that box.

We see that every maximum clique of Sn is a row, column, or box of Bn. There are n^2 rows, n^2 columns, and n^2 boxes, and so Sn has exactly $3n^2$ maximum cliques. $\square$

Corollary 4.5. Every vertex is in exactly three copies of K_{n^2}.
This follows directly from the argument above.

Proposition 4.6. The coclique number of Sn is $\alpha(Sn) = n^2$.

Proof. Let the cell in row i, column j of Bn be denoted as (i, j), where $i, j \in \{1, ..., n^2\}$. Note that rows $1, 1+n, 1+2n, ...,$ do not intersect any of the same boxes; likewise for the columns. Take cells $(1, 1), (2, 1+n), (3, 1+2n), ..., (n, 1+(n-1)n)$ from the topmost row of boxes. From the next-down row of boxes, take cells $(1+n, 2), (2+n, 2+n), (3+n, 2+2n), ..., (n+n, 2+(n-1)n)$; ...; from the lowest set of boxes, take cells $(1+(n-1)n, n), (1+(n-1)n+1, 2n), ..., (n+(n-1)n, n^2)$. For example, in $B2$ this set is $\{(1, 1), (2, 3), (3, 2), (4, 4)\}$; in $B3$ this set is $\{(1, 1), (2, 4), (3, 7), (4, 2), (5, 5), (6, 8), (7, 3), (8, 6), (9, 9)\}$. That is, begin in the top-left corner of the board; each subsequent cell is taken one row down, in the left-most column of the next (to the right) box; continue until all n of the top boxes have been visited. For each subsequent horizontal band of boxes, begin in the top row, moving down one row for each subsequent cell, and begin in the lowest numbered column not yet visited, moving n columns right for each subsequent cell. Terminate in the lower-right corner. In this manner we achieve a coclique of size n^2, and so $\alpha(Sn) \geq n^2$.

Suppose Bn has a coclique of size $n^2 + 1$. Every vertex in this coclique must be in a different row from every other vertex of this coclique. Let v be any vertex in this coclique, and consider which row v is found in: there are only $n^2 - 1$ other rows from which to choose the remaining n^2 vertices, and so some two vertices must be in the same row. By Definition 2.3, these vertices are adjacent. Hence $\alpha(Sn) \leq n^2$, and so $\alpha(Sn) = n^2$. $\square$

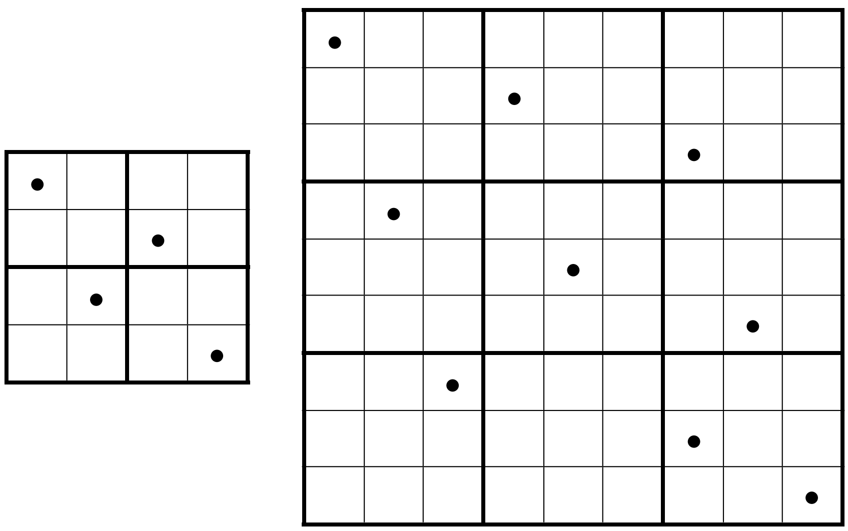

Figure 11: The coclique as described in: Left, $B2$; Right, $B3$

Proposition 4.7. Sn contains $(n!)^{2n}$ distinct cocliques of size n^2.

Proof. We will count independent sets (cocliques) by choosing cells from successive rows of Bn. In the top row, pick a stack: there are n choices. Pick a column within this stack: there are n choices. In the second row, we must choose a different stack, else the vertex being chosen will share a box with the previous vertex and thus not be independent from the previous vertex: there are $n-1$ choices. Pick a column within this stack: there are n choices. ... In the $(n-1)$th row, pick a stack that has not yet been chosen: there are 2 choices. Pick a column within this stack: there are n choices. In the nth row, there is one stack not yet chosen; within this stack there are n choices for the column.

At any row, the particular stacks chosen do not affect the number of choices remaining (only the particular choices); and the choice of a stack does not affect the number of columns within that stack; hence, we multiply these choices. Rearranging the product, we see that for the first band, the number of ways we may choose a coclique of size n is

$$\underbrace{(n(n-1)\cdots 2\cdot 1)}_{\text{choosing stacks}} \cdot \underbrace{(n\cdots n)}_{\substack{\text{columns}\\\text{within}\\\text{stacks}}} = n!\cdot n^n.$$

For the $(n+1)$th row, pick any stack. As no cells have been selected in this band, we may select any stack: there are n choices. Pick a column within this stack, avoiding the one column picked in this stack in the top band: there are $n-1$ choices. For the $(n+2)$th row, pick any other stack within this band: there are $n-1$ choices. Pick a column within this stack, avoiding the one column picked in this stack in the top band: there are $(n-1)$ choices. ... For the $(2n)$th row, there is one choice for the stack, and $n-1$ choices for the column in this stack.

So for the second band, avoiding each of the columns used in the top band, the number of ways to choose our second n cells is

$$\underbrace{(n(n-1)\cdots 2\cdot 1)}_{\text{choosing stacks}} \cdot \underbrace{((n-1)\cdots(n-1))}_{\substack{\text{columns}\\\text{within}\\\text{stacks}}} = n!\cdot(n-1)^n.$$

Proceeding this way, we see that at each band there are $n!$ ways to choose stacks; across all of the n bands, this gives $(n!)^n$ ways to choose stacks. Within band i, we see that there are $(n-i+1)^n$ ways to choose the columns. Since i ranges $1,\ldots,n$, we count the ways to choose the columns across all of the n bands by

$$n^n\cdot(n-1)^n\cdots 2^n\cdot 1^n = (n\cdot(n-1)\cdots 2\cdot 1)^n = (n!)^n.$$

Hence there are $(n!)^n\cdot(n!)^n = (n!)^{2n}$ independent sets of size n^2 in Sn. $\square$

Proposition 4.8. For all $2 \leq m \leq n$, Sn contains Sm as an induced subgraph.

Proof. For $n = 2$, $S2$ is clearly induced in itself. Consider any Bn, $n > 2$.

The first row of Bm is found as the subgraph induced on the cells

$$(1,1),\ldots,(1,m),(1,n+1),\ldots,(1,n+m),\ldots,(1,(m-1)n+1),\ldots,(1,(m-1)n+m).$$

Without difficulty we see that we are pulling sets of m cells each from distinct boxes; specifically, those boxes with leftmost column number $1, n+1, 2n+1, \ldots, (m-1)n+1$. Since $1 = (m-m)n+1$, we also see that these sets of m cells have been pulled from m boxes, for a row of length of m^2.

We continue downward within the first band of Bn, pulling cells from the same columns as chosen in the first row, now selecting from rows $2, \ldots, m$. This gives us the top band of Bm, with all of and only its necessary adjacencies, by deliberately picking these $m \times m$ subgrids within different stacks of Bn.

For the remaining bands of Bm, we continue downward, choosing those cells in the same columns as before, from the first m rows of the first m bands of Bn. As we had all of and only the necessary "horizontal" adjacencies of Bm by choosing our $m \times m$ grids from different stacks of Bn, we also have all of and only our necessary "vertical" adjacencies by choosing our $m \times m$ grids from different stacks of Bn.

By maintaining consistency with our choices of rows within bands and with our choices of columns within stacks, we thus complete our construction of an induced Bm within Bn.

From this construction, we have a more formal method of identification to find Sm: for any integers $2 \leq m \leq n$, we define $f : \mathbb{Z} \to \mathbb{Z}$ by

$$f(x) = \left\lfloor \frac{x-1}{m} \right\rfloor (n-m) + x.$$

Then each cell (i, j) in Sm corresponds to the cell $(f(i), f(j))$ in Sn; the subgraph of Sn induced on $\{(f(i), f(j)) : 2 \leq i, j \leq m^2\} \subseteq V(Sn)$ is isomorphic to Sm. $\square$

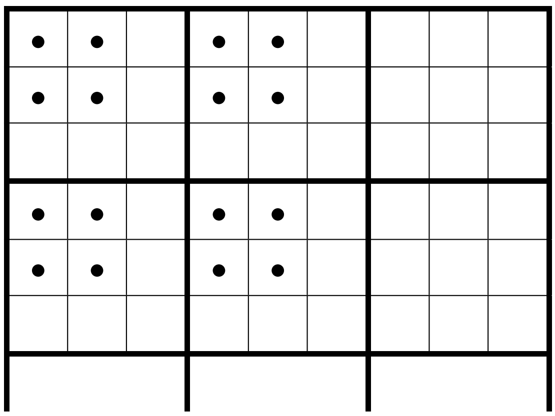

Figure 12: The described construction of an induced $B2$ within $B3$

Corollary 4.9. Sn contains at least $\binom{n}{m}^{2m+2}$ distinct induced copies of Sm, $2 \leq m \leq n$.

Proof. Relaxing the previous construction, we recognize that we must select the m bands of Bm from the n bands of Bn, for $\binom{n}{m}$ options. Within each of these m bands, we must select m rows from the n options. From band to band, these row choices are independent—that is, we have $\binom{n}{m}$ row choices for band 1, $\binom{n}{m}$ row choices for band 2, $\ldots$, $\binom{n}{m}$ row choices for band m. Our total number of row options is $\binom{n}{m}^m$, bringing our total number of "horizontal" option to $\binom{n}{m}^{m+1}$. Symmetrically and independently, we have the same number of ways to select m stacks and m columns within each stack—giving us a total of $\left(\binom{n}{m}^{m+1}\right)^2$ options, achieving the result. □

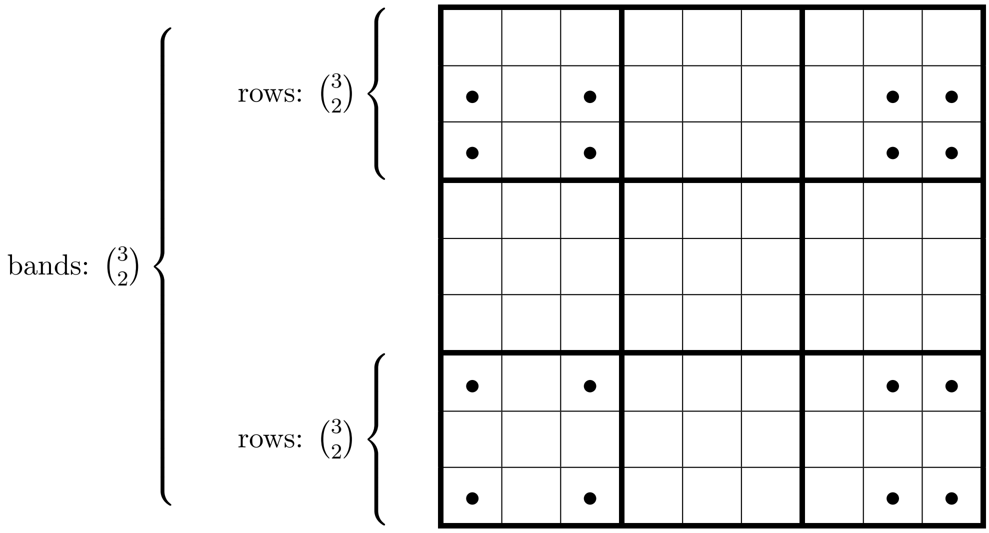

Figure 13: A different induced $B2$ within $B3$; stack/column choices are symmetric

Proposition 4.10. No Sn is planar.

Proof. Consider $S2$. The vertices of the upper-left box form an induced K_4, as do the vertices of the lower-left box. Contract the edges joining the vertices of the lower-left box and simplify the resulting graph. Call the vertex resulting from the multiple contractions x. By the original adjacencies of $S2$, each vertex of the upper-left box is adjacent to x, and so $S2$ has a K_5 minor, and is thus not planar.

By Proposition 4.8, Sn contains $S2$ as an induced subgraph, and so no Sn is planar. □

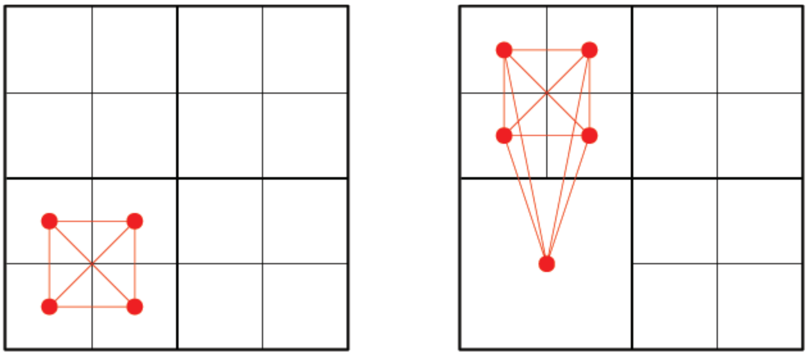

Figure 14: Left: The K_4 to be contracted; Right: The resulting K_5 minor

Proposition 4.11. No Sn is edge transitive.

Proof. Let Sn have the canonical labeling. This labeling indicates the board position of the vertices. Note that, since an edge is simply a relation between vertices, an automorphism acting on the edges of a graph can be considered to be acting on pairs of vertices.

Let $u = (1,1), v = (2,2), w = (1, n^2) \in V(Sn)$ and consider edges $e_1, e_2 \in E(Sn)$ where $e_1 = uv$ and $e_2 = uw$ (note that these edges exist in all Sn). Suppose, by way of contradiction, that Sn is edge transitive—then there exists an automorphism $\varphi \in \mathrm{Aut}(Sn)$ such that $\varphi(e_1) = e_2$. Observe that, for all Sn, u and v are in the same box of Bn. By the proof of Theorem 3.16 we have that $\varphi(u)$ and $\varphi(v)$ must be in the same box of $\varphi(Bn)$ (boxes map to boxes). But board positions $(1,1), (1,n^2)$ are in different boxes of any board—so it cannot be that one of u, v maps to $(1,1)$ and the other to $(1,n^2)$, so it cannot be that e_1 maps to e_2, and so no such φ can exist. Hence Sn is not edge transitive. □

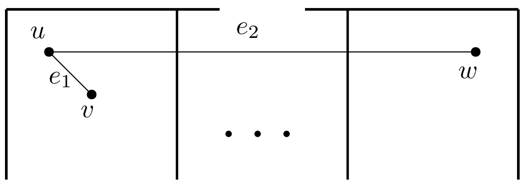

Figure 15: Non-transitive edges e_1, e_2 in any Sn

Proposition 4.12. Every Sn is Hamiltonian.

Though this proof will be rather simplistic, we include it in service to the Lovász conjecture (see [9] for more) that every finite, connected, vertex-transitive graph contains a Hamiltonian cycle.

Proof. We will consider Bn rather than Sn and condition on the parity of n.

Case 1. n is even.

Since Bn contains n^2 rows, Bn contains an even number of rows. Beginning from cell $(1,1)$, trace the rows right to left and subsequently (moving downward) left to right. The tracing of the bottom row will be right to left, and thus the last cell traced in that row will be cell $(n^2, 1)$. Since cells $(x_1, y_1) = (1, 1)$, $(x_2, y_2) = (n^2, 1)$ have the property $y_1 = y_2$, these cells lie in the same column of Bn, and hence their corresponding vertices in Sn are adjacent. Thus the cycle traced is Hamiltonian.

Case 2. n is odd.

Again begin in cell $(1, 1)$ and trace the rows alternately right to left then left to right. Every odd-numbered row will be traced left to right and every even-numbered row will be traced right to left. Stop this tracing in the second-to-last row, the row numbered $n^2 - 2$. The last cell traced in that row is cell $(n^2 - 2, n^2)$. From this point, trace an up-down track toward column 1. That is, from cell $(n^2 - 2, n^2)$, we trace cells $(n^2 - 1, n^2), (n^2, n^2)$; $(n^2, n^2 - 1), (n^2, n^2 - 2)$; .... This leftward tracing will inevitably end in either cell $(n^2, 1)$ or cell $(n^2 - 1, 1)$. However, since both of these lie in the same column as cell $(1, 1)$, the Hamiltonian cycle is achieved.

□

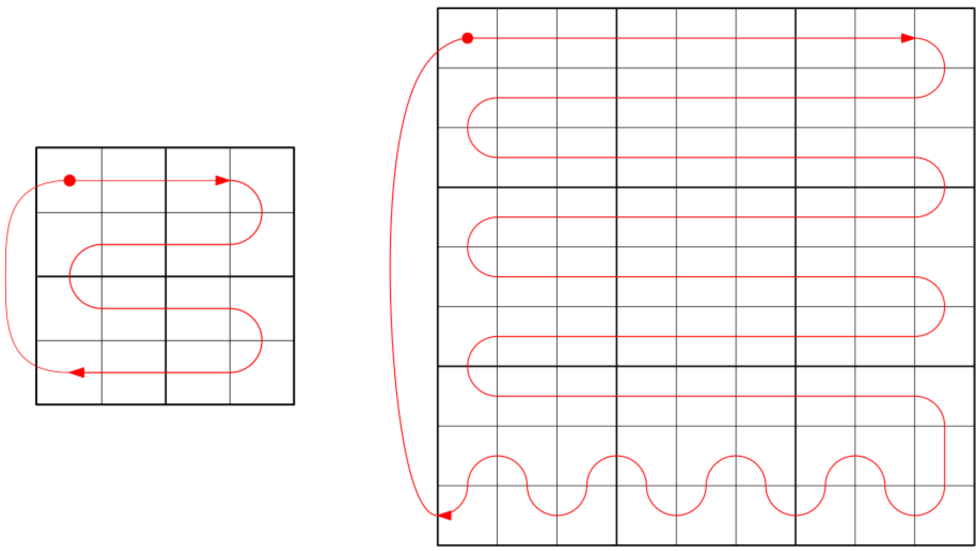

Figure 16: The Hamiltonian path as described in: Left, $B2$; Right, $B3$

Proposition 4.13. *Sn is pancyclic.*

From [1], if a graph G is Hamiltonian, G is not the complete bipartite graph $K_{|V(G)|/2,|V(G)|/2}$, and $|E(G)| \geq |V(G)|^2/4$, then G is pancyclic.

From Proposition 4.1, $|E(Sn)| = (3n^6 - 2n^5 - n^4)/2$, and $|V(Sn)|^4/4 = (n^4)^2/4 = n^6/4$.

Observe the following:

$$
\begin{aligned}
n \geq 2 \Rightarrow\, & 5n - (2 + \sqrt{14}) \geq 10 - (2\sqrt{14}) > 0, \\
& 5n - (2 - \sqrt{14}) \geq 10 - (2 - \sqrt{14}) > 0 \\
\Rightarrow\, & (5n - (2 + \sqrt{14}))(5n - (2 - \sqrt{14})) \geq 0 \\
& 5n^2 - 4n - 2 \geq 0 \\
& 5n^6 - 4n^5 - 2n^4 \geq 0 \\
& 6n^6 - 4n^5 - 2n^4 \geq n^6 \\
& \frac{3n^6 - 2n^5 - n^4}{2} \geq \frac{n^6}{4} \\
& |E(Sn)| \geq |V(Sn)|^2/4.
\end{aligned}
$$

From Proposition 4.12, Sn is Hamiltonian; and since Sn has odd cycles, Sn is not bipartite. Hence Sn is pancyclic.
Further, since Sn is vertex transitive (see Corollary 3.12), Sn is vertex-pancyclic.

Proposition 4.14. Classification of vertex pairs by number of common neighbors.
Since Sn is vertex transitive (see Corollary 3.12), we may characterize all vertices by choosing a representative. Given the canonical labeling on Bn, we let $u = (1, 1)$ be our representative vertex. We will classify all other vertices v by the number of neighbors common to u and v—i.e., $|N(u) \cap N(v)|$.

In the figures below, for $n = 2, 3$, we mark $u = (1, 1)$ by a square and u's neighboring vertices by dots. In each cell we note how many neighbors u has in common with that cell.

■	•4	•2	•2
•4	•2	4	4
•2	4	2	2
•2	4	2	2

■	•13	•13	•7	•7	•7	•7	•7	•7
•13	•7	•7	6	6	6	6	6	6
•13	•7	•7	6	6	6	6	6	6
•7	6	6	2	2	2	2	2	2
•7	6	6	2	2	2	2	2	2
•7	6	6	2	2	2	2	2	2
•7	6	6	2	2	2	2	2	2
•7	6	6	2	2	2	2	2	2
•7	6	6	2	2	2	2	2	2

Figure 17: For $u = (1, 1)$ and for each v, $|N(u) \cap N(v)|$ for: Left, $B2$; Right, $B3$

We will categorize the vertices of Sn/the cells of Bn thusly, given the position of cell v relative to cell u:
A: Different band and different stack
B: Same band, different box, different row
C: Same band, different box, same row
D: Same box, same row (different column)
E: Same box, different row, different column
Note that by the symmetries of Bn (see Theorem 3.16), categories B, C, and D have symmetric categories via band $\leftrightarrow$ stack and row $\leftrightarrow$ column. We will adjust the appropriate results accordingly.

Let $v \in A$, given $u = (1, 1)$. Since v is in a different band and different stack than u, we see that v can only be adjacent to one cell in u's row, one cell in u's column, and no cells in u's box—specifically, v is adjacent to $(1, y)$ and $(x, 1)$. Hence for $v \in A$, $|N(u) \cap N(v)| = 2$.

Let $v \in B$. Then v's shared neighbors with u are exactly those cells in u's row and v's box (of which there are n), and v's row and u's box (of which there are n), and so $|N(u) \cap N(v)| = 2n$.

Let $v \in C$. Since v is in a different box than u, v can only share those neighbors in the row of u and v. Each row contains n^2 vertices; but neither u nor v are adjacent to themselves, and so $|N(u) \cap N(v)| = n^2 - 2$.

Let $v \in D$. Since u and v do not share a column, v is not adjacent to any of the cells in u's column outside their shared box. However, both u and v are adjacent to every cell in their row and and in their box. Not including u or v, there are $n^2 - 2$ cells in their row. In their box, there are $n - 1$ rows other than their row, each containing n cells—so $|N(u) \cap N(v)| = n^2 - 2 + (n - 1) \cdot n = 2n^2 - n - 2$.

Let $v \in E$. Here, the common neighbors of u and v are precisely all the cells of their box, less u and v themselves; as each box contains a total of n^2 cells, we see that $|N(u) \cap N(v)| = n^2 - 2$.

We will now count how many cells are in each category.

In the figures below, for $n = 2, 3$, we will partition Bn by category (relative to $u = (1, 1)$): Observe the following:

$$\begin{aligned}
|A| &= \text{cells of } Bn - \text{ cells in } u\text{'s band } - \text{ cells in } u\text{'s stack} \\
&\qquad + \text{ cells in } u\text{'s box (the overlap of } u\text{'s band and stack, by inclusion-exclusion)} \\
&= (n^4 \text{ total cells }) - (n^2 \text{ columns in one band}) \cdot (n \text{ rows in one band}) \\
&\qquad - (n^2 \text{ rows in one stack}) \cdot (n \text{ cells in one stack}) + (n^2 \text{ cells in one box of overlap}) \\
&= n^4 - 2n^3 + n^2.
\end{aligned}$$

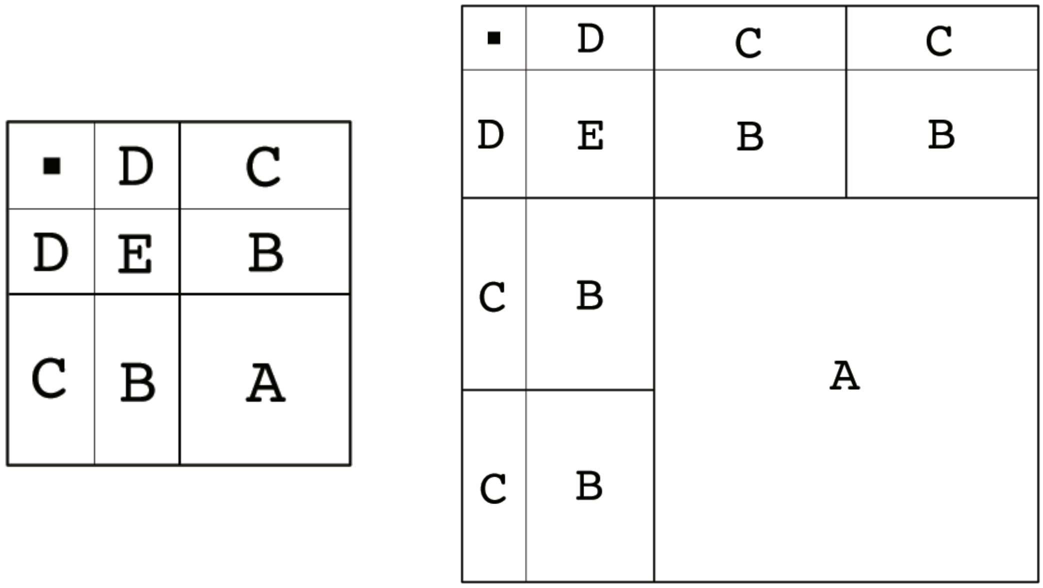

Figure 18: The partition by category based on number of common neighbors with $(1,1)$ for: Left, $B2$; Right, $B3$

$$
\begin{aligned}
|B| = &\text{ cells in } u\text{'s band } - \text{ cells in } u\text{'s box } - \text{ cells in } u\text{'s row in other boxes of } u\text{'s band} \\
&= (n^2 \text{ columns in one band}) \cdot (n \text{ rows in one band}) - (n^2 \text{ cells in one box}) \\
&\qquad - (n^2 \text{ total cells in one row less the } n \text{ cells sharing a row and a box}) \\
&= n^3 - 2n^2 + n.
\end{aligned}
$$

To account for those cells in, relative to u, the same stack, a different box, and a different column, we double the result:

$$
\begin{aligned}
&|B| = 2n^3 - 4n^2 + 2n. \\
&|C| = \text{ cells in } u\text{'s row} - \text{ cells in both } u\text{'s row and } u\text{'s box} \\
&\quad = n^2 - n.
\end{aligned}
$$

To account for those cells in, relative to u, the same stack, a different box, and the same column, we double the result:

$$
\begin{aligned}
&|C| = 2n^2 - 2n. \\
&|D| = \text{ cells in both } u\text{'s row and } u\text{'s box less } u \text{ itself} \\
&\quad = n - 1.
\end{aligned}
$$

To account for those cells in, relative to u, the same box and the same column (different row), we double the result:

$$|D| = 2n - 2.$$

$$\begin{aligned} |E| = &\text{ cells in } u\text{'s box } - \text{ cells in } u\text{'s row } - \text{ cells in } u\text{'s column} \\ &+ \text{ (by inclusion-exclusion) the single cell of overlap} \\ = &\; n^2 - 2n + 1. \end{aligned}$$

Since these categories are disjoint and no category includes u, we observe that

$$|A| + \cdots + |E| + |\{u\}| = n^4$$

—hence these categories partition the board.

Observe that those cells in C have the same number of neighbors shared with u as do those cells in E—each $n^2 - 2$; and $|C| + |E| = 2n^2 - 2n + 2n - 2 = 2n - 2$.

In summary, given any vertex u, Sn has:

- $n^4 - 2n^3 + n^2$ vertices v such that $|N(u) \cap N(v)| = 2$ (A);
- $2n^3 - 4n^2 + 2n$ vertices x such that $|N(u) \cap N(x)| = 2n$ (B);
- $2n^2 - 3n + 1$ vertices w such that $|N(u) \cap N(w)| = n^2 - 2$ (C and E); and
- $2n - 2$ vertices z such that $|N(u) \cap N(z)| = 2n^2 - n - 2$ (D).

Corollary 4.15. *Sn is not strongly regular.*

Proof. Recall that in a strongly regular graph, every two non-adjacent vertices have the same number of common neighbors. Let $u = (1, 1)$, let v be a vertex in category A, and let x be a vertex in category B (as in Proposition 4.14). Then u is not adjacent to either v or x; and $|N(u) \cap N(v)| = 2$, whereas $|N(u) \cap N(x)| = 2n \geq 4$, and so Sn is not strongly regular. ☐

Proposition 4.16. The matching number of Sn is $\lfloor n^4/2 \rfloor$.

Proof. Note that this follows from Sn being vertex transitive (Corollary 3.12) [6]; our proof will provide an algorithmic construction of a perfect matching for even n and a near-perfect matching for odd n.

Suppose n is even. Then each box of Bn has an even number of columns (let Bn have the canonical labeling and call them $y_1, \ldots, y_n$); these columns can be partitioned into consecutive pairs $\{y_1, y_2\}, \{y_3, y_4\}, \ldots, \{y_{n-1}, y_n\}$. For the rows $x_1, \ldots, x_n$ of the box, take the set of edges

$$\{(x_i, y_{2j-1})(x_i, y_{2j}) : i = 1, \ldots, n, j = 1, \ldots, n/2\}$$

(that is, $\{(x_i, y_1)(x_i, y_2), (x_i, y_3)(x_i, y_4), \ldots, (x_i, y_{n-1})(x_i, y_n)\}$ for $i = 1, \ldots, n$)—this is a perfect matching within a given box. As each box of Bn has the same form, taking this set of edges for each box gives a perfect matching for Bn (and thus for Sn). As a perfect matching includes one edge for every two vertices, the matching number is $n^4/2$.

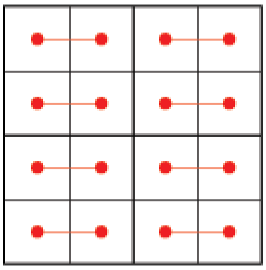

Figure 19: The described perfect matching for $S2$

Now suppose n is odd. Label the rows of Bn $x_1, \ldots, x_{n^2}$ and the columns $y_1, \ldots, y_{n^2}$. For each row x_i, first consider only the columns $y_1, \ldots, y_{n^2-1}$. This is an even number of columns and can thus be partitioned into consecutive pairs—take the edges

$$\{(x_i, y_{2j-1})(x_i, y_{2j}) : i = 1, \ldots, n, j = 1, \ldots, {}^{n^2-1}/2\}$$

(that is, $\{(x_i, y_1)(x_i, y_2), (x_i, y_3)(x_i, y_4), \ldots, (x_i, y_{n^2-2})(x_i, y_{n^2-1})\}$ for $i = 1, \ldots, n^2$). This is a perfect matching on the vertices of n^2 rows and $n^2 - 1$ columns. Taking one edge for every two vertices, this gives, so far, $n^2(n^2-1)/2$ edges. Now consider the vertices of column y_{n^2}. Rows $x_1, \ldots, x_{n^2-1}$ can be partitioned into consecutive pairs—take edges $\{(x_{2j-1}, y_{n^2})(x_{2j}, y_{n^2}) : j = 1, \ldots, \frac{n^2-1}{2}\}$ (that is, $\{(x_1, y_{n^2})(x_2, y_{n^2}), (x_3, y_{n^2})(x_4, y_{n^2}), \ldots, (x_{n^2-2}, y_{n^2})(x_{n^2-1}, y_{n^2})\}$). This leaves only vertex (x_{n^2}, y_{n^2}) unsaturated, and includes edges for the other $n^2 - 1$ vertices of the column, contributing $(n^2-1)/2$ edges to our matching, for a total of $n^2(n^2-1)/2 + (n^2-1)/2 = (n^4-1)/2$ edges, creating a near-perfect matching for odd n. $\square$

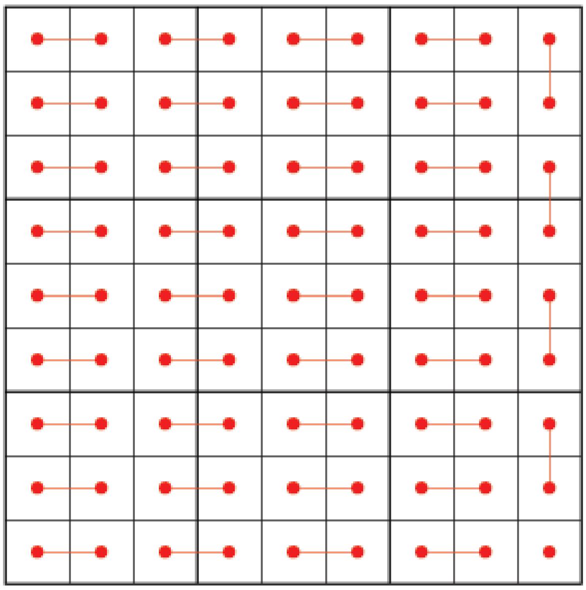

Figure 20: The described near-perfect matching for $S3$

Proposition 4.17. No Sn is a perfect graph.

Proof. By [2], a graph G is perfect iff no induced subgraph of G is an odd cycle of length at least five or the complement of an odd cycle of length at least five. We will demonstrate that each Sn contains an induced 5-cycle and the complement of an induced 5-cycle.

Let Bn have the canonical labeling. Let H be the subgraph of Sn induced on the vertices $\{u_1 = (1,1), u_2 = (1, 1+n), u_3 = (2, 2+n), u_4 = (1+n, 2+n), u_5 = (1+n, 1)\}$. We claim that H is an induced 5-cycle. First note that $2 + n \leq n^2$, and so each of these vertices is contained in $V(Sn)$. Now note the following:

$$
\begin{aligned}
& u_{1x} = 1 = u_{2x} \\
& \left\lceil \tfrac{u_{2x}}{n} \right\rceil = \left\lceil \tfrac{1}{n} \right\rceil = 1 = \left\lceil \tfrac{2}{n} \right\rceil = \left\lceil \tfrac{u_{3x}}{n} \right\rceil \\
& \left\lceil \tfrac{u_{2y}}{n} \right\rceil = \left\lceil \tfrac{1+n}{n} \right\rceil = 2 = \left\lceil \tfrac{2+n}{n} \right\rceil = \left\lceil \tfrac{u_{3y}}{n} \right\rceil \\
& u_{3y} = 2 + n = u_{4y} \\
& u_{4x} = 1 + n = u_{5x} \\
& u_{5y} = 1 = u_{1y} \\
& \Rightarrow u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_1 \in V(H),
\end{aligned}
$$

so H contains a 5-cycle. Further,

$$\begin{aligned}
u_{1x} &= 1 \neq 2 = u_{3x}, & u_{1y} &= 1 \neq 2+n = u_{3y}, & \left\lceil \tfrac{u_{1y}}{n} \right\rceil &= 1 \neq 2 = \left\lceil \tfrac{u_{3y}}{n} \right\rceil \\
u_{1x} &= 1 \neq 1+n = u_{4x}, & u_{1y} &= 1 \neq 2+n = u_{4y}, & \left\lceil \tfrac{u_{1y}}{n} \right\rceil &= 1 \neq 2 = \left\lceil \tfrac{u_{4y}}{n} \right\rceil \\
u_{2x} &= 1 \neq 1+n = u_{4x}, & u_{2y} &= 1+n \neq 2+n = u_{4y}, & \left\lceil \tfrac{u_{2x}}{n} \right\rceil &= 1 \neq 2 = \left\lceil \tfrac{u_{4x}}{n} \right\rceil \\
u_{2x} &= 1 \neq 1+n = u_{5x}, & u_{2y} &= 1+n \neq 1 = u_{5y}, & \left\lceil \tfrac{u_{2x}}{n} \right\rceil &= 1 \neq 2 = \left\lceil \tfrac{u_{5x}}{n} \right\rceil \\
u_{3x} &= 2 \neq 1+n = u_{5x}, & u_{3y} &= 2+n \neq 1 = u_{5y}, & \left\lceil \tfrac{u_{3y}}{n} \right\rceil &= 2 \neq 1 = \left\lceil \tfrac{u_{5y}}{n} \right\rceil
\end{aligned}$$

$$\Rightarrow u_1u_3, u_1u_4, u_2u_4, u_2u_5, u_3u_5 \notin V(H).$$

Hence H is a chordless 5-cycle. Observing that the complement of a 5-cycle is itself a 5-cycle, we see that these same vertices induce a 5-cycle in the complement of Sn, and so Sn is not a perfect graph. $\square$

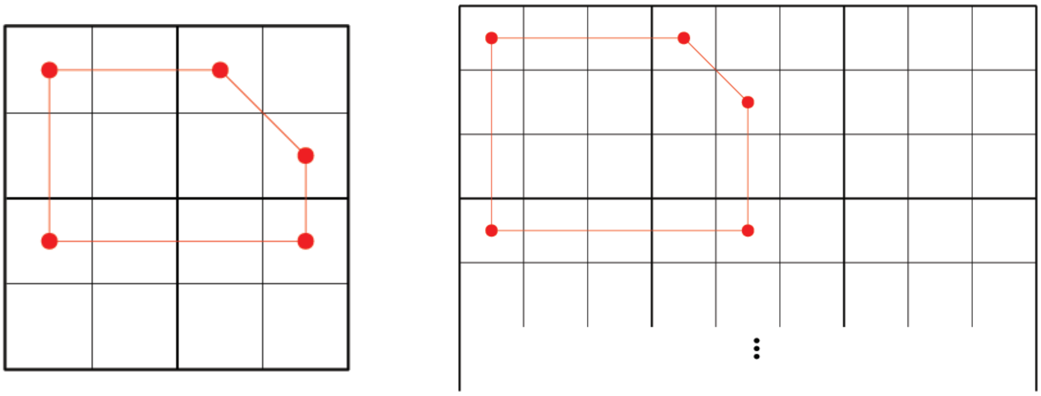

Figure 21: The described chordless 5-cycle as in: Left, $B2$; Right, $B3$

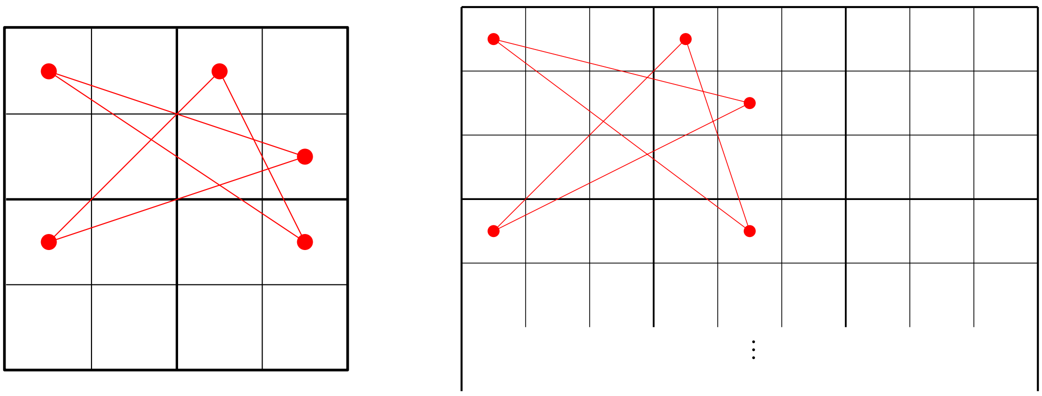

Figure 22: The subgraph induced on the same vertices in: Left, $B2^C$; Right, $B3^C$

Corollary 4.18. The distance and diameter of Sn are each 2, as is the eccentricity of each vertex.

Proof. This follows as a corollary of Proposition 4.14. Since any two distinct vertices u, v have at least one common neighbor, a 2-path exists for all $u, v \in V(Sn)$, so $d(u, v) = 2$ for all $u, v \in V(Sn)$. And since Sn is vertex transitive (see Corollary 3.12), it follows that the diameter of Sn as well as the eccentricity of each vertex are both 2. □

Proposition 4.19. Sn has edge connectivity $3n^2 - 2n - 1$.

Proof. By [6], this follows from Sn being connected and vertex-transitive with regular vertex degree $3n^2 - 2n - 1$ (see Proposition 4.1). □

For a note on vertex connectivity, see Section 5.

Proposition 4.20. $S2$, $S3$, and $S4$ are not Cayley graphs for cyclic groups.

Proof. Recall from Theorem 3.16 that the automorphism group for Sn is isomorphic to

$$[((\underbrace{S_n \times \cdots \times S_n}_{n \text{ times}}) \rtimes S_n) \times ((\underbrace{S_n \times \cdots \times S_n}_{n \text{ times}}) \rtimes S_n)] \rtimes \mathbb{Z}_2.$$

Note that given a Cayley graph for a group G, G is a subgroup of the automorphism group of the graph. Given groups H, K and respective elements h, k the element $(h, k) \in H \times K$ has order equal to $\mathrm{lcm}\{|h|, |k|\}$; and $(h, k) \in H \rtimes K$ has order at most $|h|\cdot|k|$.

Now, $\mathrm{Aut}(S2) \cong [((S_2 \times S_2) \rtimes S_2) \times ((S_2 \times S_2) \rtimes S_2)] \rtimes \mathbb{Z}_2$. Any element of S_2 has order at most 2; so any element of $S_2 \times S_2$ has order at most 2; any element of $(S_2 \times S_2) \rtimes S_2$ has order at most 4; any element of $((S_2 \times S_2) \rtimes S_2)^2$ has order at most 4; any element of $\mathrm{Aut}(S2)$ has order at most 8. Hence $\mathrm{Aut}(S2)$ has no element of order 16, and so no cyclic subgroup of order 16.

Since no element of $\mathbb{Z}_{81}$ has even order, we restrict ourselves to those elements of S_3 with order 1 or 3 when seeking a possible order-81 element of $\mathrm{Aut}(S3)$. Any such element has order, in $S_3 \times S_3 \times S_3$, at most 3. Any element of $(S_3)^3 \rtimes S_3$ has order at most 9; any element of $((S_3)^3 \rtimes S_3)^2$ has order at most 9. Hence $\mathrm{Aut}(S3)$ has no element of order 81, and so no subgroup isomorphic to $\mathbb{Z}_{81}$.

If $S4$ is Cayley for the cyclic group, then we seek an element of $\mathrm{Aut}(S4)$ with order 256. Since 256 is not divisible by 3, we restrict our search to those elements of S_4 with order 1, 2, or 4. Any element of $(S_4)^4$ can have order at most 4; any element of $(S_4)^4 \rtimes S_4$ has order at most 16; any element of $((S_4)^4 \rtimes S_4)^2$ has order at most 16; any element of $\mathrm{Aut}(S4)$ has order at most 32. So $\mathrm{Aut}(S4)$ has no element of order 256, and so no subgroup isomorphic to $\mathbb{Z}_{256}$.

Since none of $\mathrm{Aut}(S2), \mathrm{Aut}(S3), \mathrm{Aut}(S4)$ contain cyclic subgroups isomorphic to $\mathbb{Z}_{n^4}$, $n = 2, 3, 4$ respectively, we see that these Sn cannot be Cayley graphs for the cyclic groups of order n^4.

For additional support for this claim, we recall ([6] et al.) that any Cayley graph of a cyclic group is a circulant graph. From Sage, none of $S2, S3, S4$ is circulant [8], and so cannot be Cayley for the cyclic group. □

5 Future Research

As with most rich topics in mathematics, there are several points of investigation to pursue regarding sudoku graphs. For instance, we suspect that Sn is $(3n^2 - 2n - 1)$-connected: since Sn is $(3n^2 - 2n - 1)$-regular (see Proposition 4.1), Sn has connectivity at most $3n^2 - 2n - 1$; this seems provable using Menger's Theorem [12].

A subject of more particular interest, vis-à-vis our main results, follows from (in fact, informs) Proposition 4.20: we conjecture that Sn is not a Cayley Graph for the cyclic group of order n^4. As noted in the proposition, Sn is not circulant for $n = 2, 3, 4$, though it remains unclear if this is true for all n; further insight into why these particular graphs are not circulant may provide an avenue to proving or disproving this conjecture.

As the simplicity of the game of sudoku belies the combinatorial aspects of the game, so does the simplicity of the sudoku board give little hint to the potential complexity of the sudoku graph. Having shown that Sn is a Cayley graph for the direct product group $(\mathbb{Z}_n)^4$, we suggest that it may be beneficial to view the sudoku graphs in this group theoretic context.

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