Fractional Colorings and Zykov Products of Graphs

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Contents

1 Overview

The problem of efficiently coloring the vertices of a graph is a problem nearly as old as graph theory itself. Indeed, one of the most celebrated theorems of modern mathematics is the "4-color Theorem", which states that any planar graph can be properly colored with four colors. This famous result was conjectured in 1852 by Francis Guthrie and yet defied proof until 1976, when it was finally confirmed by Kenneth Appel and Wolfgang Haken. Their proof made essential use of a computer to generate and verify many subcases of the argument.

The interest in graph coloring, however, is more than historical. With direct applications to scheduling, the topic finds many real-world uses of great practical importance, ranging from busses, trains, traffic, and resource management to task sequencing, computer architecture and classroom space management. In recent years, there have been several theoretical advances in the subject that extend familiar results and generalize many of the techniques. One of the most significant of these is a generalization known as *fractional coloring*.

In this paper, we examine the results presented in the article "The Fractional Chromatic Number of Zykov Products of Graphs" by Charbit and Serini [4]. Before discussing the specific contributions of this paper, however, consider the motivation to study fractional colorings in the first place. We briefly mention two of the primary reasons.

First, as a generalization of the standard chromatic number, the factional chromatic number gives an immediate lower bound on the usual chomatic number. As we will describe later, the fractional chromatic number can be calculated using a technique known as *linear programming*, so computational packages exist that can be directly applied to this problem.

Second, and perhaps more importantly, the fractional chromatic number is of interest from a purely mathematical point of view because of its relationship to graph homomorphisms. The study of graph homomorphisms is relatively young, and many open questions remain (Hedetniemi's Conjecture, to name a famous one). A proper coloring of a graph, G, amounts to a homorphism from G to a complete graph, and so the study of graph colorings contributes directly to our understanding of graph homomorphisms. As generalizations of these, fractional colorings also correspond to a simple type of graph homomorphism, and this has generated great interest in their properties. Specifically, a fractional coloring is a homomorphism to a Kneser graph (cf. [1, p. 140]), and the fractional chromatic number determines the existance or non-existance of such a map.

In this paper, the focus is on how the fractional chromatic number behaves with respect to a certain product of graphs, the Zykov product. In 1949, Zykov [5] introduced a construction to show that the chromatic number of a triangle-free graph could be arbitrarily large. In the paper we consider, Charbit and Sereni generalize Zykov's construction and calculate the fractional chromatic number of the resulting graphs. In the process, they resolve a 2006 conjecture of Portland State University student Tony Jacobs in the affirmative.

2 Preliminaries

This section covers the basic definitions and ideas used throughout this paper. In order to discuss the topic of fractional colorings of graphs we first introduce some basic definitions from graph theory. From there we discuss linear programming and how it relates to finding the fractional chromatic number of a graph. To finish out this section we investigate the Zykov product of graphs.

2.1 Graphs

In this paper we will be working with finite simple graphs. That is, graphs with a finite number of vertices that contain no loops or multiple edges. With that in mind, we define a graph as follows. A *graph* G consists of two sets, a vertex set and an edge set. The set of *vertices*, $V(G)$, can be any finite set. The set of *edges*, $E(G)$, consists of unordered pairs of distinct elements in $V(G)$. An edge $\{x, y\}$ is usually denoted xy. We say two verticies $x, y \in V(G)$ are *adjacent*, denoted $x \sim y$, if xy is an element of $E(G)$. A graph is said to be *triangle-free* if no three vertices are mutually adjacent. The drawing in figure 1 below shows the famous Peterson graph. This is an example of a finite simple graph which also has the property of being triangle-free.

Given any graph G, a *subgraph* H of G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* has the further condition that two vertices are adjacent in H if and only if they are adjacent in G. It follows that a subset of vertices uniquely determines an induced subgraph. The drawing in Figure 1 shows a subgraph of the Peterson Graph induced by the five vertices on the outer edge of the graph.

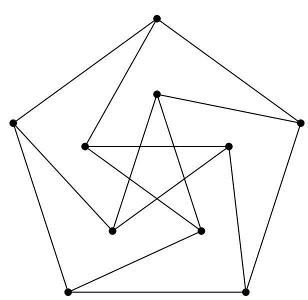


Figure 1: Peterson graph

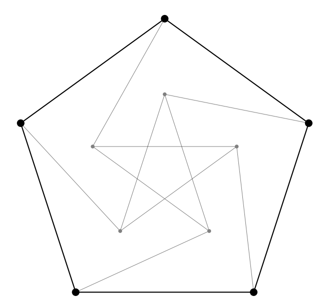


Figure 2: Induced Subgraph of Peterson graph

A set of vertices, S, is said to be *independent* if those vertices induce a graph with no edges. It follows that no two elements of an independet set are adjacent. Now, an independent set is said to be *maximal* if no other independent set properly contains it. Throughout this paper we will refer to the collection of all independent sets of a graph G using the notation $\mathscr{I}(G)$.

2.2 Fractional Chromatic Number

Before we get to the definition of fractional chromatic number, we will go over a couple definitions and examples. We use the notation given by Charbit and Sereni [4].

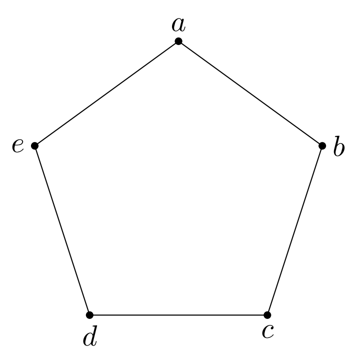
Definition 1. *A* weighting *of a set* $\mathscr{X} \subseteq \mathscr{I}(G)$ *is a function* $w : \mathscr{X} \to \mathbb{R}^{\geq 0}$. *If* $v \in V(G)$, *then*

$$w[v] = \sum_{\substack{I \in \mathscr{X} \\ v \in I}} w(I).$$

Definition 2. *A* fractional k-coloring *of* G *is a weighting of* $\mathscr{I}(G)$ *such that*

- $\sum_{S \in \mathscr{I}(G)} w(S) = k$*; and*
- $w[v] \geq 1$ *for every* $v \in V(G)$

The figure below is a drawing of a cycle on five vertices, C_5. The table on the right lists the ten independent sets contained in C_5 and an assignment of weights to each set. First, notice that for each vertex in C_5, the sum of the weights of the independent sets that contain said vertex is equal to 1. Also, the sum of the weights of all the independent sets is equal to 5/2. Hence, w is a 5/2-coloring of C_5.



Independent Sets, S	Weights, $w(S)$
{a}	0
{b}	0
{c}	0
{d}	0
{e}	0
{a,c}	1/2
{a,d}	1/2
{b,d}	1/2
{b,e}	1/2
{e,c}	1/2

Figure 3: C_5 and a corresponding weighting

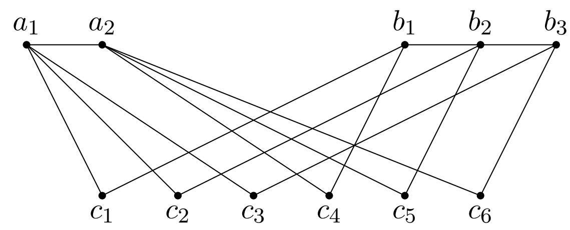
And, now, we can define the fractional chromatic number of a graph.

Definition 3. *The* fractional chromatic number, $\chi_f(G)$ *is the infimum of all positive real numbers* k *for which* G *has a fractional* k*-coloring. It is the optimum value of the linear program,*

$$\text{Minimize} \sum_{S \in \mathscr{I}(G)} w(S) \text{ where } w \text{ is a weighting of } \mathscr{I}(G) \text{ satisfying}$$
$$\forall v \in V(G), w[v] \geq 1.$$

The coloring exhibited in Figure 3 is actually an optimal coloring of C_5. So, the fractional chromatic number of the 5-cycle is 5/2.

Remark 1. *It is useful to note that the fractional chromatic number of any subgraph of a given graph,* G*, is at most* $\chi_f(G)$.



[h]

Figure 4: $\mathcal{Z}(G_1, G_2)$

2.3 Zykov Product of Graphs

The *Zykov product* $\mathcal{Z}(G_1, G_2, \ldots, G_n) = G$ of simple graphs $G_1, G_2, \ldots, G_n$ is formed as follows. Take the disjoint union of G_i and, for each possible choice of $(x_1, \ldots, x_n) \in V(G_1) \times V(G_2) \times \ldots \times V(G_n)$, add a new vertex adjacent to the set $\{x_1, x_2, \ldots, x_n\}$.

It follows that the number of veritices in $\mathcal{Z}(G_1, G_2, \ldots, G_n)$ is the sum of vertices in each G_i plus the number of ways of choosing one vertex from each graph, that is, $V(G) = \sum_{i=1}^{n} |V(G_i)| + \prod_{i=1}^{n} |V(G_i)|$. Also, we can note that the order in which the graphs are numbered makes no difference in the final construction of G.

As a example we will take two graphs and construct their Zykov product. The graphs we will start with are P_2 and P_3. The graph P_2 consists of two vertices and one edge connecting them. The graph P_3 contains three vertices and two edges. We will walk through construction the Zykov product of these graphs. Pick one vertex from P_2, say a_1 and one vertex from P_3, say b_1. Create a new vertex, c_1 adjacent to a_1 and b_1. Continue this process for each possible combination of vertices from P_2 and P_3. The figure below shows the resulting graph.

Zykov's original result concerned the following special case of the Zykov product. The *Zykov Graphs* are a family of graphs, $\{\mathcal{Z}_n\}$, where $\mathcal{Z}_1$ is a graph with one vertex and $\mathcal{Z}_{n+1} := \mathcal{Z}(\mathcal{Z}_1, \ldots, \mathcal{Z}_n)$ for $n > 1$. Zykov found that each graph $\mathcal{Z}_n$ is triangle-free and has chromatic number equal to n. In 2006, Toni Jacobs' conjectured that the fractional chromatic number of the Zykov graphs follows the formula $\mathcal{Z}_{n+1} = \mathcal{Z}_n + \frac{1}{\mathcal{Z}_n}$. The main result of this paper determines the fractional chromatic number of any Zykov product, and thereby, proves Jacob's Conjecture as a corollary.

3 Fractional Chromatic Number of Zykov Products of Graphs

In this section we present and prove the main result of the paper by Charbit and Sereni [4].

3.1 Theorem 1: Main Result

Theorem 1. *For $n \geq 2$, let $G_1, \ldots, G_n$ be graphs, and set $\chi_i = \chi_f(G_i)$. Suppose also that the graphs G_i are numbered such that $\chi_i \leq \chi_{i+1}$. Then*

$$\chi_f(\mathcal{Z}(G_1, \ldots, G_n)) = \max\left(\chi_n, 2 + \sum_{i=2}^{n} \prod_{k=i}^{n} \left(1 - \frac{1}{\chi_k}\right)\right). \tag{1}$$

Remark 2. *For the remainder of this paper we will let* $f(n) := 2 + \sum_{i=2}^{n} \prod_{k=i}^{n} \left(1 - \frac{1}{\chi_k}\right)$

3.1.1 Proof of Theorem 1: Lower Bound

In this section we prove $\chi_f(\mathcal{Z}(G_1, \ldots, G_n)) \geq \max(\chi_n, f(n))$. We begin with a lemma.

Lemma 1. *Let* G *be a graph and* w *a weighting of* $\mathscr{X} \subseteq \mathscr{I}(G)$*. Then, for every induced subgraph* H *of* G*, there exists* $x \in V(H)$ *such that*

$$w[x] \leq \frac{1}{\chi_f(H)} \sum_{S \in \mathscr{X}} w(S). \tag{2}$$

Proof. Let w_H be a weighting of $\mathscr{I}(H)$ such that

$$w_H(I) = \sum_{\substack{S \in \mathscr{X} \\ S \cap V(H) = I}} w(S)$$

for all $I \in \mathscr{I}(H)$. First we establish that $w_H[v] = w[v]$ for all $v \in V(H)$. By definition,

$$\begin{aligned} w_H[v] &= \sum_{\substack{I \in \mathscr{I}(H) \\ v \in I}} w_H(I) \\ &= \sum_{\substack{I \in \mathscr{I}(H) \\ v \in I}} \sum_{\substack{S \in \mathscr{X} \\ S \cap V(H) = I}} w(S) \\ &= \sum_{\substack{I \in \mathscr{X} \\ v \in I}} w(S) \end{aligned}$$

where the last equality follows from the fact that $v \in V(H)$ and

$$\{S \in \mathscr{X} | v \in S\} = \bigcup_{\substack{I \in \mathscr{I}(H) \\ v \in I}} \{S \in \mathscr{X} | S \cap V(H) = I\}.$$

Now, set $m := \min_{v \in V(H)} w_H[v]$. Then, it suffices to show that $m \leq \frac{1}{\chi_f(H)} \sum_{S \in \mathscr{X}} w(S)$. Now we construct a k-coloring of H. Define $w' := \frac{1}{m} \cdot w_H$ and note that

$$w'[v] = \frac{1}{m} w_H[v] = \frac{w_H[v]}{\min_{v \in V(H)} w_H[v]} \geq 1.$$

Then w' is a fractional k-coloring of H for $k = \frac{1}{m} \sum_{S \in \mathscr{I}(H)} w_H(S)$.

It follows that $k \geq \chi_f(H)$ since $\chi_f(H)$ is defined to be the minimum of all such colorings. Moreover, by definition of w_H

$$\sum_{S \in \mathscr{I}(H)} w_H(S) = \sum_{S \in \mathscr{X}} w(S).$$

Combining $k \geq \chi_f(H)$ with the previous inequality we get the desired result,

$$m \leq \frac{1}{\chi_f(H)} \sum_{S \in \mathscr{X}} w(S).$$

□

We use this result to find a lower bound of $\chi_f(G)$. First, as in Remark 1, $\chi_f(G) \geq \chi_f(G_n) = \chi_n$ since G_n is a subgraph of $\mathcal{Z}(G_1, \ldots, G_n)$. It remains to show that $\chi_f(G) \geq f(n)$. In order to do this we will define a sequence of sets contained in $\mathscr{I}(G)$ and recursively construct the desired lower bound.

Lemma 2. *If* $G \coloneqq \mathcal{Z}(G_1, ...G_n)$ *and* $f(n)$ *is as defined in Remark 2, then*

$$\chi_f(G) \geq f(n). \tag{3}$$

Proof. Let w be a $\chi_f(G)$-coloring of G and let $x_1 \in V(G_1)$. Next, set

$$\mathcal{F}_1 = \{S \in \mathscr{I}(G) : x_1 \in S\},$$

and note that, by Definition 1,

$$\sum_{S \in \mathcal{F}_1} w(S) = w[x_1] \geq 1. \tag{4}$$

We may now apply Lemma 1 with $H = G_2$ and $\mathscr{X} = \mathcal{F}_1$, to find a $x_2 \in V(G_2)$ such that

$$\sum_{\substack{S \in \mathcal{F}_1 \\ x_2 \in S}} w(S) \leq \frac{1}{\chi_2} \sum_{S \in \mathcal{F}_1} w(S).$$

Recall, from the definition of fractional chromatic number, that $w[x] \geq 1$ for all $x \in V(G)$. Combining this fact with the previous inequality gives,

$$w[x_2] - \sum_{\substack{S \in \mathcal{F}_1 \\ x_2 \in S}} w(S) \geq 1 - \frac{1}{\chi_2} \sum_{S \in \mathcal{F}_1} w(S)$$

.

Notice that the left hand side of the inequality can be written as,

$$w[x_2] - \sum_{\substack{S \in \mathcal{F}_1 \\ x_2 \in S}} w(S) = \sum_{\substack{S \in \mathscr{I}(G) \\ x_2 \in S}} w(S) - \sum_{\substack{S \in \mathcal{F}_1 \\ x_2 \in S}} w(S) = \sum_{\substack{S \in \mathscr{I}(G) \setminus \mathcal{F}_1 \\ x_2 \in S}} w(S)$$

Now, we will define a new set contained in $\mathscr{I}(G)$,

$$\mathcal{F}_2 = \{S \in \mathscr{I}(G) : S \cap \{x_1, x_2\} \neq \emptyset\},$$

and construct a new inequality as follows,

$$\begin{aligned}
\sum_{S\in\mathcal{F}_2} w(S) &= \sum_{\substack{S\in\mathscr{I}(G)\setminus\mathcal{F}_1\\ x_2\in S}} w(S) + \sum_{S\in\mathcal{F}_1} w(S)\\
&\geq 1 - \frac{1}{\chi_2}\sum_{S\in\mathcal{F}_1} w(S) + \sum_{S\in\mathcal{F}_1} w(S)\\
&\geq 1 + \left(1 - \frac{1}{\chi_2}\right)\sum_{S\in\mathcal{F}_1} w(S). \qquad (5)
\end{aligned}$$

By the same argument, we can find, $x_i \in V(G_i)$ and

$$\mathcal{F}_i = \{S \in \mathscr{I}(G) : S \cap \{x_1, \dots, x_i\} \neq \emptyset\}$$

for all $i \leq n$, such that for each $k \in \{1, 2, ..., n\}$

$$\sum_{S\in\mathcal{F}_k} w(S) \geq 1 + \left(1 - \frac{1}{\chi_k}\right)\sum_{\mathcal{F}_{k-1}} w(S). \qquad (6)$$

We can now combine these inequalities for each k so that,

$$\sum_{S\in\mathcal{F}_n} w(S) \geq 1 + \sum_{i=2}^{n}\prod_{k=i}^{n}\left(1 - \frac{1}{\chi_k}\right) = f(n) - 1.$$

By the way G is constructed, there exists a vertex $x \in V(G)$ that is adjacent to all the vertices $x_1, x_2, \dots, x_n$. So, an independent set in G that contains x cannot be in $\mathcal{F}_n$ by construction. And, since $w[x] \geq 1$, we can conclude,

$$\sum_{I\in\mathscr{I}(G)} w(S) \geq w[x] + \sum_{I\in\mathcal{F}_n} w(I) \geq 1 + (f(n) - 1).$$

Therefore, $\chi_f(G) \geq f(n)$. □

We have established that both $\chi_f(G) \geq f(n)$ and $\chi_f(G) \geq \chi_n$. Hence $\chi_f(G) \geq \max(\chi_n, f(n))$, as desired.

3.1.2 Proof of Theorem 1: Upper Bound

To prove the upper bound, we will construct a k-coloring of G with $k = \max(\chi_n, f(n))$, where $f(n)$ is defined as in Remark 2 . Then we may immediately conclude that $\chi_f(G) \leq \max(\chi_n, f(n))$.

To get started, let us set $V_i \coloneqq V(G_i)$ for $i \in \{1, 2, \dots, n\}$ and $V_0 = V(G) \setminus \cup_{i=1}^{n} V_i$. Also, set $\mathscr{I}_i \coloneqq \mathscr{I}(G_i)$.

The weighting that will define our factional k-coloring will assign nonzero weight to only a subset of all the independent sets of G. The following lemma describes some important properties of such sets.

Lemma 3. *Let $\mathscr{M}(G) \subset \mathscr{I}(G)$ be the set of all maximal independent sets of G. If*

$$\mathcal{F}_i := \{S \in \mathscr{M}(G) | S \cap V_i = \emptyset \text{ if and only if } j < i \ , \forall j \in \{1, \ldots, n\}\}$$

and

$$\mathcal{F} := \bigcup_{i=1}^{n} \mathcal{F}_i$$

then

(i) $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$ if $i \neq j$

(ii) There is a one to one correspondence between $\mathcal{F}_j$ and $\cup_{i>j} \mathscr{I}_i \backslash \{\emptyset\}$.

Proof. To prove (i), let us suppose to the contrary, that there exists $S \in \mathcal{F}_i \cap \mathcal{F}_j$ for $i > j$. Then $S \in \mathcal{F}_i$ implies $S \cap V_k = \emptyset$ for all $k < i$. In particular, $S \cap V_{j+1} = \emptyset$. It follows that $S \notin \mathcal{F}_j$, a contradiction. Therefore, the elements of $\{\mathcal{F}_i\}$ are pairwise disjoint. Now, to prove (ii), note that, by construction of G, V_0 is an independent set and that no edges join V_i and V_j if $i \neq j$ and $i, j \in \{1, 2, \ldots, n\}$. So, every maximal independent set S of G is determined by its intersection with the sets V_i, that is, $S \cap V_0$ consists of vertices that are not adjacent to any vertices in $\cup_{i=1}^{n} (S \cap V_i)$. It follows that there is a one-to-one correspondence between $\mathcal{F}_j$ and $\cup_{i>j} \mathscr{I}_i \backslash \{\emptyset\}$. □

We will now define a weighting p of $\mathcal{F}$ as a product of weightings of each G_i. Let w_i be a χ_i-coloring of G_i such that $w_i[v] = 1$ for every $v \in V_i$ and $w(\emptyset) = 0$. Now define a new weighting $p_i : \mathscr{I}_i \to \mathbb{R}^{\geq 0}$ such that $p_i(S) = w_i(S)/\chi_i$ for all nonempty S and $p_i(\emptyset) := 1$. It is advantegeous to note that

$$\sum_{S \in \mathscr{I}_i \backslash \{\emptyset\}} p_i(S) = \frac{1}{\chi_i} \sum_{S \in \mathscr{I}_i \backslash \{\emptyset\}} w_i(S) = \chi_i / \chi_i = 1$$

and

$$\forall x \in V_i, \quad p_i[x] = \frac{1}{\chi_i}.$$

We can now define p as

$$\begin{array}{rcl} p : \mathcal{F} & \longrightarrow & \mathbb{R}^{\geq 0} \\ S & \longmapsto & \prod_{i=1}^{n} p_i(S \cap V_i). \end{array}$$

As a lemma, we state some useful properties of p,

Lemma 4. *Let $i, j \in \{1, \ldots, n\}$. The weighting p satisfies the following.*

(i)

$$\sum_{S \in \mathcal{F}_i} p(S) = 1.$$

(ii) For each $x \in V_j$,

$$\sum_{\substack{S \in \mathcal{F}_i \\ x \in S}} p(S) = \begin{cases} \frac{1}{\chi_j} & , \ i \leq j \\ 0 & , \ i > j \end{cases}$$

(iii) For each $(x_1, x_2, \ldots, x_n) \in V_1 \times V_2 \times, \ldots \times V_n$,

$$\sum_{\substack{S \in \mathcal{F}_i \\ S \cap \{x_1, x_2, \ldots, x_n\} = \emptyset}} p(S) = \prod_{k=i}^{n} \left(1 - \frac{1}{\chi_k}\right).$$

Proof. To prove (i) we first note that

$$\begin{aligned} \sum_{S \in \mathcal{F}_i} p(S) &= \sum_{S \in \mathcal{F}_i} \prod_{k=1}^{n} p_k(S \cap V_k) \\ &= \sum_{S \in \mathcal{F}_i} \prod_{k=i}^{n} p_k(S \cap V_k) \end{aligned}$$

since $S \cap V_k = \emptyset$ for $k < i$ and $p_k(\emptyset) = 1$. It follows from part (ii) of Lemma 3 that

$$\sum_{S \in \mathcal{F}_i} \prod_{k=i}^{n} p_k(S \cap V_k) = \prod_{k=i}^{n} \sum_{S \in \mathscr{I}_k \setminus \{\emptyset\}} p_k(S) = 1$$

which proves (i).

To prove (ii), let $x \in V_j$ and $i \leq j$. By the reasoning given above we can swap the product and sum sign to get,

$$\begin{aligned}
\sum_{\substack{S\in\mathcal{F}_i\\ x\in S}} p(S) &= \sum_{\substack{S\in\mathcal{F}_i\\ x\in S}} \prod_{k=1}^{n} p_k(S\cap V_k)\\
&= \sum_{\substack{S\in\mathcal{F}_i\\ x\in S}} \prod_{k=i}^{n} p_k(S\cap V_k)\\
&= \left(\sum_{\substack{S\in\mathscr{I}_j\\ x\in S}} p_j(S)\right) \prod_{\substack{k=i\\ k\neq j}}^{n} \sum_{S\in\mathscr{I}_k\setminus\{\emptyset\}} p_k(S)\\
&= \frac{1}{\chi_j}
\end{aligned}$$

Furthermore, if $i > j$, then no element of $\mathcal{F}_i$ intersects V_j. That is, the set $\{S \in \mathcal{F}_i : x \in S\}$ is empty. Hence $\sum_{\substack{S\in\mathcal{F}_i\\ x\in S}} p(S)$ is equal to zero.

To prove (iii), let $(x_1, x_2, \ldots, x_n) \in V_1 \times V_2 \times, \ldots \times V_n$. To begin, note that by part (i) and (ii) of this lemma, for any $k \in \{1, \ldots, n\}$,

$$\sum_{\substack{S\in\mathscr{I}_k\setminus\{\emptyset\}\\ x_k\notin S}} p_k(S) = \sum_{S\in\mathscr{I}_k\setminus\{\emptyset\}} p_k(S) - \sum_{\substack{S\in\mathscr{I}_k\setminus\{\emptyset\}\\ x_k\in S}} p_k(S) \tag{7}$$

$$= 1 - \frac{1}{\chi_k}. \tag{8}$$

It follows from similar reasoning as in the proofs of part (i) and (ii) that

$$\begin{aligned}
\sum_{\substack{S\in\mathcal{F}_i\\ S\cap\{x_1,x_2,\ldots,x_n\}=\emptyset}} p(S) &= \sum_{\substack{S\in\mathcal{F}_i\\ S\cap\{x_1,x_2,\ldots,x_n\}=\emptyset}} \prod_{k=1}^{n} p_k(S\cap V_k)\\
&= \prod_{k=i}^{n}\left(\sum_{\substack{S\in\mathscr{I}_k\setminus\{\emptyset\}\\ x_k\notin S}} p_k(S)\right)\\
&= \prod_{k=i}^{n}\left(1-\frac{1}{\chi_k}\right).
\end{aligned}$$

where the last line follows from Equation 7. □

Now, we will define our final weighting w of $\mathscr{I}(G)$ and show that it is a $\max(\chi_n, f(n))$-coloring of G. Since $\chi_f(G)$ is the minimum value of such colorings, we will have established that $\chi_f(G) \leq \max(\chi_n, f(n))$ as desired.

It is useful to note that $\mathscr{I}(G)$ can be partitioned, by Lemma 3, so that $\mathscr{I}(G) = \mathcal{F}_1 \cup \ldots \mathcal{F}_n \cup V_0 \cup \mathscr{I}(G) \backslash (\mathcal{F} \cup V_0)$ where the sets in the union are pairwise disjoint. Let $w : \mathscr{I}(G) \longrightarrow \mathbb{R}^{\geq 0}$ such that

$$w(S) = \begin{cases} (\chi_i - \chi_{i-1})p(S), & S \in \mathcal{F}_i, \forall i \in \{1, ..., n\} \\ \max(0, f(n) - \chi_n), & S = V_0 \\ 0, & \text{otherwise} \end{cases}$$

Next we will prove the w satisfies the properties of a $\max(\chi_n, f(n))$-coloring.

By lemma $4(i)$, we have

$$\begin{aligned} \sum_{S \in \mathscr{I}(G)} w(S) &= \sum_{S \in \mathcal{F}} w(S) + w(V_0) + 0 \\ &= \sum_{S \in \mathcal{F}} (\chi_i - \chi_{i-1})p(S) + w(V_0) \\ &= \sum_{i=1}^{n} \sum_{S \in \mathcal{F}_i} (\chi_i - \chi_{i-1})p(S) + w(V_0) \\ &= \sum_{i=1}^{n} (\chi_i - \chi_{i-1}) \left(\sum_{S \in \mathcal{F}_i} p(S) \right) + \max(0, f(n) - \chi_n) \\ &= \chi_n + \max(0, f(n) - \chi_n) \\ &= \max(\chi_n, f(n)). \end{aligned}$$

The next property that we will show is that $w[x] \geq 1$ for all $x \in V(G)$. We split the proof into two cases. First, for each $x \in V_j$, Lemma $4(ii)$ gives us that,

$$\begin{aligned} w[x] &= \sum_{\substack{S \in \mathscr{I}(G) \\ x \in S}} w(S) \\ &= \sum_{\substack{S \in \mathcal{F} \\ x \in S}} (\chi_i - \chi_{i-1})p(S) \\ &= \sum_{i=1}^{j} (\chi_i - \chi_{i-1}) \sum_{\substack{S \in \mathcal{F}_i \\ x \in S}} p(S) \\ &= \frac{1}{\chi_i} \cdot \sum_{i=1}^{j} (\chi_i - \chi_{i-1}) \\ &= 1. \end{aligned}$$

Next, if $x \in V_0$ then we can find its set of neighbors in G, $(x_1, x_2, \ldots, x_n) \in V_1 \times V_2 \times, \ldots \times V_n$. So, by Lemma 4(iii),

$$
\begin{aligned}
w[x] &= w(V_0) + \sum_{\substack{S \in \mathcal{F} \\ S \cap \{x_1, \ldots, x_n\} = \emptyset}} w(S) = w(V_0) + \sum_{i=1}^{n} \left((\chi_i - \chi_{i-1}) \sum_{\substack{S \in \mathcal{F}_i \\ S \cap \{x_1, \ldots, x_n\} = \emptyset}} p(S) \right) \\
&= w(V_0) + \sum_{i=1}^{n} \sum_{\substack{S \in \mathcal{F}_i \\ S \cap \{x_1, \ldots, x_n\} = \emptyset}} w(S) \\
&= w(V_0) + \sum_{i=1}^{n} \left((\chi_i - \chi_{i-1}) \prod_{k=i}^{n} \left(1 - \frac{1}{\chi_k}\right) \right) \\
&= w(V_0) + \sum_{i=1}^{n} \left(\chi_i \prod_{k=i}^{n} \left(1 - \frac{1}{\chi_k}\right) \right) - \sum_{i=1}^{n} \left((\chi_{i-1} - 1) \prod_{k=i}^{n} \left(1 - \frac{1}{\chi_k}\right) \right) - \sum_{i=1}^{n} \prod_{k=i}^{n} \left(1 - \frac{1}{\chi_k}\right) \\
&= w(V_0) + \sum_{i=1}^{n} \left((\chi_i - 1) \prod_{k=i+1}^{n} \left(1 - \frac{1}{\chi_k}\right) \right) - \sum_{i=1}^{n} \left((\chi_{i-1} - 1) \prod_{k=i}^{n} \left(1 - \frac{1}{\chi_k}\right) \right) - \sum_{i=1}^{n} \prod_{k=i}^{n} \left(1 - \frac{1}{\chi_k}\right) \\
&= w(V_0) + \chi_n - 1 - \sum_{i=2}^{n} \prod_{k=i}^{n} \left(1 - \frac{1}{\chi_k}\right) \\
&= \max(0, f(n) - \chi_n) + \chi_n + 1 - f(n) \\
&\geq 1.
\end{aligned}
$$

Therefore, w is a fractional $\max(\chi_n, f(n))$-coloring of G, and the proof is complete.

3.2 Jacobs' Conjecture Proved

We now use Theorem 1 to prove the Conjecture made by Jacobs concerning the Zykov graphs.

Corollary 1. *For every $n \geq 2$*

$$\chi_f(\mathcal{Z}_{n+1}) = \chi_f(\mathcal{Z}_n) + \frac{1}{\chi_f(\mathcal{Z}_n)}$$

Proof. As before, for $n \geq 1$ we set $\chi_n := \chi_f(\mathcal{Z}_n)$ and $f(n) := 2 + \sum_{i=2}^{n} \prod_{k \geq i} (1 - \frac{1}{\chi_k})$. Notice that $f(n)$ can be written as follows,

$$
\begin{aligned}
f(n) &= 2 + \sum_{i=2}^{n} \prod_{k \geq i} \left(1 - \frac{1}{\chi_k}\right) \\
&= 2 + \left(1 - \frac{1}{\chi_n}\right) \cdot \sum_{i=2}^{n-1} \prod_{k \geq i} \left(1 - \frac{1}{\chi_k}\right) + \left(1 - \frac{1}{\chi_n}\right) \\
&= 2 + \left(1 - \frac{1}{\chi_n}\right) \cdot \left(\sum_{i=2}^{n-1} \prod_{k \geq i} \left(1 - \frac{1}{\chi_k}\right) + 1\right) \\
&= 2 + \left(1 - \frac{1}{\chi_n}\right) \cdot (f(n-1) - 2 + 1) \\
&= 2 + \left(1 - \frac{1}{\chi_n}\right) \cdot (f(n-1) - 1)
\end{aligned}
$$

for all $n \geq 2$. We procede by induction on $n \geq 2$ to show that $\chi_n = f(n-1) = \chi_{n-1} + \chi_{n-1}^{-1}$. We first consider the base case where $n = 2$. We can immediately note from our previous work and from definitions that $\chi_1 = 1$ and $f(1) = 2 = \chi_2$. Now for the induction hypothesis, suppose that $\chi_n = f(n-1)$ for any $n \geq 2$. It follows that

$$
\begin{aligned}
f(n) &= 2 + (1 - \frac{1}{\chi_n}) \cdot (f(n-1) - 1) \\
&= 2 + (1 - \frac{1}{\chi_n}) \cdot (\chi_n - 1) \\
&= \chi_n + \frac{1}{\chi_n}
\end{aligned}
$$

Since $\chi_{n+1} \geq \chi_n$, we can conclude, by Theorem 1, that $\chi_{n+1} = f(n)$. Hence, for all $n \geq 2$

$$
\chi_{n+1} = \chi_n + \frac{1}{\chi_n}.
$$

□

References

[1] C. D. Godsil and G. Royle. *Algebraic Graph Theory*. Springer, New York, 2001.

[2] D. B. West. *Introduction to Graph Theory* Prentice Hall, Second Edition, 2000

[3] T. Jacobs. *Fractional Colourings and the Mycielski graphs*. Master thesis, Protland State University, 2006.

[4] P. Charbit and J. Serini. The fractional chromatic number of Zykov products of graphs. *Applied Math. Letters*. 24, (4):432-437, 2011.

[5] A. A. Zykov. On some properties of linear complexes (russian). *Mat. Sbornik,* 24(66):163-188, 1949.