Spring 6-16-2021

Spanning Trees of Complete Graphs and Cycles

Minjin Enkhjargal

Spanning Trees of Complete Graphs and Cycles

by

Minjin Enkhjargal

An undergraduate honors thesis submitted in partial fulfillment of the

requirements for the degree of

Bachelor of Science

in

University Honors

and

Mathematics

Thesis Adviser

John Caughman

Portland State University

2021

Abstract

Spanning trees are typically used to solve least path problems for finding the minimal spanning tree of a graph. Given a number $t \geq 3$ what is the least number $n = \alpha(t)$ such that there exists a graph on n vertices having precisely t spanning trees? Specifically, how will the factoring of t with the use of cycles connected by one vertex affect $\alpha(t)$? Lower and upper bounds of $\alpha(t)$ are graphed by using properties of cycles and complete graphs. The upper bound of $\alpha(t)$ is then improved by constructing a graph of connected cycles $\{C_{p1}, C_{p2}, C_{p3}, \dots, C_{pn}\}$ where p1, p2, p3 ... pn belong to the prime factorization of t. The bounds of $\alpha(t)$ are significantly improved.

1 Introduction

Graph theory is one of the many areas of study that has grown popular in modern mathematics. Ideas that originated from graph theory are used in applications of Mathematics and in other areas of studies as well. Graphs are structures that consist of vertices and edges. One property of graphs that is often studied is its spanning trees. A spanning tree of a graph is a subgraph that contains the least number of edges needed to create a path from one vertex to another until all vertices are connected [1].

The discussion of least spanning trees or the more commonly discussed problem of finding the shortest path is popular among computer science and computer networking. It is an important theoretical foundation for many algorithms used in graph data structures in computer science, civil network planning, and computer network routing. [2] These applications use minimal spanning trees to find the least path or most efficient subgraph containing no cycles, which can provide information on finding the most efficient and economical solution. In many modern technologies, efficiency is a crucial demand for applications using mathematics and computer science.

A function related to spanning trees of graphs is $\alpha(t) = n$, which takes in the spanning trees of some graph G, denoted $\tau(G)$, as the parameter. The output n equals the vertices of graph G, where G contains the least number of vertices that is able to construct a graph with t spanning trees. In 1970, graph theorist J. Sedlacek [3] proved some properties about the asymptotic behavior of $\alpha(t)$. He showed that $\alpha(t) \leq \frac{t+6}{3}$ if $t \equiv 0 \pmod 3$ and $\alpha(t) \leq \frac{t+4}{3}$ if $t \equiv 2 \pmod 3$. Later, the known upper bound of $\alpha(t)$ is improved by J. Azarija and R. Skrekovski. They prove [4] that

$\alpha(t) \leq \frac{t+9}{4}$ and $\beta(t) \leq \frac{t+13}{4}$, if $t \notin \{3, 4, 5, 6, 7, 9, 10, 13, 18, 22\}$. They also show that the known fixed points of α are 3, 4, 5, 6, 7, 9, 10, 13, 18, 22, which is the subset of Euler's Idoneal numbers. Additionally, Azarija and Skrekovski also considers β(t) = k, the spanning trees resulting for graphs with *t* edges. A formula for finding the number of spanning trees for all vertices and edges have not been found yet. A conjecture has been made [5] in the math community that $\alpha(n)=o(\log n)$. Given a number $t \geq 3$ what is the least number $n = \alpha(t)$ such that there exists a graph on n vertices having precisely t spanning trees? Let there exist a graph G with *t* spanning trees. Then α(t) is the graph with the least number of vertices to produce such a graph.

1.1 Definitions

Graph: A graph is a mathematical representation of a network. It represents the relationship between lines and points. A graph is given as G = (V,E), where V is the set of vertices and E is the set of edges.

Spanning Tree: A spanning tree is a subgraph of any graph that contains the least number of edges needed to create a path, with no cycles, from one vertex to another until all vertices are connected. All spanning trees have (n-1) edges, for a graph with n vertices.

Complete Graph: A complete graph is a simple undirected graph with each distinct vertices connected by a unique edge. The set of complete graphs on n vertices is denoted by Kn.

Cycle: A cycle is a simple graph with n vertices, where n >2, with edges connecting each vertex in a cycle of length n. The set of cycles on n vertices is denoted by Cn.

1.2 Spanning Tree Counting Theorems

There are two common techniques for counting spanning trees. Let G be any graph, then τ(G) outputs the number of spanning trees for graph G.

Deletion-Contraction Formula

This technique uses a deletion-contraction formula in a recursive form to count spanning trees of a graph. For a graph G, τ(G) = τ(H-e) + τ(H●e).

Kirchhoff's Matrix-Tree Theorem

This technique uses the adjacency matrix of a graph to create the Laplacian matrix denoted L. With matrix L, a row and column is crossed off to get the smaller matrix L*. Then, for a given graph G, $\tau(G) = \det(L^*)$ [6].

Cayley's Tree Formula

Cayley's tree formula [7] is based off the matrix-tree theorem. Let $T_{n,k}$ be the number of labelled forests on n vertices with k connected components, then $T_{n,k} = kn^{n-k-1}$. For the spanning trees of complete graphs, $\tau(K_n) = n^{(n-2)}$.

2 Bounds of α(n) using cycles and complete graphs

In a related discussion of this topic, the spanning trees of circulant graphs are also worth mentioning. The spanning trees of this subclass of regular graphs correlates closer to the applications of computer network routing. A circulant graph is defined [8] as "a graph of n graph vertices in which the *i*-th graph vertex is adjacent to the (*i+j*)-th and (*i-j*)-th graph vertices for each *j* in a list l." When visualized, these graphs belong to C_n with contained adjacent edges.

1. The spanning trees of the circulant graph, *square of the cycle* $C^{1,2}{}_n$ [9], is $\tau(C^{1,2}{}_n) = nF^2{}_n$, where F_n is the sequence of the *Fibonacci* numbers.

N	Fibonacci Sequence	$\tau(C^{1,2}{}_n)$
1	1	1
2	1	2
3	2	12
4	3	36
5	5	125
6	8	384
7	13	1183
8	21	3528
9	34	10404
10	55	30250

2. The number of spanning trees for complete graphs is $\tau(K_n) = n^{(n-2)}$

n	$\tau(K_n)$
1	1
2	1
3	3
4	16
5	125
6	1296
7	16807
8	262144
9	4782969
10	100000000

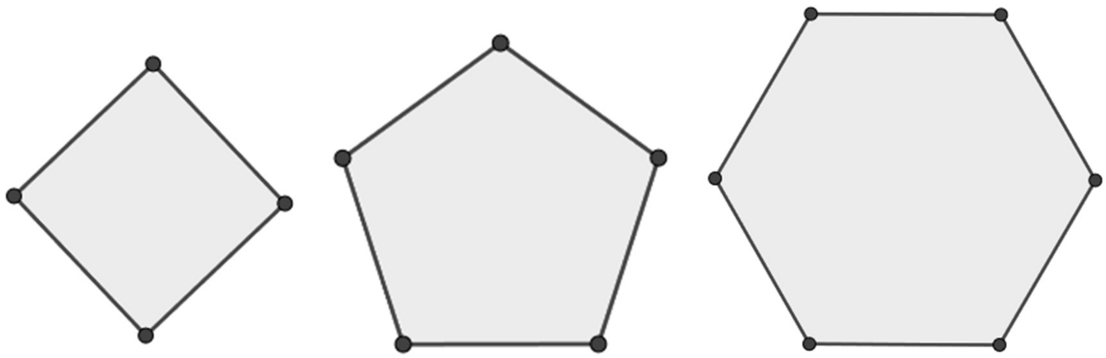
2.1 Fixed Points of $\alpha(t)$

By Azarija and Skrekovski, $\alpha(t)$ is fixed for the subset of Euler's Idoneal numbers: {3, 4, 5, 6, 7, 9, 10, 13, 18, 22} For this set, the $\alpha(t)$ cannot be reduced and is fixed.

n	$\alpha(t) = n$
3	3
4	4
5	5
6	6
7	7
9	9
10	10
13	13
18	18
22	22

2.2 Upper Bounds of $\alpha(t)$

Upper Bound for a Cycle: A cycle with *n* vertices has exactly *n* spanning trees. Each spanning tree of a cycle is found by removing a single edge to produce a unique spanning tree. Then the maximum upper bound of $\alpha(t)$ for the set of cycles C_n is $\tau(C_n) = n$. Therefore, $\alpha(\tau(C_n)) \leq n$.



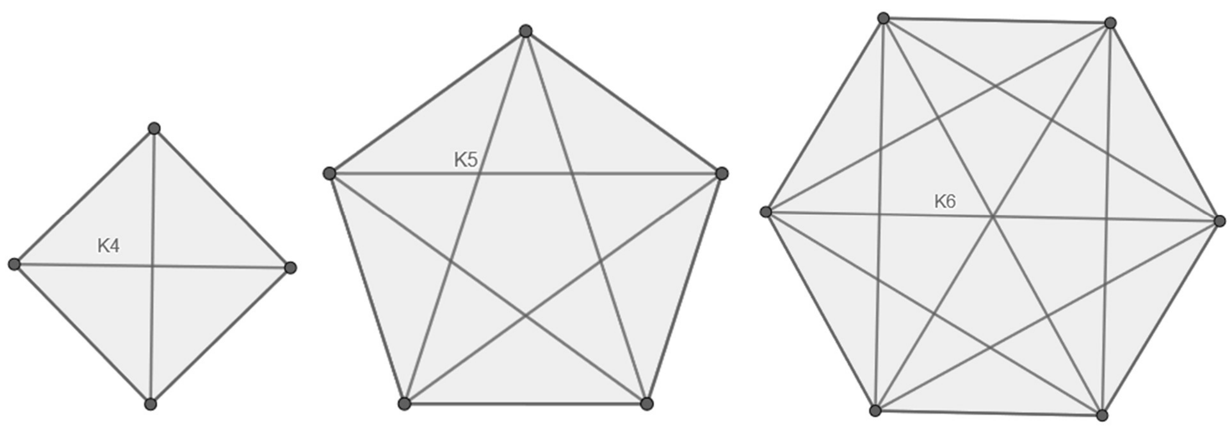
$n = 4$ $\quad$ $n = 5$ $\quad$ $n = 6$

$\tau(C_4) = 4$ $\quad$ $\tau(C_5) = 5$ $\quad$ $\tau(C_6) = 6$

$\alpha(4) \leq 4$ $\quad$ $\alpha(5) \leq 5$ $\quad$ $\alpha(6) \leq 6$

Figure 1: For cycles C_4, C_5, and C_6 the number of spanning trees is equal to the number of vertices of that cycle and $\alpha(\tau(C_n))$ is less than or equal to it's vertices.

Upper bound for complete graphs: The set of complete graphs denoted by K_n is one that contains edges from each vertex to another. For any given cycle with n vertices > 4, adding additional edges between vertices will allow the graph to produce a greater number of spanning trees. Then a complete graph K_n will produce the most spanning trees for n vertices, since K_n contains edges from every vertex to all other vertices. Therefore $\alpha(\tau(K_n)) \leq n$.



$n=4$ $\quad$ $n=5$ $\quad$ $n=6$

$\tau(K_4) = 16$ $\quad$ $\tau(K_5) = 125$ $\quad$ $\tau(K_6) = 1296$

$\alpha(16) \leq 4$ $\quad$ $\alpha(125) \leq 5$ $\quad$ $\alpha(1296) \leq 6$

Figure 2: For complete graphs K_4, K_5, and K_6 the number of spanning trees is equal to the number of vertices of that cycle and $\alpha(\tau(C_n))$ is less than or equal to it's vertices. $\tau(K_4)$ is higher than that of C_n since it contains more edges.

For each *n,* C_n and K_n represent the cycles and complete graphs, respectively, with n vertices. Let $t = \tau(K_n)$. Then taking $\alpha(\tau(K_n))$ allows us to determine some upper and lower bound for $\alpha(t)$. Clearly, the maximum upper bound is $\alpha(t) \leq t$ so, $\alpha(\tau(K_n)) \leq \tau(K_n)$. The lower bound is determined by the result of $\alpha(\tau(K_n))$.

Let $n = 4$. Note that: $\tau(K_4) = 16$ and $\alpha(16) \leq 4$

Any graph with $n = 4$ vertices can have at most $\tau(K_4) = 16$ spanning trees.

For any graph with $16 + 1$ spanning trees, since $n = 4$ has at most 16 spanning trees, $\alpha(17)$ must be a graph with 5 or more vertices.

$$5 \leq \alpha(17)$$

Then it is true that $4 < \alpha(17)$

It is also true that $4 \leq \alpha(16)$

Then the bounds are $n \leq \alpha(\tau(K_n)) \leq \tau(K_n)$

n	$\tau(C_n)$	$\tau(K_n)$	Bounds of α
3	3	3	$3 \leq \alpha(3) \leq 3$
4	4	16	$4 \leq \alpha(16) \leq 16$
5	5	125	$5 \leq \alpha(125) \leq 125$
6	6	1296	$6 \leq \alpha(1296) \leq 1296$
7	7	16807	$7 \leq \alpha(16807) \leq 16807$
8	8	262144	$8 \leq \alpha(262144) \leq 262144$
9	9	4782969	$9 \leq \alpha(4782969) \leq 4782969$
10	10	100000000	$10 \leq \alpha(100000000) \leq 100000000$
11	11	2357947691	$11 \leq \alpha(2357947691) \leq 2357947691$
12	12	61917364224	$12 \leq \alpha(61917364224) \leq 61917364224$
13	13	1.79216E+12	$13 \leq \alpha(1792160394037) \leq 1792160394037$
14	14	5.66939E+13	$14 \leq \alpha(56693912375296) \leq 56693912375296$
15	15	1.9462E+15	$15 \leq \alpha(1946195068359370) \leq 1946195068359370$

These upper bounds can be improved by taking the prime factorization of the upper bound. Since the upper bound is the result of the number of spanning trees for that cycle (i.e. $C_{16807} \Rightarrow \tau(C_{16807}) = 16807$) Then, taking the prime factorization of 16807, a graph of connected cycles can be constructed, where each sub-cycle contains the same number of vertices as each of the prime factorization.

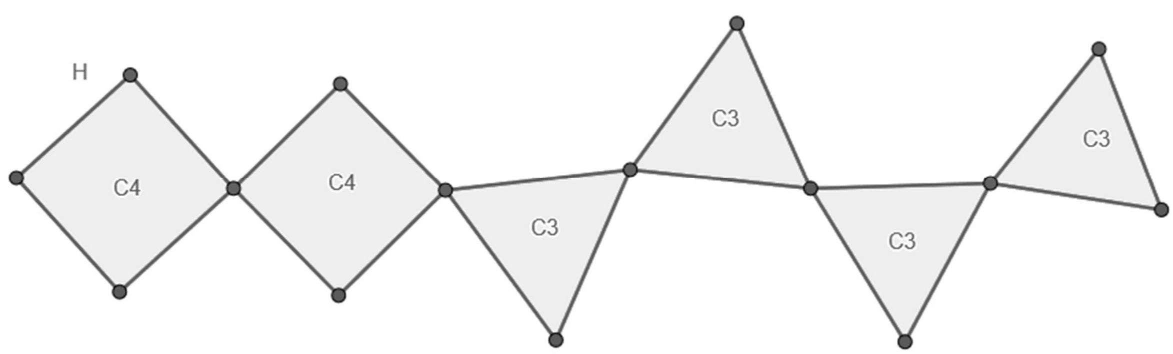
Let G be the cycle C_t. Let H be the graph constructed of cycles from the set $\{C_{p1}, C_{p2}, C_{p3}, ..., C_{pn} \mid p1, p2, p3, ..., pn \in \text{prime factorization of } t\}$. If the prime factorization includes any powers of 2, it must be converted to a power of 4, since there cannot be a cycle of 2 vertices, without requiring duplicate edges.

t	Prime Factorization of t
16	4 x 4
125	5 x 5
1296	4 x 4 x 3 x 3 x 3 x 3
16807	7 x 7 x 7 x 7 x 7
262144	4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4
4782969	3 x 3 x 3 x 3 x 3 x 3 x 3 x 3 x 3 x 3 x 3 x 3 x 3 x 3
100000000	4 x 4 x 4 x 4 x 5 x 5 x 5 x 5 x 5 x 5 x 5 x 5
2357947691	11 x 11 x 11 x 11 x 11 x 11 x 11 x 11 x 11
61917364224	4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 3 x 3 x 3 x 3 x 3 x 3 x 3 x 3 x 3 x 3
1792160394037	13 x 13 x 13 x 13 x 13 x 13 x 13 x 13 x 13 x 13 x 13

Example.

let $G = C_{1296}$. The prime factorization of $1296 = 2^4 * 3^4 = 4 \times 4 \times 3 \times 3 \times 3 \times 3$.

Let H be the graph constructed of cycles $C_4, C_4, C_3, C_3, C_3, C_3$. Then H is such a graph:



H contains the same number of spanning trees as G. To see this, choose one spanning tree from each sub-cycle in H. The outcome is one spanning tree of H. To find the second spanning tree of H, the chosen spanning trees for the sub-cycles shall remain the same. Now choose a different spanning tree for only the first sub-cycle C_4. To find the third spanning tree of H, choose a different spanning tree of C_4. This can be repeated for each sub-cycle. C_4 has 4 choices of spanning trees and C_3 has 3 choices of spanning trees. Then this becomes a counting problem where the total number of spanning trees is equal to 4 x 4 x 3 x 3 x 3 x 3, which is the same as the factorization of 1296.

Thus, $G = C_t$ and H both have the same number of spanning trees. While C_t has t vertices, H has less vertices since each sub-cycle shares a vertex. The number of vertices for a set of sub-cycles $\{C_{p1}, C_{p2}, C_{p3}, ..., C_{pn}\}$ is calculated by taking the sum $p1 + p2 + p3 + ... + pn - n$ (the number of sub-cycles in the graph). For graph H, the number of vertices is $4+4+3+3+3+3-6 = 14$. Then there now exists a graph with significantly less vertices than the original cycle. The updated bounds for this set of graphs is:

t	Vertices	Bounds of α
16	6	$4 \leq \alpha(16) \leq 6$
125	8	$5 \leq \alpha(125) \leq 8$
1296	14	$6 \leq \alpha(1296) \leq 14$
16807	30	$7 \leq \alpha(16806) \leq 30$
262144	27	$8 \leq \alpha(262144) \leq 27$
4782969	28	$9 \leq \alpha(4782969) \leq 28$
100000000	44	$10 \leq \alpha(100000000) \leq 44$
2357947691	90	$11 \leq \alpha(2357947691) \leq 90$
61917364224	50	$12 \leq \alpha(61917364224) \leq 50$
1792160394037	132	$13 \leq \alpha(1792160394037) \leq 32$

The first two powers of the prime factors 3 and 4 ($4=2^2$, since a 2-cycle cannot be constructed) is calculated in the table below. The number of spanning trees is equal to the power of the 3 and 4. The number of vertices for a graph constructed of 3-cycles begins with 4 and continues to increase by a constant value of 2. The number of vertices for a graph constructed of 4-cycles begins with 6 and continues to increase by a constant value of 3. Since the vertices for connected graphs is the sum of each sub-cycle minus the number of sub-cycles in the graph, the pattern is expected to be similar for powers of other prime numbers (5, 7, 11...)

Exp	Powers	Prime Factorization	Spanning Trees	# vertices	Bounds of α
3^2	9	3*3	9	4	$\alpha(9) \leq 4$
3^3	27	3*3*3	27	6	$\alpha(27) \leq 6$
3^4	81	3*3*3*3	81	8	$\alpha(81) \leq 8$
3^5	243	3*3*3*3*3	243	10	$\alpha(243) \leq 10$
3^6	729	3*3*3*3*3*3	729	12	$\alpha(729) \leq 12$
4^2	16	4*4	16	6	$\alpha(16) \leq 6$
4^3	64	4*4*4	64	9	$\alpha(64) \leq 9$
4^4	256	4*4*4*4	256	12	$\alpha(256) \leq 12$
4^5	1024	4*4*4*4*4	1024	15	$\alpha(1024) \leq 15$
4^6	4096	4*4*4*4*4*4	4096	18	$\alpha(4096) \leq 18$

The bounds proven by Sedlacek $\alpha(t) \leq \frac{t+6}{3}$ if $t \equiv 0 (\mathrm{mod}\ 3)$ and $\alpha(t) \leq \frac{t+4}{3}$ if $t \equiv 2 (\mathrm{mod}\ 3)$ and J. Azarija and R. Skrekovski $\alpha(t) \leq \frac{t+9}{4}$ could possibly be represented by a table such as this one, although it does not make sense to plot out powers of every prime number. One thing to note about graphs constructed by connected cycles is that the bound for $\alpha(t)$ will be improved if the sum of the factors used for the cycle is close to each other.

Example.

For C_{36}, the factors of 36 are 1, 2, 3, 4, 6, 9, 12, 18, 36.

Note that:

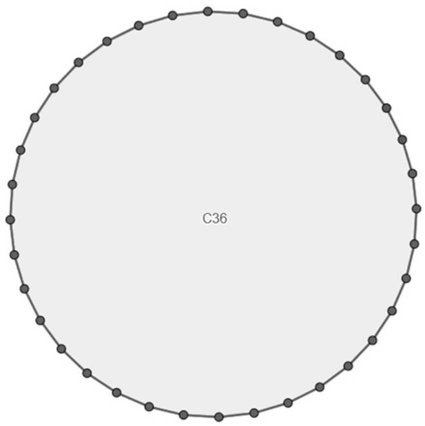
2 * 18 = 2 * 2 * 9 = 2 * 2 * 3 * 3 = 4 * 3 * 3

3 * 12 = 4 * 3 * 3

4 * 9 = 4 * 3 * 3

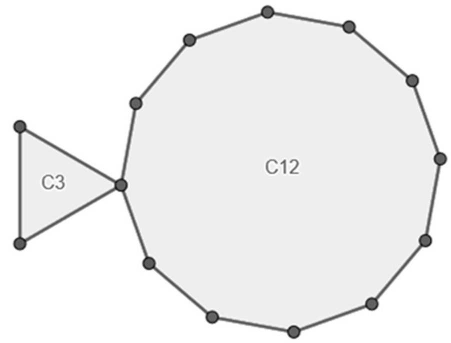
6 * 6 = 3 * 2 * 3 * 2 = 4 * 3 * 3

For C_{36} and each of the graphs constructed with connected cycles, the number of spanning trees will remain the same: $\tau(C_{36}) = \tau(C_2 * C_{18}) = \tau(C_6 * C_6) = 36$ and so forth for each connected cycle.



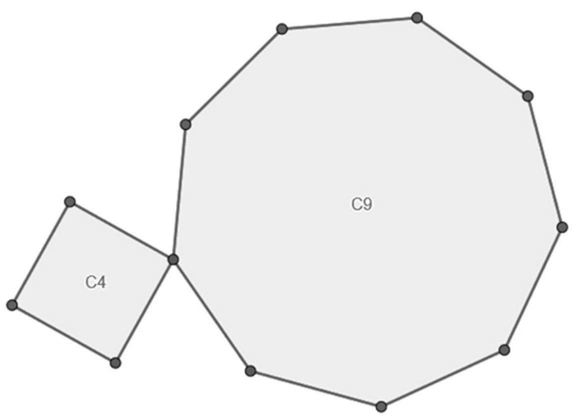
C_{36}

$\alpha(\tau(C_{36})) \leq 36$



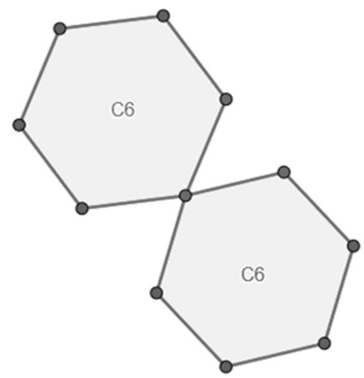
$C_3 * C_{12}$

$\alpha(\tau(C_3 * C_{12})) \leq 14$



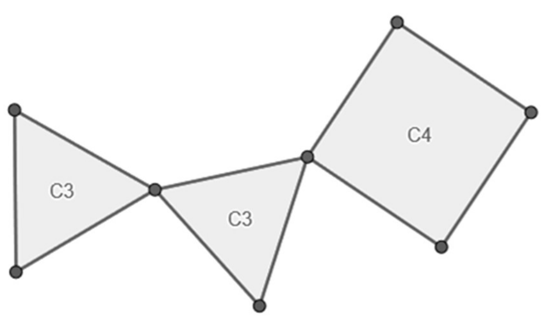
$C_4 * C_9$

$\alpha(\tau(C_4 * C_9)) \leq 12$



$C_6 * C_6$

$\alpha(\tau(C_6 * C_6)) \leq 11$



$C_3 * C_4 * C_3$

$\alpha(\tau(C_3 * C_4 * C_3)) \leq 8$

3 Conclusion

The behavior of $\alpha(t)$ is observed by finding the upper bounds using properties known about $\tau(C_n)$ and $\tau(K_n)$ and improving that bound significantly by creating a graph constructed of connected cycles by one vertex. Since constructed cycles may have different factorizations, it is best to break out the cycle into multiple connected cycles that have factors that are close in number.

References

[1] “Data Structure & Algorithms - Spanning Tree.” Tutorialspoint. Retrieved from www.tutorialspoint.com/data_structures_algorithms/spanning_tree.htm

[2] Eppstein, David, "Spanning trees and spanners”, Sack, J.-R.; Urrutia, J. (eds.), Handbook of Computational Geometry, Elsevier, (1999) pp. 425–461.

[3] J. Sedlacek, On the minimal graph with a given number of spanning trees, Canad. Math. Bull. 13 (1970) 515-517.

[4] J. Azarija, R. Skrekovski, Euler's idoneal numbers and an inequality concerning minimal graphs with a prescribed number of spanning trees, IMFM preprints 49 (2011) 3. G. Baron, F. Boesch, H. Prodinger, R. Tichy and J. Wang, The Number of Spanning Trees in the Square of Cycle, The Fibonacci Quart., 23 (1985) 258–264.

[5] https://mathoverflow.net/questions/93656/minimal-graphs-with-a-prescribed-number-of-spanning-trees

[6] Moore, Cristopher (2011). The nature of computation. Oxford England New York: Oxford University Press.

[7] Aigner, Martin; Ziegler, Günter M. (1998). Proofs from THE BOOK. Springer-Verlag. pp. 141–146.

[8] Weisstein, Eric W. "Circulant Graph." From MathWorld--A Wolfram Web Resource. https://mathworld.wolfram.com/CirculantGraph.html

[9] Golin M.J., Leung Y.C., Wang Y. (2004) Counting Spanning Trees and Other Structures in Non-constant-jump Circulant Graphs. In: Fleischer R., Trippen G. (eds) Algorithms and Computation. ISAAC 2004. Lecture Notes in Computer Science, vol 3341. Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-540-30551-4_45

[10] Minimal graphs with a prescribed number of spanning trees. (2012, April 22). Retrieved from http://www.openproblemgarden.org/op/minimal_graphs_with_a_prescribed_number_of_spanning_trees