Destroying Automorphisms in Trees

By

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1. Introduction

Graph theory is a study of structure of objects (vertices) and connections between them (edges). The study began in 1736 when Euler wrote a paper on the seven bridges of Königsburg addressing the problem of whether or not a person could cross all seven bridges visiting every quarter of town, and return home without retracing his or her steps. Seemingly disparate problems about coloring maps, orienting molecules, connecting computer systems and tracking the spread of disease have all been addressed using the tools of graph theory. One of the main avenues of study, algebraic graph theory, considers the symmetries of a given graph and the group that these mappings form. The ideas in this paper arise from this field and address a simple problem that surprisingly has not been studied in much depth.

The question is simply to determine how many vertices of a graph you would have to fix in order to destroy any symmetry that the graph may have. The answer has been used in some surprising ways. In [**10**] Lynch attempts to find a minimal set of colored dots on the surface of a sphere so that given a two dimensional image of the sphere, one can tell the orientation with certainty. The dots can be drawn as a graph (a platonic solid) and the question becomes how many vertices do you need to fix in order for the graph to have no symmetries. Symmetry destruction is also closely related to distinguishing vertices and resolving location in a graph. Chartrand et. al. [**3**] relate the resolvability of a graph to drug design and Khuller et. al. [**9**] use these ideas to program robots to navigate in euclidean space.

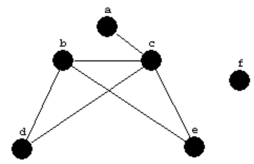
The present work is based on the paper "Destroying Automorphisms by Fixing Nodes" by David Erwin and Frank Harary [**4**] which was published in 2006. Harary dealt with similar topics in [**8**] published in 1976 and it seems he revisited the problem shortly before his death in 2005. I have followed the structure of [**4**] with some added background information, proofs, and examples.

The first section is necessarily mostly definitions and preliminary material in which the basics of graph theory are introduced, as well as the idea of fixing sets and the fixing number of a graph. Once we have these definitions, some results are immediate. The fixing number of disconnected graphs relies on the fixing number of its connected components, and the relationship is simple, albeit hard to express in a formula. It is also easy to see some bounds on the fixing number of a graph by relating it to a metric basis, to the orbits, and to the chromatic number of that graph. We then consider some examples of graphs with small fixing number, and give a construction for a graph with any given fixing number and arbitrarily large automorphism group. Then, narrowing our focus to examine graphs that contain no cycles (trees), we find that by considering the "center" of a tree, we can deduce several interesting facts about fixing relationships among vertices. In particular, we refine the orbit partition of vertices by introducing interchange equivalence classes (IECs). This allows us to formulate and prove the main theorem, that the fixing number of a tree is completely determined by sets that dominate the IECs in the fixing digraph. Once this is proved, we put it to some use in order to relate, for trees, the fixing number to the size of the automorphism group, and we consider what information we need in order to compute the fixing number of a graph using our main result.

2. Preliminaries

Definition 2.1. A graph G consists of a finite nonempty set $V(G)$ of *vertices* together with a set $E(G)$ (possibly empty) of *edges*, which are unordered pairs of distinct vertices. We say that two vertices u and v are *adjacent* (denoted $u \sim v$) if the pair $\{u, v\} \in E(V)$.

Figure 1: A graph with $V(G) = \{a, b, c, d, e, f\}$ and
$E(G) = \{\{a, c\}, \{b, c\}, \{c, e\}, \{b, e\}, \{b, d\}, \{d, c\}\}$



DEFINITION 2.2. A *path* in a graph G is a sequence P of distinct vertices $v_1, v_2, ..., v_n$ such that for each $i = 1, ...n - 1$, v_i is adjacent to v_{i+1}. We refer to v_1 and v_n as the *endpoints* of P, and P is referred to as a v_1v_n*-path.* The *length* of a path P is the number of consecutive edges that the path contains (the number of vertices in the path minus one). We denote the graph consisting of a path of length n by P_n. In Figure 1, the sequence a, c, d is an ad-path of length 2.

DEFINITION 2.3. A *cycle* is a path that begins and ends at the same vertex (here the condition that a path consist of all distinct vertices is weakened slightly). We denote the graph consisting of a cycle of length n by C_n. In Figure 1, the sequence b, c, d, b is a cycle.

DEFINITION 2.4. Given two vertices u and v in a graph G, the *distance* between u and v (denoted $d(u, v)$) is the minimum length of a path containing u and v. In Figure 1 the distance between a and b is two ($d(a, b) = 2$). The distance between a vertex and a set S of vertices (denoted $d(v, S)$) is the minimum of the distances between v and vertices in S.

DEFINITION 2.5. A graph G is *connected* if for every $u, v \in V(G)$, there is a uv-path in G. A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *component* of a graph G is a maximal connected subgraph of G. The graph in Figure 1 is not connected; its two components are the subgraph with vertex set $\{f\}$ and empty edge set and the subgraph with vertices $\{a, b, c, d, e\}$ with edge set $E(G)$.

Given two graphs G and H, a bijection ϕ from $V(G)$ to $V(H)$ is an *isomorphism* if $\{u, v\} \in E(G)$ if and only if $\{\phi(u), \phi(v)\} \in E(H)$ (ϕ preserves adjacency). Two graphs G and H are *isomorphic* (denoted $G \cong H$) if there exists an isomorphism from G to H.

DEFINITION 2.6. An *automorphism* of a graph G is an isomorphism from G to G. The set of automorphisms of a graph G form a group under the composition operation with identity *id*, the automorphism that sends every vertex to itself. The automorphism group of G is denoted $\Gamma(G)$.

THEOREM 2.7. *Every automorphism is also an isometry. That is for* $u, v \in V(G)$ *and* $\phi \in \Gamma(G)$, $d(u, v) = d(\phi(u), \phi(v))$.

PROOF. Let G be a graph, let $u, v \in V(G)$, let P be a uv-path of length $d = d(u, v)$, and let $\phi \in \Gamma(G)$. Then ϕ is a bijection and preserves adjacency, so $\phi(P)$ is a $\phi(u)\phi(v)$-path of length d. Suppose that $d(\phi(u), \phi(v)) < d$. Then there is a $\phi(u)\phi(v)$-path P' with length less than d. Since $\Gamma(G)$ is a group, ϕ^{-1} is an automorphism of G, so $\phi^{-1}(P')$ is a uv-path with length less than d. This is a contradiction, so $d(\phi(u), \phi(v)) \geq d(u, v)$. The reverse inequality is obtained similarly and the result follows. $\square$

DEFINITION 2.8. Given a graph G, let $S \subseteq V(G)$ and $\phi \in \Gamma(G)$. The automorphism ϕ is said to *fix* the set S if for every $v \in S$, we have $\phi(v) = v$. The set of automorphisms that fix S is called the *stabilizer* of S and is denoted $\Gamma_S(G)$. The stabilizer of a single vertex $\Gamma_{\{v\}}(G)$ is written $\Gamma_v(G)$.

THEOREM 2.9. *For any $S \subseteq V(G)$, $\Gamma_S(G)$ is a subgroup of $\Gamma(G)$ and $\Gamma_S(G) = \bigcap_{v \in S} \Gamma_v(G)$.*

PROOF. To show that $\Gamma_S(G)$ is a subgroup, note that $id \in \Gamma_S(G)$ since id fixes every vertex in G. Further, $\Gamma_S(G)$ is closed under compositions since for any $\phi, \psi \in \Gamma_S(G)$ and any $v \in S$,

$$(\phi \circ \psi)(v) = \phi(\psi(v)) = \phi(v) = v,$$

so $\phi \circ \psi \in \Gamma_S(G)$. Finally to see that $\Gamma_S(G)$ is closed under taking inverses, let $\phi \in \Gamma_S(G)$. Then $\phi^{-1}(\phi(v)) = v$ for all $v \in V(G)$, so if $v \in S$ then $v = \phi^{-1}(\phi(v)) = \phi^{-1}(v)$. Therefore ϕ^{-1} fixes v. Since this holds for all $v \in S$, it follows that $\phi^{-1} \in \Gamma_S(G)$.

To show that $\Gamma_S(G) = \bigcap_{v \in S} \Gamma_v(G)$, let $\phi \in \Gamma_S(G)$. Then for all $v \in S$, $\phi(v) = v$, so $\phi \in \Gamma_v(G)$, thus $\Gamma_S(G) \subseteq \bigcap_{v \in S} \Gamma_v(G)$. For the other inclusion let $\phi \in \bigcap_{v \in S} \Gamma_v(G)$. Then for every $v \in S$, $\phi \in \Gamma_v(G)$ so ϕ fixes every vertex in S and hence $\phi \in \Gamma_S(G)$. This shows that $\bigcap_{v \in S} \Gamma_v(G) \subseteq \Gamma_S(G)$, so $\Gamma_S(G) = \bigcap_{v \in S} \Gamma_v(G)$. □

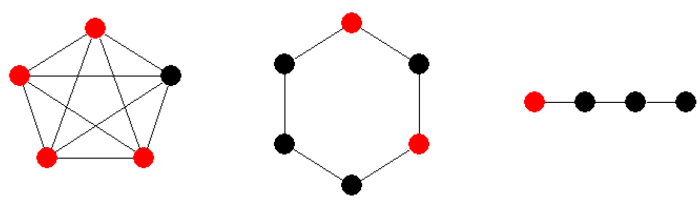
DEFINITION 2.10. If S is a set of vertices for which $\Gamma_S(G) = \{id\}$ then S *fixes* the graph G, and we say that S is a *fixing set* of G. The minimum cardinality of a set of vertices that fixes G is the *fixing number*, denoted fix(G). A fixing set containing fix(G) vertices is a *minimum fixing set* of G (or a fix(G)-set).

EXAMPLE 2.11. For every positive integer n,

$$\begin{aligned} \text{fix}(K_n) &= n-1 \\ \text{fix}(P_n) &= 1, \quad n \geq 2 \\ \text{fix}(C_n) &= 2, \quad n \geq 3 \end{aligned}$$

Where K_n is the complete graph (the graph with all possible edges), P_n is the path and C_n is the cycle, each with n vertices.

The graphs K_5, C_6, and P_4 with fixing sets colored red:



DEFINITION 2.12. A *coloring* of a graph G is a function c from $V(G)$ to some finite set of colors. A *proper coloring* of G is a coloring such that no two adjacent vertices are mapped to the same color.

DEFINITION 2.13. Let c be a (not necessarily proper) coloring of $V(G)$ and $\phi \in \Gamma(G)$. The automorphism ϕ is said to *fix* the coloring c if for every $v \in V(G)$, we have $c(\phi(v)) = c(v)$.

THEOREM 2.14. *The set of automorphisms that fix a coloring c is a subgroup of $\Gamma(G)$.*

PROOF. The proof is not difficult and similar to the proof of Theorem 2.9. □

DEFINITION 2.15. We denote the group of automorphisms that fix a coloring c by $\Gamma(G, c)$. If c is a coloring for which $\Gamma(G, c) = \{id\}$ then c *fixes* G, and we say that c is a *fixing coloring* of G. The minimum number of colors in a coloring that fixes G is the *chromatic fixing number* $\chi_{\text{fix}}(G)$.

The chromatic fixing number was defined independently by Albertson and Collins [**2**] and Harary [**7**] and is further studied in [**2, 13**]. We will make use of the concept in section 4.

3. The Fixing Number of Disconnected Graphs

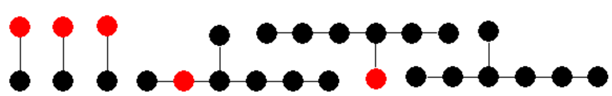
Some results on the fixing number of a graph come easily. As may be guessed, the fixing number of a disconnected graph is determined by the fixing numbers of its components. We develop a formula for this and show an example to explain its slightly complicated appearance.

DEFINITION 3.1. For any graph G and positive integer t, let tG denote the graph consisting of t pairwise disjoint copies of G. A maximal set of pairwise isomorphic components of a graph G is a *component class* of G, and if H is a component of G and ϕ an automorphism of G, then ϕ *acts nontrivially* on H if there is some $v \in V(H)$ with $\phi(v) \neq v$. A component of G that is isomorphic to H is an *H-component*.

EXAMPLE 3.2. Let T be the graph obtained from P_6 by adding a new vertex v and joining v to the third vertex of P_6. Consider the graph $G = 3K_2 \cup 3T$. Notice that $|\Gamma(K_2)| = 2$ and $|\Gamma(T)| = 1$. Let $\phi \in \Gamma(G)$ and let H be a component of G on which ϕ acts nontrivially. Then either i) $\phi(V(H)) = V(H)$, or ii) $\phi(V(H)) = V(H')$ where H' is another H-component of G. If H is isomorphic to K_2 then i) or ii) could be true, while if H is isomorphic to T, then only ii) can be true since there are no nontrivial automorphisms of T. Thus if $S \subseteq V(G)$ fixes G, then every K_2 component has at least one ($= \text{fix}(K_2)$) vertex in S, and two of the three components isomorphic to T must contain a vertex of S in order to prevent an automorphism that swaps two of these components.

Consequently, $\text{fix}(G) \geq 5$. To show that $\text{fix}(G) = 5$, one needs only to choose vertices x_1, x_2, x_3 from different K_2-components and y_1, y_2 from different T-components and verify that the set $S' = \{x_1, x_2, x_3, y_1, y_2\}$ fixes G. so $\text{fix}(G) = 5$.

The graph $G = 3K_2 \cup 3T$ and a fixing set (red)



The reasoning used in the above example can be easily generalized. This gives us a formula for the fixing number of a disconnected graph in terms of the fixing numbers of its components.

THEOREM 3.3. *Let G be a graph, $\mathcal{A}$ be the set of components X of G satisfying $|\Gamma(X)| = 1$ and let $\mathcal{B}$ be the set of components Y of G satisfying $|\Gamma(Y)| > 1$. Let k be the number of component classes in A. Then*

$$\text{fix}(G) = \sum_{Y \in \mathcal{B}} \text{fix}(Y) + |\mathcal{A}| - k.$$

Proof. Let G be as above. To fix G we construct a set S that fixes every component of G. We begin by taking a fixing subset of Y for each component Y with non-trivial automorphism group ($Y \in \mathcal{B}$). Thus far, we have $\sum_{Y \in \mathcal{B}} \text{fix}(Y)$ vertices in S. This set will fix all automorphisms that send all components to themselves. Next we need to prevent automorphisms that map isomorphic components to each other. Notice that for the components with nontrivial automorphism group we already have some vertices in S for each component. So no automorphism can swap two of these components, since such an automorphism would have to move one of our vertices in S. In other words, these components are already fixed.

To fix the other components (those in $\mathcal{A}$), given a component class, we could include in S one vertex per component this will indeed prevent any automorphisms that swap isomorphic components. This is overkill, though, since if we fix all but one component in a class, the remaining component will have no one to map to. So in each component class in $\mathcal{A}$, we need only pick one vertex in all but one component in that class. Thus, adding up all the component classes with trivial automorphism group, we see that we need $|\mathcal{A}| - k$ vertices in S in order to fix $\mathcal{A}$ Combining the two pieces we see that we need $\sum_{Y \in \mathcal{B}} \text{fix}(Y) + |\mathcal{A}| - k$ vertices in S to fix G and no fewer would suffice.. $\square$

4. Bounds on Fixing Numbers

Some upper bounds on $\text{fix}(G)$ are found easily as we will see in this section.

Definition 4.1. The number of vertices in a graph is its *order*.

Definition 4.2. Two vertices u and v in a graph G are *similar* if there exists an automorphism ϕ of G such that $\phi(u) = v$. A maximal set of vertices in a graph that are similar to each other is an *orbit*.

Lemma 4.3. *Let S be a set constructed by choosing from each orbit of G every vertex except one. Then S fixes G. Hence for every graph G having order n and α orbits under the action of* $\Gamma(G)$, $\text{fix}(G) \leq n - \alpha$.

Proof. Let S be as given and pick any $v \in V(G) \backslash S$. Then v is in some orbit Ω. If $\phi \in \Gamma(G)$ and $\phi(v) \neq v$, then $\phi(v) \in \Omega$ which implies that $\phi(v) \in S$ (there can only be one vertex in each orbit that is not in S). So $\phi(\phi(v)) = \phi(v)$. But ϕ must preserve distances, and $d(v, \phi(v)) > 0$ so $d(\phi(v), \phi(\phi(v))) > 0$. This is a contradiction, so v must be fixed by ϕ. $\square$

Since the minimum number of orbits is $\alpha = 1$, the above lemma implies that $\text{fix}(G) \leq |\Gamma(G)| - 1$ for any graph G.

Another upper bound on $\text{fix}(G)$ is given by an invariant that has been previously studied.

Definition 4.4. Let $S = \{s_1, s_2, ..., s_k\}$ be a k-subset of $V(G)$ and, for each vertex $v \in V(G)$, define the k-tuple by $r(v|S) := (d(v, s_1), d(v, s_2), ..., d(v, s_k))$. A k-set S is a *resolving set* for G if for every pair u, v of distinct vertices of G, $r(u|S) \neq r(v|S)$. The *(metric) dimension* $\dim(G)$ is the smallest cardinality of a resolving subset $S \subseteq V(G)$. A resolving set of minimum cardinality is a *metric basis* for G.

Lemma 4.5. *If S is a metric basis for G, then $\Gamma_S(G)$ is trivial.*

Proof. Let $S = \{s_1, s_2, ..., s_k\}$ and suppose, to the contrary, that there is some node u and some $\phi \in \Gamma_S(G)$ for which $u \neq \phi(u)$. Since S is a resolving set for G, there is some integer i with $1 \leq i \leq k$ such that $d(u, s_i) \neq d(\phi(u), s_i)$. However, since ϕ fixes s_i

and automorphisms preserve distance, $d(u, s_i) = d(\phi(u), \phi(s_i)) = d(\phi(u), s_i)$. This is a contraction so no such automorphism ϕ exists and $\Gamma_S(G)$ is trivial. □

THEOREM 4.6. *For every connected graph G.*

$$\chi_{\text{fix}}(G) - 1 \leq \text{fix}(G) \leq \dim(G).$$

PROOF. The previous lemma implies $\text{fix}(G) \leq \dim(G)$. For the other inequality let S be a fixing set for G with $S = \{v_1, v_2, ..., v_n\}$. Then if we color each v_i a different color and all the remaining vertices a separate color, any isomorphism that preserves colors must fix S and hence G so $\chi_{\text{fix}}(G) \leq \text{fix}(G) + 1$. □

5. Graphs With Small Fixing Number

A graph G has fixing number zero if and only if it has trivial automorphism group. Graphs with fixing number 1 can be algebraically categorized almost as easily. The following result was essentially proved by Albertson and Collins in [**1**] and relies on the well known orbit-stabilizer theorem, which we state below for the context of graph automorphisms (see Godsil [**5**] for more about this result).

THEOREM 5.1. (**Orbit-Stabilizer**) *Let G be a graph, $u \in V(G)$, and $\mathcal{O}_u$ be the orbit of $\Gamma(G)$ containing u. Then*

$$|\Gamma(G)| = |\Gamma_u(G)|\,|\mathcal{O}_u|.$$

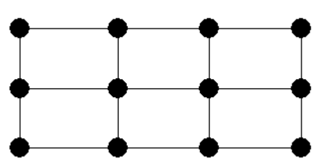
THEOREM 5.2. *Let G be a graph with $\Gamma(G) \neq \{id\}$. Then $\text{fix}(G) = 1$ if and only if G has an orbit of cardinality $|\Gamma(G)|$.*

PROOF. Suppose that G is a graph, $\Gamma(G) \neq \{id\}$ and $\text{fix}(G) = 1$. Let $u \in V(G)$ such that $\{u\}$ is a fixing set of G, let $\mathcal{O}_u$ be the orbit containing u and $\Gamma_u(G)$ be the stabilizer of u. Suppose by contradiction that $|\mathcal{O}_u| < |\Gamma(G)|$. Then by the orbit-stabilizer theorem, $\Gamma_u(G) > 1$. This means that there is some non-trivial automorphism ϕ of G that fixes u. This contradicts the fact that $\{u\}$ is a fixing set, so $|\mathcal{O}_u| = |\Gamma(G)|$.

For the converse let $\mathcal{O}$ be an orbit of G and $|\mathcal{O}| = |\Gamma(G)|$. Then by the orbit-stabilizer theorem, for any $u \in \mathcal{O}$, $|\Gamma_u(G)| = 1$. So necessarily $\Gamma_u(G) = \{id\}$ and by Definition 2.10, u fixes G. Since $\Gamma(G) \neq \{id\}$, $\text{fix}(G) \neq 0$. Thus $\text{fix}(G) = 1$. □

NOTATION 5.3. The *grid* $P_n \times P_m$ is the graph with vertex set $V(P_n \times P_m) = \{(u, v) : u \in V(P_n)\ v \in V(P_m)\}$ and edge set determined by $(u_1, v_1) \sim (u_2, v_2)$ if $u_1 = u_2$ and $\{v_1, v_2\} \in E(P_m)$ or $\{u_1, u_2\} \in E(P_n)$ and $v_1 = v_2$.

The grid $P_4 \times P_3$



THEOREM 5.4. *For every pair s, t of integers with $s, t \geq 2$,*

$$\text{fix}(P_s \times P_t) \quad = \quad \begin{cases} 2 & \text{if } s = t = 2 \text{ or } s = t = 3, \\ 1 & \text{otherwise.} \end{cases}$$

PROOF. If $s \neq t$, then $\Gamma(P_s \times P_t) = \mathbb{Z}_2 \times \mathbb{Z}_2$. The automorphisms are just the vertical and horizontal flips. The four corners (nodes of degree two) are similar, so there is an orbit of size four. Noticing that $|\Gamma(P_s \times P_t)| = |\mathbb{Z}_2 \times \mathbb{Z}_2| = 4$, Theorem 5.2 gives that the fixing number of $P_s \times P_t$ is one.

If $s = t \geq 4$ then $\Gamma(P_s \times P_t) = D_4$ (where D_4 is the dihedral group, also known as the octic group, or the group of symmetries of the square). Since $|\Gamma(P_s \times P_t)| = |D_4| = 8$, we need to find an orbit of size eight in order to use Theorem 5.2. Looking at the graph, we see that the eight vertices adjacent to the corners are similar and hence constitute the desired orbit (notice that this is not true in the case where $s = t = 2$ or $s = t = 3$, since the vertices adjacent to the corners are either corners themselves or midpoints of edges, of which there are only four). Hence by Theorem 5.2, $\text{fix}(P_s \times P_t) = 1$.

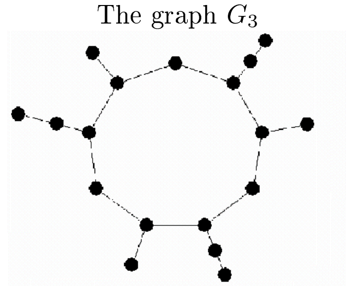
Finally to see the case where $s = t = 2$ or $s = t = 3$ we note that $\Gamma(P_s \times P_t) = D_4$, but there is no orbit of size eight. The largest orbits have size four. Hence by the "only if" statement in Theorem 5.2, $\text{fix}(P_s \times P_t) \neq 1$. We can easily observe, however, that if we fix a corner vertex and a vertex adjacent to it in $P_2 \times P_2$ or $P_3 \times P_3$, then the graph must be fixed. Hence $\text{fix}(P_s \times P_t) = 2$ in these two cases. □

We now give a construction of graphs that have arbitrarily large automorphism groups and fixing number one.

LEMMA 5.5. *For every positive integer t , there is a graph G_t with fixing number 1 and $|\Gamma(G_t)| = t$.*

PROOF. We construct a class of graphs G_t. Let $G_1 = K_1$, the graph of a single vertex and $G_2 = K_2$, the graph with two adjacent vertices. Then $|\Gamma(G_1)| = |\Gamma(G_2)| = 1$. For $t \geq 3$, let G_t be constructed as follows: start with the cycle C_{3t} labeled by $u_0, u_1, ..., u_{3t-1}$, join to each vertex u_i with $i \equiv 1 (\text{mod } 3)$ or $i \equiv 2 (\text{mod } 3)$ a new vertex w_i (this introduces $2t$ new vertices). Now for each vertex u_i with $i \equiv 1 (\text{mod } 3)$ add a new vertex by subdividing the edge $u_i w_i$ (see the figure below). The graph constructed has order $6t$ and automorphism group $\mathbb{Z}_t$. To see this, note that you start with the symmetries of the cycle C_{3t} and then eliminate the "flips" or degree two automorphisms by adding the additional vertices at an odd spacing. Also the rotations of one or two vertices are no longer automorphisms since the u_i with $i \equiv 1 (\text{mod } 3)$ are not similar to the u_i with $i \equiv 2 (\text{mod } 3)$ or $i \equiv 0 (\text{mod } 3)$. Noting that the set of vertices $\{u_i : i \equiv 0 (\text{mod } 3)\}$ is an orbit of size t, Theorem 5.2 gives that $\text{fix}(G_t) = 1$. □

The graph G_3

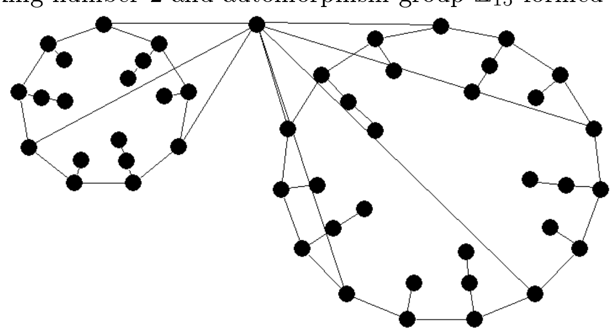


While every graph in the class constructed in Lemma 5.5 has fixing number one and cyclic automorphism group, not every graph with cyclic automorphism group has fixing number one. We give a similar construction of a family of graphs with cyclic automorphism group and arbitrary fixing number.

FACT 5.6. *For every positive integer k, there are infinitely many connected graphs with fixing number k and cyclic automorphism group.*

PROOF. When $k = 1$, Lemma 5.5 gives the desired result. Let $k \geq 2$ and $p_1, p_2, ..., p_k$ be distinct prime numbers. For each p_i with $i = 1, ..., k$ construct G_{p_i} as in Lemma 5.5. We form a new graph H from $G_{p_1}, G_{p_2}, ..., G_{p_k}$ using the following process: introduce a new vertex z, and for each G_{p_i} constructed from the cycle C_{3p_i} with the vertices on the cycle labeled $u_0, u_1, ..., u_{3p_i-1}$ connect u_j to z if $j \equiv 0 \pmod 3$. Thus we have p_i edges from z to G_{p_i}. The symmetries of each individual G_{p_i} remain intact and no symmetries between the G_{p_i} exist since $p_1, p_2, ..., p_k$ are all distinct. Thus the automorphism group of the graph H is $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times ... \times \mathbb{Z}_{p_k}$, which is isomorphic to $\mathbb{Z}_{p_1 p_2 ... p_k}$(which is cyclic). In order to fix this graph H you need to fix each G_{p_i}. Since $\text{fix}(G_{p_i}) = 1$ for every p_i, $\text{fix}(H) = k$ as desired. □

A graph with fixing number 2 and automorphism group $\mathbb{Z}_{15}$ formed from G_3 and G_5.



6. The Fixing Number of a Tree

DEFINITION 6.1. A *tree* is a connected graph that contains no cycles.

The main result of the paper is a formula for the fixing number of a tree. The formula involves some previously studied invariants, so we need some preliminary results and definitions.

DEFINITION 6.2. In a tree, an *end-vertex* is a vertex of degree one. An orbit that consists of end-vertices is called an *endorbit*. Note that every vertex similar to an end-vertex is also an end-vertex.

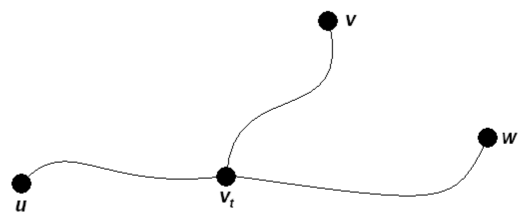
LEMMA 6.3. *Let Tbe a tree and $S \subseteq V(T)$. Then S fixes T if and only if S fixes the end-vertices of T.*

PROOF. Clearly if S fixes T, then S fixes the end-vertices of T. For the other direction, suppose by contradiction that S fixes the end-vertices of T and doesn't fix T. Then there exists a $\phi \in \Gamma_S(G)$ and $u, v \in V(G)$ with $u \neq v$ and $\phi(u) = v$. Let P be a uv-path with maximal length. The path P contains an end-vertex u' with $d(u, u') < d(v, u')$, and since S fixes the end-vertices of T, $\phi(u') = u'$. This gives that $d(\phi(u), \phi(u')) = d(v, u') > d(u, u')$ which is a contradiction since as an isometry, ϕ must preserve distances. □

DEFINITION 6.4. The *eccentricity* of a vertex u in a graph G is defined by $e(u) = \max\{d(u,v) : v \in V(G)\}$. The *radius* of G is defined by $\text{rad}(G) = \min\{e(u) : u \in V(G)\}$ and the *center* of G is the subgraph $\text{Cen}(G)$ induced by those vertices with eccentricity equal to the radius. A set S of vertices *lie between* two vertices u and v if some minimum length uv-path has nonempty intersection with S.

LEMMA 6.5. *Let u, v, w be three vertices of a tree T. If $d(u,v) = d(u,w)$, then $d(v,w)$ is even (and hence $vw \notin E(T)$).*

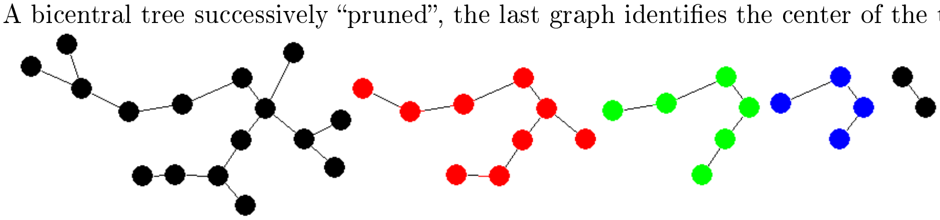
PROOF. This is obvious if the path from u to v and path from u to w are disjoint. If this is not the case, we consider the disjoint "pieces" of these paths, that must also be of equal length. Let $P : u = v_0, v_1, ..., v_k = v$ be the uv-path in the tree T. (Note that P is unique since T is a tree). Let Q be the uw-path and t be the largest integer $(0 \leq t \leq k-1)$ for which $v_t \in V(P) \cap V(Q)$. Then $d(u,v) = d(u,v_t) + d(v_t,v)$ and $d(u,w) = d(u,v_t) + d(v_t,w)$ so $d(v_t,v) = d(v_t,w)$. Since there is only one path from v to w (otherwise T would contain a cycle and would not be a tree), $d(v,w) = d(v,v_t) + d(v_t,w) = 2d(v,v_t)$ and $d(v,w)$ is even as desired. □



THEOREM 6.6. *(Jordan see* [**6**]*) The center of a tree T is either a single vertex or two connected vertices.*

PROOF. Clearly this is true of the graphs K_1 and K_2. Let T be a tree; we show that if we remove all the end-vertices of T, then the resulting tree has the same center. Let T' be the subgraph of T induced by the vertices of T that are not end-vertices. Given a vertex $u \in V(T)$, for any vertex $v \in V(T)$ if $d(u,v) = e(u)$ then v must be an end-vertex. This is true because (in a tree) there is only one uv-path and if v is not an end-vertex, then this path can be lengthened by adding some vertex v' that is adjacent to v and not in the uv-path. Thus for any vertex $u \in T'$, the eccentricity of u in T' is one less than the eccentricity of u in T. The center of a graph is defined as those vertices with minimum eccentricity so $\text{Cen}(T) = \text{Cen}(T')$. We are only dealing with finite trees so if we iterate the process of "pruning" our tree T, eventually we get down to a single point or two connected points since the only connected graphs for which each vertex has degree less than or equal to one are $K_1$and K_2. (Note that a tree has no cycles so every vertex is eventually an end-vertex). This graph will have the same center as T, so the center of T is either a single vertex or two connected vertices.

A bicentral tree successively "pruned", the last graph identifies the center of the tree.



□

DEFINITION 6.7. If the center of a tree T is a single vertex, we say that T is *central.* Otherwise T is *bicentral.* For brevity we write C for $\mathrm{Cen}(T)$.

There is a certain sense of balance that a tree has about its center. We make note of this relationship in the following lemma.

LEMMA 6.8. *Let C be the center of a tree T, $u \in V(T)$ and $d(u, V(C)) = n$. Then there exists a vertex $w \in T$ with $d(w, V(C)) = n$ such that the path between u and the center is disjoint from the path between w and the center.*

PROOF. Let T be a tree, $u \in T$ with $d(u, C) = n$ and P_u be the path of length n between u and C. Suppose by contradiction that there is no vertex $w \in T$ such that $d(w, V(C)) = n$ and the path P_w between w and C is disjoint from P_u. We use the pruning process shown in Theorem 6.6 to arrive at a contradiction. Let $T^{(0)}$ be T and let $T^{(i+1)}$ denote the subgraph of $T^{(i)}$ obtained by removing all of the end-vertices of $T^{(i)}$ for $i = 0, ..., n-1$. Consider the path P_u with vertices $u = v_0, v_1, ..., v_n$ where $v_n \in V(C)$. In $T^{(1)}$ there can be no vertex distance $n-1$ from the center with a path disjoint from P_u because if this were the case, such a vertex could not have been an end-vertex in T so there would be a vertex in T distance n from u on a path disjoint from P_u. Continuing for all $i = 0, ..., n-1$ we see that there can be no vertex disjoint from $P_u$with distance $n-i$ from the center. The end result is that in $T^{(n-1)}$ there is no vertex disjoint from P_u with distance 1 from the center. This means that if we "prune" once more, we get the graph $T^{(n)}$ which is non-empty and doesn't contain the center! This contradicts the process for finding the center outlined in Theorem 6.6. □

LEMMA 6.9. *Let u, v be adjacent vertices of a tree T. If $v \notin V(C)$, then $d(u, V(C)) \neq d(v, V(C))$. In fact $d(u, V(C))$ and $d(v, V(C))$ differ by one.*

PROOF. Again we use the fact that in a tree, there is only one path between any two vertices. v is not in the center, so there is a unique path P with length $d(v, V(C))$ from v to C, this path either contains u or not. If $u \notin P$ then the path created by adjoining the edge uv to P is a path from u to the center. This must be the only path so $d(u, V(C)) = d(v, V(C)) + 1$. On the other hand, if $u \in P$, then the path created by deleting the edge uv from P is the unique path from u to $V(C)$, so $d(u, V(C)) = d(v, V(C)) - 1$. □

LEMMA 6.10. *Let $u, v \in V(T)$ with $d(u, v) = e(u)$. Then $V(C)$ lies between u and v. (Note that if T is bicentral this only means that at least one of the central vertices must lie between u and v).*

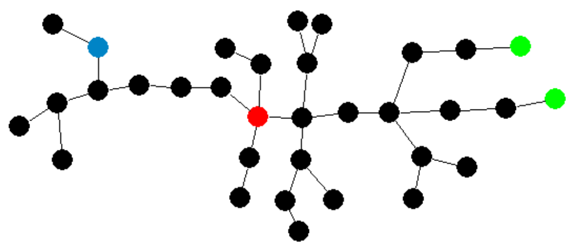
PROOF. Let $P : u = v_0, v_1, ..., v_k = v$ be the uv-path in T, P_u be the unique path between u and the center and P_v be the path between v and the center. P_u and P_v cannot be disjoint or P, P_u, P_v would be a cycle. Let $z \in P_u \cap P_v$ be the vertex in both paths that has minimum distance from u. We want z to be in P. Suppose that $z \notin P$. Then the path from z to u, along P to v and back to z contains a cycle, or the paths from z to u and from z to v overlap. In the later case, z would not have minimal distance to u. Hence z is the unique vertex in P that lies between u and the center and between v and the center. Let $j = d(u, z)$, $c \in C$, $t = d(z, V(C))$ and r be the radius of T. Then $e(c) = r$ and $d(v, c) \leq r$ (this is true for all vertices in T), so

$$d(v, c) = t + (k - j) \leq r$$

Now Lemma 6.8 gives that there is some vertex w in T with distance r from the center such that the path from w to the center is disjoint from P_u. We assumed that $e(u) = d(u, v)$ (v is as far from u as possible) so $d(u, w) \leq d(u, v)$. Adding up all the pieces we see that

$d(u,w) = j+t+r$ if T is central and $d(u,w) = j+t+r+1$ if T is bicentral. But $d(u,v) = k$, so in either case we have $j+t+r \leq k$. Combining this inequality with above equation we get $2t+r = t+(t+r) \leq t+(k-j) \leq r$, so $2t \leq 0$ and $t = 0$. Recall that t is the distance from C to z which is in P, so $z \in C$ and $V(C)$ lies between u and v as desired.

A graph with center in red, the vertex u in blue and two vertices with distance equal to $e(u)$ in green. Notice that the path from the blue vertex to either of the green vertices must go through the center.



□

Corollary 6.11. *Let u be a vertex in a tree T. Then $e(u) = rad(T) + d(u, V(C))$.*

Proof. Let u, v be vertices in a tree T with $d(u,v) = e(u)$, let P be the (unique) path between u and v. We consider two cases, depending on whether T is central or bicentral.

<u>Case I:</u> T is central. Let c be the central vertex. By Lemma 6.10 P must contain c. So we can divide P into two pieces, P_1 between u and c and P_2 between c and v. The length of V_1 is $d(u, V(C))$. By definition the length of P_2 can't be larger than $\mathrm{rad}(T)$. If the length of P_2 is less than $\mathrm{rad}(T)$ then there would be a vertex further from the center than v and hence further from u. There is one nuance here; it could conceivably be the case that any vertex with distance $\mathrm{rad}(T)$ from c shares part of the path P_1. Luckily Lemma 6.8 eliminates this possibility. By Lemma 6.8, the fact that T is central means that there are at least two vertices distance $\mathrm{rad}(T)$ from c, and they lie on disjoint paths from c. So if one of them shares part of its path to the center with u, the other won't.

<u>Case II:</u> T is bicentral. Let c_1 and c_2 be the two central vertices. By Lemma 6.10 P must contain either c_1 or c_2. Assume without loss of generality that P contains c_1 and $d(u, c_1) = d(u, V(C))$. Again we split up the path P into P_1 from u to c_1 and P_2 from c_1 to v. The length of P_1 is $d(u, V(C))$. If the length of P_2 is less than the radius, we get a contradiction because by Lemma 6.8 there must be a vertex lying on a path disjoint from P_1 that is distance $\mathrm{rad}(T)$ from the center. Such a vertex would be further from u than v and contradict the assumption that $d(u,v) = e(u)$. On the other hand, if P_2 is longer than the radius, then $d(c_1, v) > e(c_1)$ since $e(c_1)$ is equal to the radius. This contradicts the definition of eccentricity. □

Corollary 6.12. *Let u and v be vertices in a tree T. If $d(u, V(C)) < d(v, V(C))$, then $e(u) < e(v)$.*

Proof. This follows directly from Corollary 6.11. Since $d(u, V(C)) < d(v, V(C))$,

$$e(u) = \mathrm{rad}(T) + d(u, V(C)) < d(v, V(C)) + \mathrm{rad}(T) = e(v).$$

□

LEMMA 6.13. *Let u and v be two nodes of a tree T with $e(u) = e(v)$.*

i) If $d(u, v)$ is odd, then T is bicentral and both central nodes of T lie between u and v.

ii) If $d(u, v)$ is even, then T may be central or bicentral, and the middle node of the uv-path P lies between u and the center C (the middle of the uv-path also lies between v and the center).

PROOF. Let P be the path between u and v. We consider two cases, depending on whether $d(u, v)$ is even or odd.

Case I: $d(u, v)$ is odd. By Corollary 6.11, $e(u) = e(v)$ implies that $d(u, V(C)) = d(v, V(C))$. If T is central, then the path from u to the center and then to v would either be a uv-path (of even length), or contain some overlap which when removed would give a path of even length. Similarly if T is bicentral and only one of the central vertices is between u and v, then the same process would give a path of even length between u and v.

Case II: $d(u, v)$ is even. Again from Corollary 6.11, $d(u, V(C)) = d(v, V(C))$. If T is central then, as in Case I, the path from u to the center and then to v (removing any overlap) is P. If T is bicentral then the same central vertex, say c_1 must be the closest central vertex to both u and v. Otherwise, the edge between the two central vertices would be in the uv-path and since $d(u, V(C)) = d(v, V(C))$, P would have odd length. We use the same trick as above, taking the path from u to $c_1$and then to v and removing any overlap to get P. In either case, removing overlap is done evenly to both the path from u to the $c_1$and from v to c_1, so the middle of the path P must lie between both u and the center and v and the center. □

If u and v are similar vertices of a graph G, then since every automorphism preserves distances, we must have $e(u) = e(v)$.

LEMMA 6.14. *Let u, v, w be three similar vertices in a tree T and P the uv-path in T. If P has odd order, z is the middle vertex of P, and w is in the component of $T \backslash \{z\}$ containing u, then $d(u, w) < d(v, w)$. Similarly, if P has even order, e is the middle edge of P, and w is in the component of $T \backslash e$ containing u, then $d(u, w) < d(v, w)$.*

PROOF. Case I: P has odd order. Then the length of P is even and $d(u, v)$ is even so by Lemma 6.13 the middle vertex z of the uv-path lies between u and the center and between v and the center. Since w is in the component of $T \backslash \{z\}$ that contains u, z must lie between w and the center and the path between w and u must not contain z. Thus

$$d(u, w) \leq d(u, z) + d(w, z) - 2 = 2d(v, z) - 2 < 2d(v, z).$$

But, w is in the component of $T - z$ that contains u, so z is between w and the center. Also since w and v are similar, $d(w, V(C)) = d(v, V(C))$, $d(v, z) = d(w, z)$ and $d(v, w) = 2d(v, z)$. Thus the above equation gives $d(u, w) < 2d(v, z) = d(v, w)$ as desired.

Case II: P has even order. Then the length of P is odd, so by Lemma 6.13, T is bicentral and the center edge e lies between u and v. Since w is in the component of $T \backslash e$ that contains u,

$$d(u, w) \leq d(u, V(C)) + d(w, V(C)) = 2d(u, V(C)).$$

Also, $d(v, w) = d(v, V(C)) + 1 + d(w, V(C))$, so combining the two equations gives $d(u, w) < d(v, w)$ as desired. □

7. Interchange Equivalence Classes

DEFINITION 7.1. An automorphism ϕ of a graph G *interchanges* two vertices u and v if $\phi(u) = v$ and $\phi(v) = u$. Let $P : u_0, u_1, ..., u_k$ be a path in G and ϕ an automorphism of G. If for every integer $i = 1, 2, ..., k$, $\phi(u_i) = u_{k-i}$ then ϕ *flips* the path P.

LEMMA 7.2. *If an automorphism ϕ interchanges two vertices u and v in a graph G, then ϕ flips every uv-path in G.*

PROOF. This follows directly from the fact that automorphisms preserve adjacency. Suppose we are given an automorphism ϕ of a graph G that interchanges two vertices u and v and a path $P : u = u_0, u_1, ..., u_k = v$. Since $u_1$is adjacent to u, $\phi(u_1)$ must be adjacent to v. Likewise, for any $i = 1, .., k-1$, u_i adjacent to u_{i+1} implies that $\phi(u_i)$ must be adjacent to $\phi(u_{i+1})$. We know that $\phi(u_k) = \phi(v) = u$, so the image of the path P is a path between v and u. □

THEOREM 7.3. *(Prins* [**12**]*) For every pair of similar vertices u and v in a tree T, there is an automorphism of T that interchanges u and v.*

We want to refine orbits into smaller groups of vertices that are even more alike than vertices in the same orbit.

DEFINITION 7.4. Let u and v be similar vertices in an orbit $\mathcal{O}$ of a graph G. An automorphism that interchanges u and v and fixes every other vertex in $\mathcal{O}$ is a *uv-interchange.*

We want to define an equivalence relation on vertices using this concept of a uv-interchange.

DEFINITION 7.5. Define a relation $R_{\mathcal{O}}$ on vertices in an orbit $\mathcal{O}$ of a graph G by $uR_{\mathcal{O}}v$ if there is a uv-interchange.

THEOREM 7.6. *$R_{\mathcal{O}}$ is an equivalence relation on an orbit $\mathcal{O}$ of a graph G.*

PROOF. We need $R_{\mathcal{O}}$ to be reflexive, symmetric and transitive.

1) $R_{\mathcal{O}}$ is reflexive: We must show that for any vertex u in $\mathcal{O}$, $uR_{\mathcal{O}}u$. The identity automorphism is technically a uu-interchange, so $R_{\mathcal{O}}$ is reflexive

2) $R_{\mathcal{O}}$ is symmetric: Suppose that $uR_{\mathcal{O}}v$. Then there is a uv-interchange ϕ such that $\phi(u) = v$, $\phi(v) = u$ and ϕ leaves every other vertex in $\mathcal{O}$ fixed. Hence by definition ϕ is also a vu-interchange.

3) $R_{\mathcal{O}}$ is transitive: Suppose that f is a uv-interchange and g is a vw-interchange. Consider the automorphism fgf. Notice that $fgf(u) = w$, $fgf(v) = v$,$fgf(w) = u$ and all other vertices in $\mathcal{O}$ are fixed, so fgf is a uw-interchange. □

DEFINITION 7.7. Two vertices u and v in an orbit $\mathcal{O}$ are in the same *interchange equivalence class (IEC)* if $uR_{\mathcal{O}}v$. Every vertex u in a graph G is in some orbit $\mathcal{O}_u$ and hence in some interchange equivalence class. Let $\bar{u}$ denote the IEC containing u. The set of IECs of $\mathcal{O}$ under the relation $R_{\mathcal{O}}$ is written $\mathcal{O}/R_{\mathcal{O}}$.

LEMMA 7.8. *For all $\phi \in \Gamma(G)$, $aR_{\mathcal{O}}b$ if and only if $\phi(a)R_{\mathcal{O}}\phi(b)$.*

PROOF. For the "only if" direction of the proof, assume that $aR_{\mathcal{O}}b$ in a tree T and f is an ab-interchange. Consider the automorphism $\phi f \phi^{-1}$. Notice that $\phi f \phi^{-1}(\phi(a)) = \phi(b)$, $\phi f \phi^{-1}(\phi(b)) = \phi(a)$ and if c is a vertex in $\mathcal{O}$ with $c \neq \phi(a)$ and $c \neq \phi(b)$ then $\phi f \phi^{-1}(c) = c$ since $f(\phi^{-1}(c)) = \phi^{-1}(c)$. Hence $\phi f \phi^{-1}$ is an $\phi(a)\phi(b)$-interchange and $\phi(a)R_{\mathcal{O}}\phi(b)$.

For the other direction of the proof, assume that $\phi(a)R_{\mathcal{O}}\phi(b)$. Then there exists a $\phi(a)\phi(b)$-interchange g. The automorphism $\phi^{-1}g\phi$ will then be an ab-interchange since

$\phi^{-1}g\phi(a) = b$, $\phi^{-1}g\phi(b) = a$ and all other vertices in the orbit $\mathcal{O}$ will remain fixed by $\phi^{-1}g\phi$. $\square$

The following corollary describes the refinement of the orbit partition into IECs.

COROLLARY 7.9. *Let G be a graph, $\mathcal{O}$ an orbit in G, $A, B \in \mathcal{O}/R_{\mathcal{O}}$. Then $|A| = |B|$ and for every $\phi \in \Gamma(G)$, either $\phi(A) = B$ or $\phi(A) \cap B = \emptyset$.*

PROOF. Since $A, B \in \mathcal{O}/R_{\mathcal{O}}$ if $a \in A$, $b \in B$ then $a, b \in \mathcal{O}$, so there is an automorphism ϕ with $\phi(a) = b$. We want to show that $\phi(A) = B$. If $c \in A$ then $cR_{\mathcal{O}}a$, so by Lemma 7.8 $\phi(c)R_{\mathcal{O}}\phi(a) = b$ so $\phi(A) \subseteq B$. If $d \in B$, then since ϕ is onto, there exists a $d' \in V(G)$ such that $\phi(d') = d$. Since $d \in B$, $dR_{\mathcal{O}}b$, so $\phi(d')R_{\mathcal{O}}\phi(a)$ and Lemma 7.8 gives $d'R_{\mathcal{O}}a$. Thus $B \subseteq \phi(A)$ and $|A| = |B|$.

To prove the second claim let $\phi \in \Gamma(G)$. If $\phi(A) \cap B \neq \emptyset$, then there exists an $a \in A$ and a $b \in B$ such that $\phi(a) = b$. The above argument then gives that $\phi(A) = B$. $\square$

COROLLARY 7.10. *For every vertex v in an orbit $\mathcal{O}$ of a graph G, $|\bar{v}|$ divides $|\mathcal{O}|$ divides $|\Gamma(G)|$.*

PROOF. Let v be in an orbit $\mathcal{O}$ of a graph G then if $u \in \mathcal{O}$, $|\bar{v}| = |\bar{u}|$ and $\bar{v} \cap \bar{u} = \emptyset$ so $\mathcal{O}$ can be divided into pieces of size $|\bar{v}|$. The fact that $|\mathcal{O}|$ divides $|\Gamma(G)|$ is the orbit-stabilizer theorem (see for example [**5**]). $\square$

DEFINITION 7.11. Let u, v be vertices in a connected graph G. Then we say that u *fixes* v (denoted $u \mapsto v$) if, for all $\phi \in \Gamma(G)$, we have $\phi(u) = u$ implies that $\phi(v) = v$. Equivalently we can say that u fixes v if and only if $\Gamma_u(G) \subseteq \Gamma_v(G)$.

THEOREM 7.12. *The relation $\mapsto$ defined above is reflexive and transitive, but not necessarily symmetric.*

PROOF. Clearly $u \mapsto u$ since $\Gamma_u(G) \subseteq \Gamma_u(G)$. Transitivity follows from the transitivity of $\subseteq$. To see an example of a graph where the symmetry condition doesn't hold consider P_3. This graph consists of three vertices a, b and c such that $a \sim b$ and $b \sim c$ but $a \nsim c$. Then $a \mapsto b$, but b doesn't fix anything, including a. $\square$

DEFINITION 7.13. A *digraph* is a directed graph; that is, a graph with a vertex set $V(G) = \{u_1, ..., u_k\}$ and an edge set consisting of ordered pairs (instead of unordered pairs). The edges of a digraph are referred to as *arcs* (denoted u_iu_j). We say that u_j is *adjacent from* u_i. Digraphs are normally pictured as a graph with arrows on the edges pointing away from the first vertex in the ordered pair.

DEFINITION 7.14. Given a graph G, the *fixing digraph* $F(G)$ is constructed as follows: $V(F(G)) = V(G)$ and $uv \in E(F(G))$ if and only if u fixes v (in G). Notice that the reflexivity of the relation $\mapsto$ gives that for every vertex u in $V(G)$ the arc uu is in $E(F(G))$.

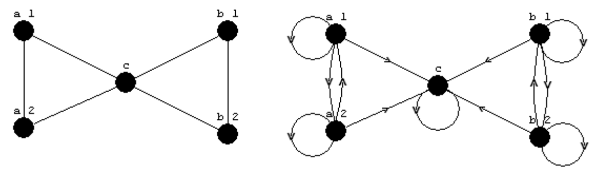
EXAMPLE 7.15. Let G be the graph with $V(G) = \{a_1, a_2, b_1, b_2, c\}$ and

$$E(G) = \{\{a_1, a_2\}, \{a_1, c\}, \{a_2, c\}, \{b_1, b_2\}, \{b_1, c\}, \{b_2, c\}\}.$$

Then a_1 and a_2 fix each other, b_1 and b_2 fix each other and every vertex fixes c. Hence the fixing digraph of G has $V(F(G)) = V(G)$ and

$$E(F(G)) = \{a_1a_1, a_1a_2, a_2a_1, a_2a_2, a_1c, a_2c, b_1b_1, b_1b_2, b_2b_1, b_2b_2, b_1c, b_2c\}.$$

The graph G and its fixing digraph $F(G)$

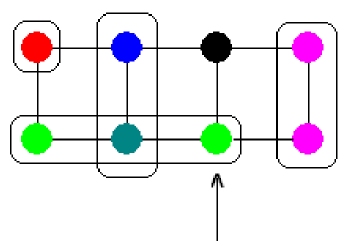


Definition 7.16. Let G be a graph (or digraph), the *domination* number $\gamma(G)$ is the smallest cardinality of a set $S \subseteq V(G)$ such that every vertex in $V(G) \backslash S$ is adjacent to a vertex in S.

Further refining this concept (and actually getting to what we will use) given a collection $\mathcal{P} = X_1, X_2, ..., X_k$ of sets vertices of a graph G, the *$\mathcal{P}$-defective domination* number $\tilde{\gamma}_{\mathcal{P}}(G)$ is defined to be the minimum cardinality of a set $S \subseteq V(G)$ such that for every integer $i = 1, ..., k$ at most one vertex in $(V(X_i) \backslash S)$ is not adjacent to a vertex in S.

Example 7.17. Let G be the grid $P_2 \times P_4$ with $V(G) = \{a_{i,j} : 1 \leq i \leq 2 \text{ and } 1 \leq j \leq 4\}$ and $E(G) = \{(a_{i,j}, a_{i',j'}) : |i - i'| = 1 \text{ or (but not both) } |j - j'| = 1\}$. Let $X_1 = \{a_{1,1}\}$, $X_2 = \{a_{1,2}, a_{2,2}\}$, $X_3 = \{a_{2,1}, a_{2,2}, a_{2,3}\}$ and $X_4 = \{a_{1,4}, a_{2,4}\}$ and $\mathcal{P} = \{X_1, X_2, X_3, X_4\}$. Then $\{a_{2,3}\}$ is a minimum $\mathcal{P}$-defective dominating set, so $\tilde{\gamma}_{\mathcal{P}}(G) = 1$.

The graph G with X_1 in red, X_2 in blue, X_3 in green and X_4 in purple:



8. The Main Theorem

We can now state and prove the main result, which relates what we have shown about the fixing number of a tree to the $\mathcal{P}$-defective domination number of its fixing graph.

Theorem 8.1. *Let T be a tree, let Θ be the set of end-orbits of T and let $\mathcal{P} = \bigcup_{\mathcal{O} \in \Theta} \mathcal{O}/R_{\mathcal{O}}$. Then $\operatorname{fix}(T) = \tilde{\gamma}_{\mathcal{P}}(F(T))$ where $F(T)$ is the fixing digraph of T.*

Proof. We begin by noting that $\operatorname{fix}(T) \geq \tilde{\gamma}_{\mathcal{P}}(F(T))$ is trivial. If we have some fixing set $S \subseteq V(T)$ with $|S| = \operatorname{fix}(T)$, then every vertex in T is fixed by S so in the fixing digraph $F(T)$, every vertex must be adjacent from some vertex in S. This is stronger than the $\mathcal{P}$-defective dominating condition; here, every vertex is dominated by S, regardless of the choice of the sets $X_i \in \mathcal{P}$, and ignoring the "every vertex in X_i except one" condition.

It remains to show that $\operatorname{fix}(T) \leq \tilde{\gamma}_{\mathcal{P}}(F(T))$. Let S be a $\mathcal{P}$-defective dominating set of $F(T)$. By Lemma 6.3 it is sufficient to show that every end vertex of T is fixed by S. Suppose not; then there exists an endorbit $\mathcal{O}$, a vertex $u \in \mathcal{O}$, and an automorphism $\phi \in \Gamma_S(G)$ such that $\phi(u) \neq u$. Notice that $\phi(\phi(u)) \neq \phi(u)$ because we could apply ϕ^{-1} to both sides of this equation to obtain $\phi(u) = u$. By our choice of S, we see that

$\phi(u)$ and u can't be in the same IEC (only one vertex in each IEC is not fixed by S), so $\overline{u} \neq \overline{\phi(u)}$.

Claim: $\overline{u} = \{u\}$. Suppose not. Let $u' \in \overline{u}$ with $u' \neq u$. By our choice of S, since ϕ doesn't fix u it follows that ϕ fixes u'. Then since $u \in \overline{u}$, $uR_{\mathcal{O}}u'$ Lemma 7.8 gives that $\phi(u)R_{\mathcal{O}}\phi(u')$ and $\phi(u') = u'$, so $\phi(u)R_{\mathcal{O}}u$. This contradicts the above statement that $\overline{u} \neq \overline{\phi(u)}$, so our claim is proved.

Let v be a vertex in $\mathcal{O}$ distinct from u for which $d(u,v)$ is a minimum, and P the uv-path in T. We consider two cases.

Case 1: P has odd order (even length). Let z be the central vertex of P and T_u, T_v the components of $T\backslash\{z\}$ containing u and v respectively. T_u and T_v are isomorphic since u and v are in the same orbit and automorphisms preserve adjacency. Let ϕ'' be an automorphism that sends u to v. Using Prins 7.3, we can construct an automorphism ϕ' that flips the path P and sends u to v. If we then let ϕ be the restriction of ϕ' to $T_u \cup T_v$, then ϕ will fix every vertex in $T\backslash(T_u \cup T_v)$. We now see that ϕ must map T_v onto T_u, and ϕ^2 must equal the identity. We want ϕ to be a uv-interchange, so we need to know that there are no elements of $\mathcal{O}\backslash\{u,v\}$ in T_u or T_v. Lemma 6.14 and our choice of v gives that there can be no vertices other than u in $\mathcal{O}$ in T_u. The fact that T_u and T_v are isomorphic then gives that there are no vertices other than v in $\mathcal{O}$ and in T_v. Thus ϕ is a uv-interchange so $v \in \overline{u}$ which is a contradiction since we've shown that $\overline{u} = \{u\}$.

Case 2: P has even order (odd length). Lemma 6.13 gives that T is bicentral and since $u, v \in \mathcal{O}$ implies that $e(u) = e(v)$, both central vertices, z_1, z_2 lie between u and v. Thus every automorphism that moves u to v must move every vertex in T (since the center must be flipped). Let T_u and T_v be the components of $T\backslash\{z_1, z_2\}$ containing u and v respectively. Again similarly to Case 1, Lemma 6.14 gives that there can be no elements of $\mathcal{O}\backslash\{u\}$ in T_u. The symmetry between T_u and T_v then gives that $T_v \cap \mathcal{O} = \{v\}$ so $\mathcal{O} = \{u, v\}$ and $v \in \overline{u}$ which is again a contradiction.

This completes this direction of the proof; since every line of reasoning ended in a contradiction, our assumption that $\text{fix}(T) \not\leq \tilde{\gamma}_{\mathcal{P}}(F(T))$ is false. □

9. An Alternate Characterization of the IECs

In order to use the previous theorem to compute that fixing number of a tree, we must find the end-orbits, compute the IECs and determine the $\mathcal{P}$-defective domination number of the associated fixing digraph. Computing the end-orbits of a graph is computationally easy, see for example [**1**]. We now show that once we have the end-orbits it is relatively easy to compute the IECs and we don't have to find the whole automorphism group to do so.

We then apply this result to obtain a characterization of trees with fixing number 1. In the next section we note that the process of determining $\text{fix}(T)$ can be simplified as well.

LEMMA 9.1. *Let T be a tree, $\mathcal{O}$ an orbit of T and $u, v \in \mathcal{O}$ such that $d(u,v)$ is odd. Then every automorphism that sends u to v moves every vertex of T.*

PROOF. By Lemma 7.2, T is bicentral and both central vertices lie between u and v. Let ϕ be an automorphism that sends u to v and let $c_1$and c_2 be the central vertices. Assume without loss of generality that c_1 is closer to u than c_2. Then since c_1 and c_2 are between u and v, $\phi(c_1)$ is closer to $\phi(u) = v$ than $\phi(c_2)$. Any automorphism must fix the center, so $\phi(c_1) = c_2$ and $\phi(c_2) = c_1$ (ϕ flips the center). Now let a be any vertex in T, a must be closer to either $c_1$or c_2, and ϕ preserves distances, so $\phi(a) \neq a$. □

The following theorem gives a clear picture of the IECs in a tree. For a given vertex, its IEC either has just one other vertex, or it consists of the closest vertices in the orbit.

THEOREM 9.2. *Let T be a tree, $\mathcal{O}$ be an orbit of T and $u, v \in \mathcal{O}$ with $u \neq v$.*

i) If $d(u, v)$ is odd, then $uR_{\mathcal{O}}v$ if and only if $\mathcal{O} = \{u, v\}$.

ii) If $d(u, v)$ is even then $uR_{\mathcal{O}}v$ if and only if $d(u, v) = d(u, \mathcal{O}\backslash\{u\})$

PROOF. Let $u, v \in \mathcal{O} \subseteq T$ with $u \neq v$.

i) $d(u, v)$ is odd: Clearly if $\mathcal{O} = \{u, v\}$ then $uR_{\mathcal{O}}v$. For the other direction, assume that $uR_{\mathcal{O}}v$, $d(u, v)$ is odd and ϕ is a uv-interchange. Then by Lemma 9.1, ϕ moves every vertex in T. By definition of uv-interchange, ϕ must fix every other vertex of $\mathcal{O}$, so $\mathcal{O} = \{u, v\}$.

ii) $d(u, v)$ is even: For the "if" direction, assume that $d(u, v) = d(u, \mathcal{O}\backslash\{u\})$. Let P be the uv-path in T and z the central vertex of P. Let T_u and T_v be the components of $T\backslash\{z\}$ containing u and v respectively. Then T_u and T_v are isomorphic and there is an automorphism ϕ'' that maps u to v. Prins 7.3 gives that there exists a $\phi' \in \Gamma(T)$ that interchanges u and v. Let ϕ be the restriction of ϕ' to $T_u \cup T_v$, then ϕ fixes every vertex in $T\backslash(T_u \cup T_v)$. Furthermore, since $d(u, v)$ is a minimum among $\mathcal{O}\backslash\{u\}$, Lemma 6.14 gives that $\mathcal{O} \cap (T_u \cup T_v) = \{u, v\}$. Hence ϕ is a uv-interchange, so $uR_{\mathcal{O}}v$.

For the converse, let $uR_{\mathcal{O}}v$ and assume by contradiction that $d(u, v) > d(u, \mathcal{O}\backslash\{u\})$ (notice that $d(u, v) < d(u, \mathcal{O}\backslash\{u\})$ is impossible since $u \neq v$). Let u' be such that $d(u, \mathcal{O}\backslash u) = d(u, u')$. Since $uR_{\mathcal{O}}v$, there is a uv-interchange ϕ such that $\phi(u') = u'$ and

$$d(u, u') = d(\phi(u), \phi(u')) = d(v, u').$$

Let P be the uv-path in T and z the central vertex of P. Then since ϕ moves all of $P\backslash\{z\}$, $u' \notin P$ or $u' = z$. By the above equation (and since paths between two vertices are unique in trees) we see that z must lie between u and u' and between v and u'. We assumed that $d(u, u') < d(u, v)$ and since $d(u, u') = d(u, z) + d(z, u')$ and $d(u, v) = 2d(u, z)$, we see that $d(z, u') < d(u, z)$. Now by ii) of Lemma 6.13, z is between u and the center, so

$$d(u', V(C)) \leq d(u', z) + d(z, V(C)) < d(u, z) + d(z, V(C)) = d(u, V(C)).$$

This is a contradiction since u and u' are in the same orbit, so $e(u) = e(u')$ and by Corollary 6.11 $d(u, V(C)) = d(u', V(C))$. $\square$

DEFINITION 9.3. An IEC $\overline{v}$ is called *trivial* if $\overline{v} = \{v\}$ and *nontrivial* otherwise.

COROLLARY 9.4. *Let $\mathcal{O}$ be an orbit in a tree T. If $|\mathcal{O}| \geq 3$, then $\mathcal{O}$ contains a nontrivial IEC.*

PROOF. Let $\mathcal{O}$ contain three or more vertices. We consider separately whether T is central or bicentral and, in ether case, we will find vertices in $\mathcal{O}$ that are even distance apart. If T is central, then by Lemma 6.13 the distance between any two vertices in $\mathcal{O}$ is even. If T is bicentral let e be the central edge of T, then since $|\mathcal{O}| \geq 3$ some component of $T\backslash\{e\}$ must contain two (or more) vertices $u, v \in \mathcal{O}$. Now Lemma 6.13 gives that $d(u, v)$ is even. Notice that the vertex in $\mathcal{O}$ that is the closest to u must be in the same component of $T\backslash\{e\}$ as u. In either case we obtain vertices $u, v \in \mathcal{O}$ with $d(u, v) = d(u, \mathcal{O}\backslash\{u\})$ even. Part ii) of Theorem 9.2 then gives that $uR_{\mathcal{O}}v$. Thus $\mathcal{O}$ has a nontrivial IEC. $\square$

THEOREM 9.5. *Let T be a tree. Then* $\operatorname{fix}(T) = 1$ *if and only if* $\Gamma(T) \cong \mathbb{Z}_2$.

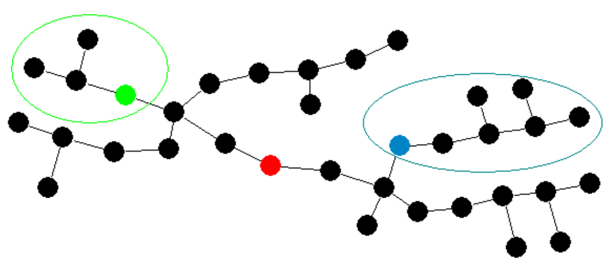
PROOF. If $\Gamma(T) = \mathbb{Z}_2$ then there are only two automorphisms, the identity and one other, say ϕ. In order to fix the graph T we just fix one vertex that ϕ moves. Thus any vertex moved by ϕ will be a fixing set for T.

For the converse, let $\text{fix}(T) = 1$ and let $S = \{u\}$ be a fixing set for T. Suppose by contradiction that $|\Gamma(T)| > 2$. Then $\{u\}$ is a fixing set so every automorphism must move u. Thus the orbit $\mathcal{O}$ containing u has cardinality $|\mathcal{O}| \geq 3$. By Corollary 9.4 $\mathcal{O}$ must contain a nontrivial IEC and by Corollary 7.9 all the IECs contained in $\mathcal{O}$ are the same size. This means if $|\mathcal{O}| \geq 3$ either $\overline{u}$ has more than two elements and there is an interchange for two vertices other than u, or there is an IEC in $\mathcal{O}$ disjoint from $\overline{u}$ and again there is an interchange for two vertices other than u. In either case, this is a contradiction because an automorphism that is an interchange of two vertices other than u in $\mathcal{O}$ must fix u, so $\{u\}$ is not fixing set. □

10. The Minimum Fixing Set

NOTATION 10.1. If v is a non-central vertex in a tree T, then there is a unique edge e incident with v that lies between v and the center of T. Let $T(v)$ denote the component of $T\backslash\{e\}$ that contains v.

A tree with center in red, and some branches in blue and green:



Notice that if an automorphism moves the blue point, all the vertices in that branch must also be moved.

LEMMA 10.2. *Let v be a non-central vertex in a tree T and $\phi \in \Gamma(T)$. If ϕ moves v, then ϕ moves every vertex in $T(v)$.*

PROOF. Let T be a tree, $\phi \in \Gamma(T)$, $v \in V(T)$ and $\phi(v) \neq v$. Suppose by contradiction that there is a vertex $v' \in V(T(v))$ with $\phi(v') = v'$. Then since ϕ is an isometry, $d(v', v) = d(v', \phi(v))$ and thus v does not lie between v' and $\phi(v)$. Since $v' \in V(T(v))$, this means that $\phi(v) \in V(T(v))$ and so v lies between $\phi(v)$ and the center of T and the distance from v to the center is less than the distance from $\phi(v)$ to the center. This contradicts Corollary 6.11 since $e(v) = e(\phi(v))$. □

COROLLARY 10.3. *If v is a non-central vertex in a tree T and $u \in V(T(v))$, then u fixes every vertex that is fixed by v.*

LEMMA 10.4. *If S is a $\text{fix}(T)$-set and $v \in S$, then v fixes every vertex in $T(v)$.*

PROOF. The result holds vacuously if v is an end-vertex. Suppose by contradiction that S is a $\text{fix}(T)$-set, $v \in S$ and there is a vertex $u \in T(v)$ and an automorphism ϕ such that $\phi(v) = v$ and $\phi(u) \neq u$. Since S fixes T, $S \cap T(v) \neq \emptyset$ (otherwise ϕ restricted to $T(v)$ would fix S and move u). Let $v' \in S \cap V(T(v))$. Then v'fixes v, by Corollary 10.3. Let $\hat{S} = S\backslash\{v\}$ then $\Gamma_{\hat{S}}(T) \subseteq \Gamma_S(T) = \{id\}$. This gives that $\hat{S}$ fixes T which is a contradiction since $|\hat{S}| < |S|$. □

THEOREM 10.5. *For every tree T, there is a* $\text{fix}(T)$*-set consisting of only end-vertices of* T.

PROOF. Let T be a tree and S a $\mathcal{P}$-defective dominating set of $F(T)$ (S is a $\tilde{\gamma}_{\mathcal{P}}(F(T))$-set) with the maximal amount of end-vertices. Assume that $\Gamma(T)$ is nontrivial, we claim that every vertex in S is an end-vertex. Suppose by contradiction that there is a vertex $w \in S$ that is not an end-vertex. If w is central then in order for w to be moved by an automorphism, T must be bicentral. Any automorphism that moves w will move every vertex in T, so w can be removed from S and replaced by any vertex in T. This contradicts our choice of S. If w is non-central, let x be an end-vertex in $T(w)$ then by Corollary 10.3 x fixes every vertex that is fixed by w. Again this contradicts our choice of S. Hence we see that S must consist of all end-vertices. $\square$

11. Conclusion

In this paper I have examined the fixing number of a graph, noted that we could restrict our study to connected graphs, given some bounds on fixing numbers, constructed graphs with a given fixing number, and used the $\mathcal{P}$-defective domination number to find a way to compute the fixing number of a tree.

A problem is that the $\mathcal{P}$-defective domination number of a fixing graph, where $\mathcal{P}$ is a set of IECs, is not a number that we come across often. It is shown in [**1**] that determining the end-orbits of a tree is not computationally hard. Finding the IECs is also relatively easy using Theorem 9.2 (luckily we don't need to find the whole automorphism group). However, little is known about the $\mathcal{P}$-defective domination number, and in particular, it isn't known how hard it is to compute $\tilde{\gamma}_{\mathcal{P}}$ for a given graph and collection $\mathcal{P}$.

Also, the results presented in this paper deal mostly with trees; computing the fixing number of graphs in general has not been studied much, and many of the tactics used here will not work when cycles are present. In particular, the center of a graph becomes a lot more complicated.

One final possible avenue of study could be expanding the idea of automorphism group to include any group acting on the vertices. What restrictions, if any, would the structure of the group place on the value of the fixing number?

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