The Independence Fractal of a Graph

By

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Section 1: Introduction

In 1736, the great mathematician Leonhard Euler visited the city of Königsburg, in an area of Prussia near what is now Poland. He encountered there a puzzle which had stumped the residents of the city for generations: Could a person walk from their house, cross all seven bridges across the river Pregel exactly once, visiting all four sections of the city, and return home without retracing their steps? Euler solved the problem not just for the city of Königsburg, but in general for any number of bridges and islands, and in the process set the spark from which was born the branch of geometry known as *geometris situs* ("the geometry of location"), now known as graph theory.

Graph theory is the study of interconnectedness. Euler was concerned with a set of objects, the separate landmasses of Königsburg, and the connections between them, in Euler's case the bridges. Modern graph theorists refer to the set of objects as vertices, and the connections between them as edges. This simple starting point has let to applications in a variety of fields, from computer science (a network of computers, connected by cables), to mapmaking (countries connected by borders), to social science (high school students connected by a "relationship" in the more literal sense).

The study of graph theory is devoted in large measure to finding similarities between graphs. While there is a natural way to form a picture of a graph, with dots for the vertices and lines for the connections, when analyzing a graph the picture can be misleading. The graph G, shown below, can be drawn in many ways based on where the vertices are placed. Even though each picture of the graph is different, they are all the same graph, and so should exhibit characteristics which are common to all of the pictures.

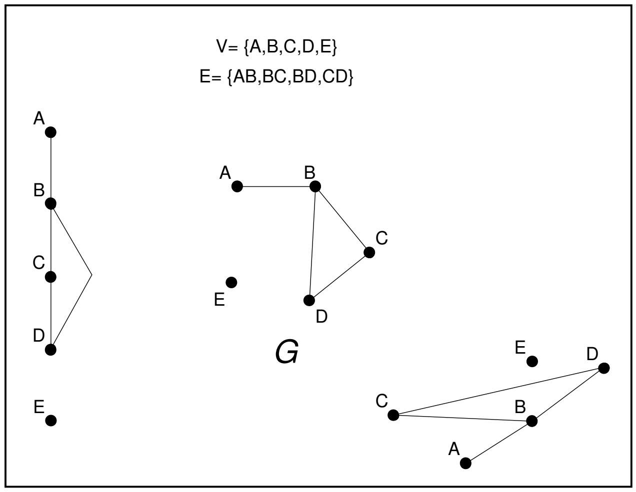


Fig. 1.1. Three drawings of the graph *G*.

This is where a graph theorist's bread is buttered – in finding measurable characteristics which are inherent in a graph. The graph *G* can then be described by the collection of identifying characteristics, some of which are shown below, which do not change no matter how the graph is drawn.

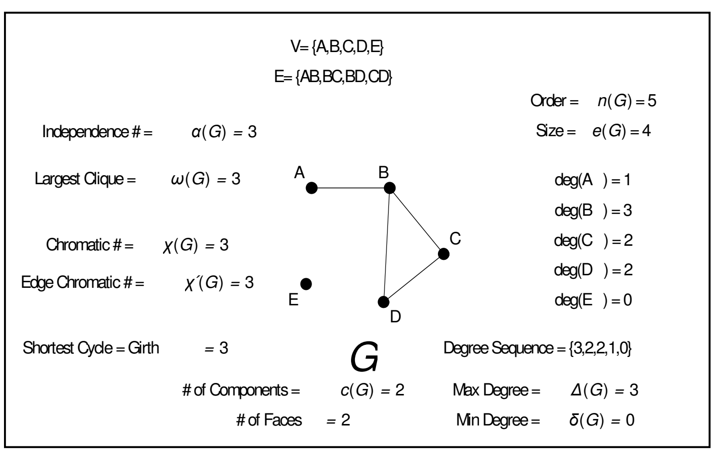


Fig. 1.2. The graph *G* and some characteristics.

The authors of the paper “The independence fractal of a graph” wade into this soup of alphas, deltas, and chis with an ambitious plan: to create another piece of identification for *G* not by counting vertices or trying to color the graph, but by associating a fractal with *G*. The hope is that there will be a connection between the graph-theoretic properties of *G* and the structure of the fractal.

What is a fractal? A **fractal** is an object or quantity that displays self-similarity on all scales. The object does not need to exhibit exactly the same structure at all scales, but the same "type" of structures. This somewhat hazy definition is typical in the context of fractals, as there are as many alternative definitions as there are authors (see [Barnsley, 1988], [Devaney, 1992], [Weisstein, 1999] etc…). Barnsley has this to say on the definition of fractals: “It is too soon to be formal about the exact meaning of ‘a fractal’. At the present stage of development of science and mathematics, the idea of a fractal is most useful as a broad concept. Fractals are not defined by a short legalistic statement, but by the many pictures and contexts which refer to them…more meaning is suggested than is formalized.”

Fortunately for us, we can be a bit more precise in our definitions. After we cover some basic graph theory terminology, we will use some results from iteration theory in Section 3 to formulate a precise definition of the independence fractal of a graph. Once we have defined this object, the remainder of the paper will be devoted to using existing iteration theory to analyze the graph-fractal connections.

Section 2: Graph Theory

First, a discussion of the graph theory we will need to get started. A **graph** G consists of a set $V(G)$ of vertices along with an edge set $E(G)$, where each edge consists of a pair of vertices. If a pair of vertices (x, y) is in $E(G)$, then we say x is **adjacent** to y and write $x \sim y$. The number $n(G)$ (or just n) is the order of G, the number of vertices in the vertex set $V(G)$.

Our goal is to associate a fractal with each graph. To do this we need two things: a polynomial which we can associate with our graph, and a mechanism to iterate the graph to create the sequences necessary for our fractal.

There are several candidates to choose from for the polynomial, among them the **chromatic polynomial**, which denotes the number of k-colorings of the graph G, or the characteristic polynomial formed by the eigenvalues of the adjacency matrix. For our purposes however, we will use a modified version of the independence polynomial as the basis for our fractals.

An **independent set** in a graph G is a vertex subset $S \subseteq V(G)$ that contains no edge of G (that is, the subgraph induced by S has no edges). The **independence number** of a graph is the maximum size of an independent set of vertices. For a graph G and non-negative integer k, let i_k be the number of independent sets of vertices in G of cardinality k.

The **independence polynomial** of G is the generating polynomial $i_G(x) = \sum_{k=0}^{\alpha} i_k x^k$ for the sequence $\{i_k\}$, where α is the independence number of G.

Independence polynomials are used in other applications, in particular the study of **well-covered graphs**, which are graphs with the property that every maximal independent set has the same cardinality α. For an example, see [Brown, 2000].

Example 2.1. For the graph G, pictured at right, we compute the independence polynomial. There is one subset of G with size zero, namely the empty set. Since this set contains no edges in G, it is independent and $i_0 = 1$. There are five subsets of G with size one, {A}, {B}, {C}, {D}, and {E}.

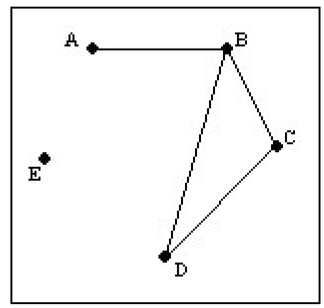


Fig. 2.1. The graph G for Example 2.1.

These sets contain no edges, so they are all independent and $i_1 = 5$. The two-element independent subsets of G are {A,C}, {A,D}, {A,E}, {B,E}, {C,E}, and {D,E}, and $i_2 = 6$. There are two independent sets of order three, {A,C,E} and {A,D,E}, so $i_3 = 2$. There are no independent sets of order greater than three, so $\alpha = 3$ and the independence polynomial of G is:

$$i_G(x) = \sum_{k=0}^{\alpha} i_k x^k = 2x^3 + 6x^2 + 5x^1 + 1x^0 = 2x^3 + 6x^2 + 5x + 1 \qquad \blacktriangleleft$$

Our example leads us to some observations about the independence polynomial. First, this polynomial will always have the constant term 1, since for any graph the empty set is the only subset of cardinality zero, and is by definition an independent set. Second, i_1, the number of independent sets of size one, is simply the number of vertices, so $i_1 = | V(G) | = n(G)$.

The **lexicographic product** (or **composition**) of two graphs G and H, denoted $G[H]$, is the graph with vertex set $V(G) \times V(H)$, with $(g, h) \sim (g', h')$ iff $[g \sim g']$ or $[g = g'$ and $h \sim h']$.

Example 2.2. The graphs K_3 and P_2 are shown at right. Taking the lexicographic product $P_2[K_3]$ we generate the following graph:

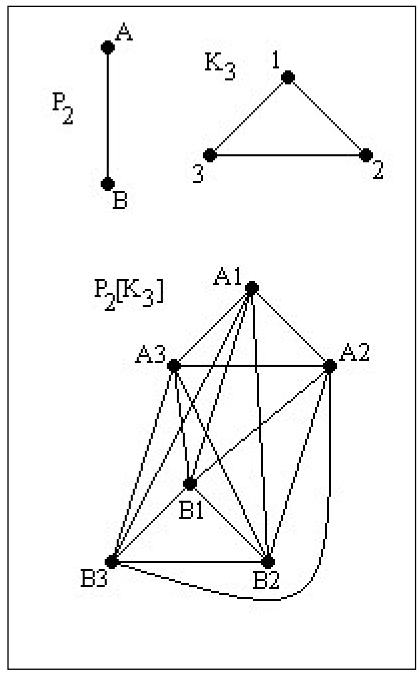


Fig. 2.2. The graphs K_3, P_2 , and $P_2[K_3]$.

$V(P_2[K_3]) = \{(A,1), (A,2), (A,3), (B,1), (B,2), (B,3)\}$

$E(P_2[K_3]) = \{$ [(A,1), (B,1)], [(A,1), (B,2)], [(A,1), (B,3)],
[(A,2), (B,1)], [(A,2), (B,2)], [(A,2), (B,3)],
[(A,3), (B,1)], [(A,3), (B,2)], [(A,3), (B,3)],
[(A,1), (A,2)], [(A,1), (A,3)], [(A,2), (A,3)],
[(B,1), (B,2)], [(B,1), (B,3)], [(B,2), (B,3)] $\}$

◄

As the name implies, the lexicographic product can be thought of in the same way as a dictionary ordering. Two books A1 and B1 would be shelved next to each other since A is adjacent to B. But for books A1 and A2, they would also be shelved next to each other, since A=A and 1 is adjacent to 2. Another way to picture the lexicographic product $G[H]$ is by forming a graph by replacing each vertex of G with a copy of H. Figure 2.2 illustrates this technique for $P_2[K_3]$.

The lexicographic product has a number of desirable properties that make it suitable to "iterate" a graph, the first of which is that it is an associative operation.

Theorem 2.1. *Lexicographic product is associative.*

Proof. Let G, H, and K be graphs, and let $f: G[(H[K])] \to (G[H])[K]$ such that $f\{\ (g,(h,k))\ \} = ((g,h),k)$. We will show that f is an isomorphism. First, f is clearly a bijection from $V(G) \times (V(H) \times V(K))$, the vertex set of $G[(H[K])]$ to $(V(G) \times V(H)) \times V(K)$, the vertex set of $(G[H])[K]$.

Now assume $(g_1,(h_1,k_1)) \sim (g_2,(h_2,k_2))$ in $G[(H[K])]$.

$\Leftrightarrow$ [$g_1 \sim g_2$ in G] or [$g_1 = g_2$ in G and $(h_1,k_1) \sim (h_2,k_2)$ in $H[K]$]

$\Leftrightarrow$ [$g_1 \sim g_2$ in G] or [$g_1 = g_2$ in G and {[$h_1 \sim h_2$ in H] or [$h_1 = h_2$ in H and $k_1 \sim k_2$ in K]}]

$\Leftrightarrow$ [$g_1 \sim g_2$ in G] or [$g_1 = g_2$ in G and $h_1 \sim h_2$ in H] or [$g_1 = g_2$ in G, $h_1 = h_2$ in H and $k_1 \sim k_2$ in K]

$\Leftrightarrow$ [$g_1 \sim g_2$ in G] or [$g_1 = g_2$ in G and $h_1 \sim h_2$ in H] or [$(g_1,h_1) = (g_2,h_2)$ in $G[H]$ and $k_1 \sim k_2$ in K]

$\Leftrightarrow$ [$(g_1,h_1) \sim (g_2,h_2)$ in $G[H]$] or [$(g_1,h_1) = (g_2,h_2)$ in $G[H]$ and $k_1 \sim k_2$ in K]

$\Leftrightarrow$ $((g_1,h_1),k_1)) \sim ((g_2,h_2)k_2)$ in $(G[H])[K]$

$\Leftrightarrow f\,(g_1,(h_1,k_1)) \sim f\,(g_2,(h_2,k_2))$.

$\therefore\ G[(H[K])] \cong (G[H])[K] \Rightarrow$ lexicographic product is associative. ◄

Because lexicographic product is associative, we can use the notation G^k to denote "lexicographic powers" of a graph G by setting $G^1 = G$ and $G^k = G[G[G[\ldots]]_k$, for $k = 2, 3, 4\ldots$ Using this technique, we can now iterate our graph by taking lexicographic powers.

We will be interested in the roots of the independence polynomial for these powers of G, and so a few theorems are in order.

Theorem 2.2. *The independence polynomial of* $G[H]$ *is given by* $i_{G[H]}(x) = i_G(i_H(x) - 1)$.

Proof. By definition, the polynomial $i(G, i(H, x) - 1)$ is given by

$$\sum_{k=0}^{\alpha_G} i_k^G \left(\sum_{j=1}^{\alpha_H} i_j^H x^j \right)^k, \tag{1}$$

where i_k^G is the number of independent sets of cardinality k in G (similarly for i_k^H). Now, an independent set in $G[H]$ of cardinality l arises by choosing an independent set in G of cardinality k, for some $k \in \{0, 1, \ldots, l\}$, and then, within each associated copy of H in $G[H]$, choosing a non-empty independent set in H, in such a way that the total number of vertices chosen is l. But the number of ways of actually doing this is exactly the coefficient of x^l in (1), which completes the proof. ◄

Example 2.3. The graph $P_3[C_4]$ is pictured below. The independence polynomial for this graph is $i_{P_3[C_4]}(x) = i_{P_3}(i_{C_4}(x) - 1) = 4x^4 + 16x^3 + 22x^2 + 12x + 1$ where

$i_{P_3}(x) = x^2 + 3x + 1$, and $i_{C_4}(x) = 2x^2 + 4x + 1$. Using the notation from Theorem 2.2 we can use a counting argument to find the coefficient of x^2 in $i_{P_3[C_4]}(x)$.

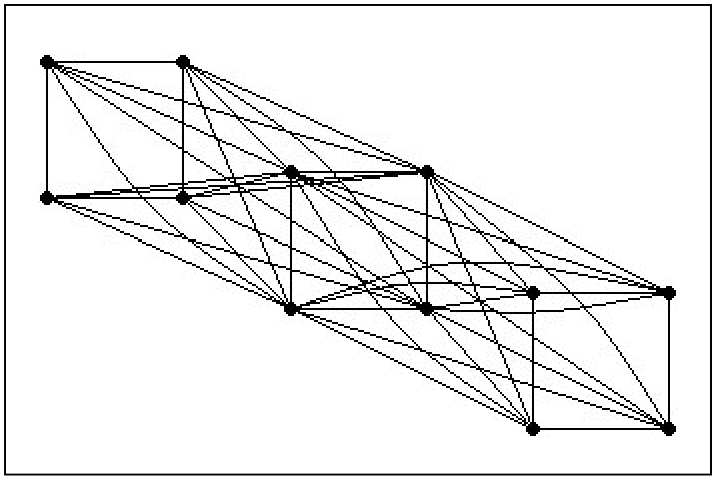


Fig. 2.3. The graph $P_3[C_4]$.

When $k = 0$ there is nothing to count, so we start with $k = 1$ and count the following:

A = ways to choose an independent set of order one from $P_3 = 3$

B = the number of independent sets of order two from $C_4 = 2$

For $k = 2$ we have:

C = ways to choose an independent set of order two from $P_3 = 1$

D = the number of independent sets of order one from $C_4 = 4$

E = the number of independent sets of order one from $C_4 = 4$

The total number of independent sets of order two in $P_3[C_4]$ is then found by:

$A * B + C * D * E = 3 * 2 + 1 * 4 * 4 = 6 + 16 = 22$

To find the coefficient of x^3 in $i_{P_3[C_4]}(x)$, for $k = 0$ or 1 there is nothing to count, so beginning with $k = 2$ we count the following:

A = ways to choose an independent set of order two from P_3 = 1

B = the number of independent sets of order two from C_4 = 2

C = the number of independent sets of order one from C_4 = 4

D = ways to choose an independent set of order two from P_3 = 1

E = the number of independent sets of order one from C_4 = 4

F = the number of independent sets of order two from C_4 = 2

The total number of independent sets of order three in P_3 [C_4] is then found by:

$A*B*C+D*E*F=1*2*4+1*4*2=8+8=16$ ◄

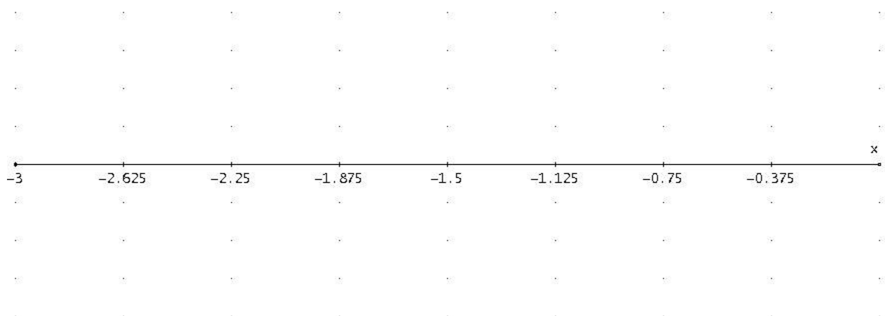
Theorem 2.2 simplifies the task of finding the independence polynomial of a lexicographic product, reducing the problem to elementary function composition. A small modification to the independence polynomial will simplify this task even further. Since every graph has one and only one independent subset of size 0 (the empty set), every independence polynomial has constant term 1. Define the **reduced independence polynomial** of G as the function $f_G(x)=i_G(x)-1$. By removing the constant term, Theorem 2.2 reduces to:

Corollary 2.3. $f_{G[H]}(x)=f_G(f_H(x))$.

This result makes it feasible to analyze the roots of the reduced independence polynomial for powers of a graph G. Recall that we defined $G^1 = G$ and $G^k = G[G[G[\ldots]]_k$, for

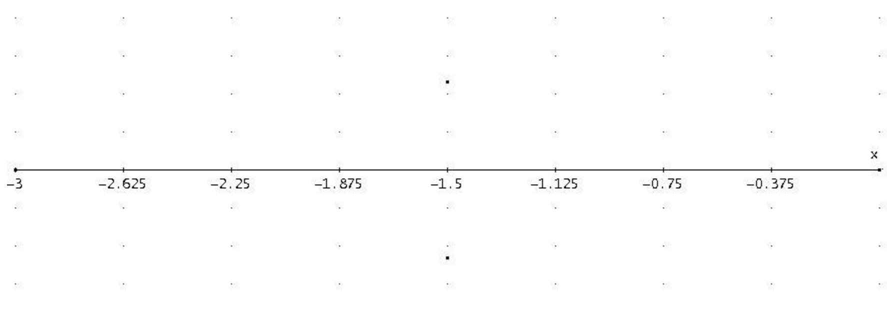
$k = 2, 3, 4\ldots$ Corollary 2.3 tells us that the reduced independence polynomials are closed under lexicographic powers: $f_{G^k}(x) = f_G(f_G(\ldots f_G(x)\ldots))_k$ for $k = 2, 3, 4\ldots$

Using the lexicographic powers to iterate the graph, we now examine the behavior of the roots of the reduced independence polynomial as $k \to \infty$. For the graph P_3, the following plots (on the complex plane) show the roots of the reduced independence polynomial $f_{P_3}(x) = x^2 + 3x$.



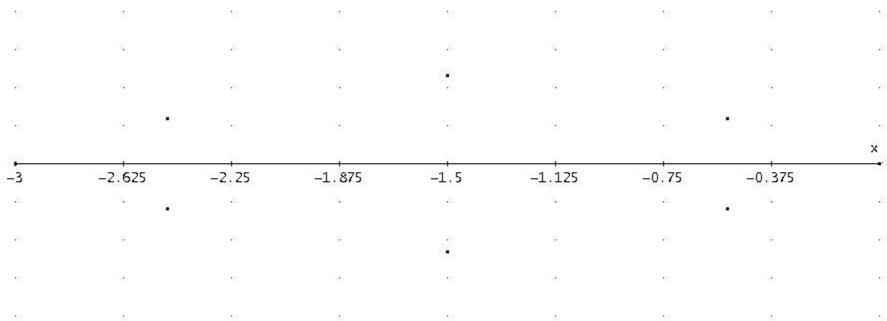
$$f_{P_3}(x) = x^2 + 3x,$$

roots: {0, 3}



$$f_{P_3^2}(x) = f_{P_3}(f_{P_3}(x)) = x^4 + 6x^3 + 12x^2 + 9x,$$

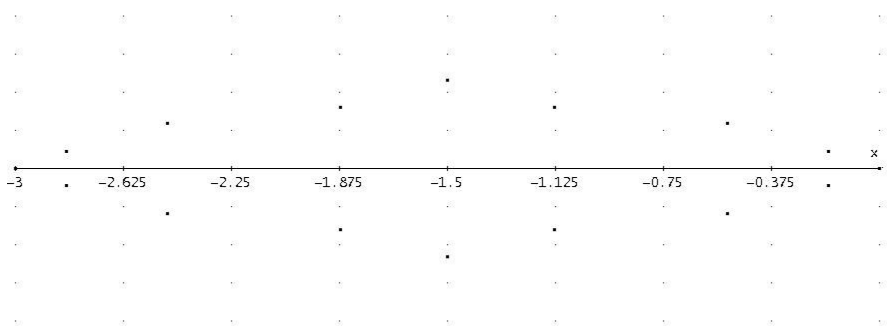
roots: {0, 3, -1.5-0.866*i*, -1.5+0.866*i*}



$$f_{P_3^3}(x) = f_{P_3}(f_{P_3}(f_{P_3}(x))) = x^8 + 12x^7 + 60x^6 + 162x^5 + 255x^4 + 234x^3 + 117x^2 + 27x,$$

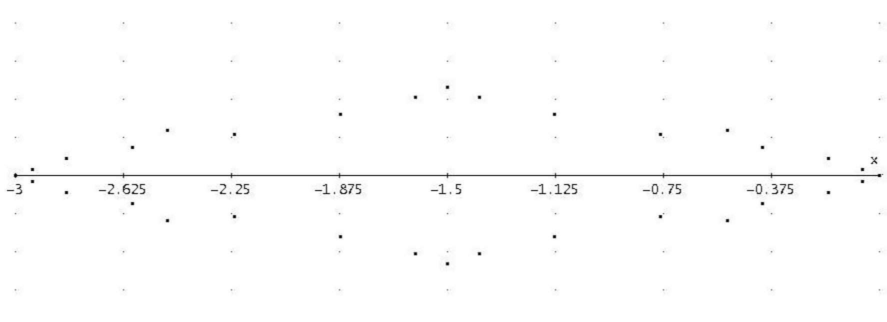
roots: {0, 3, -1.5-0.866*i*, -1.5+0.866*i*, -2.47-0.445*i*,

-2.47+0.445*i*, -0.526-0.445*i*, -0.526+0.445*i*}

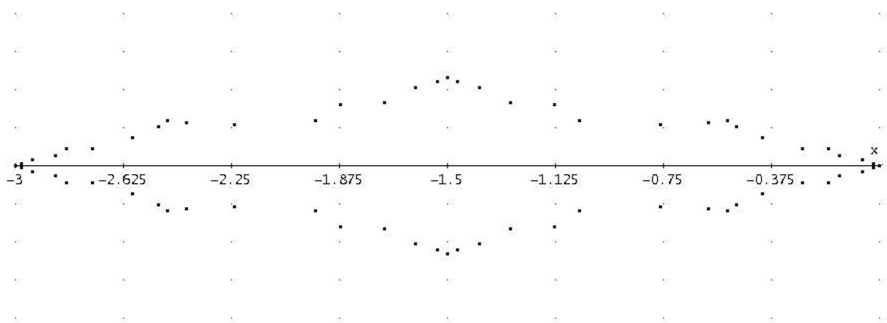


$f_{P_3^4}(x)$

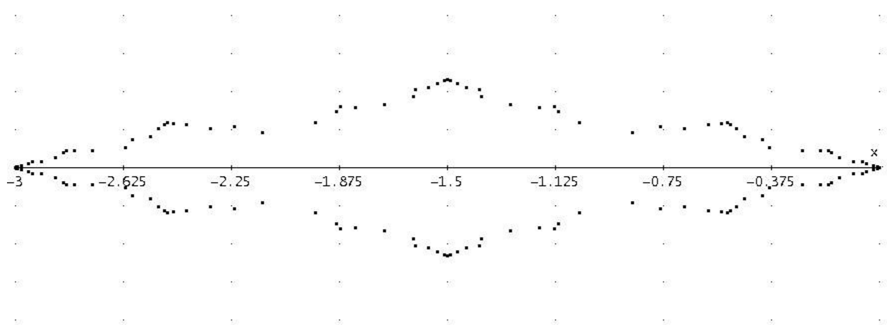
roots: (see Appendix [1] for derive output)



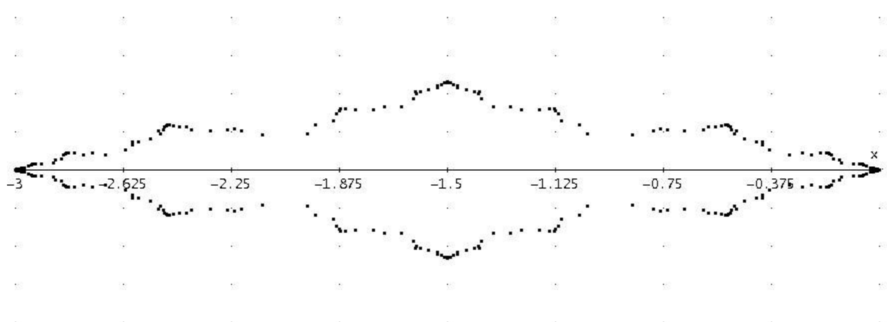
$f_{P_3^5}(x)$



$f_{P_3^6}(x)$



$f_{P_3^7}(x)$



$f_{P_3^8}(x)$

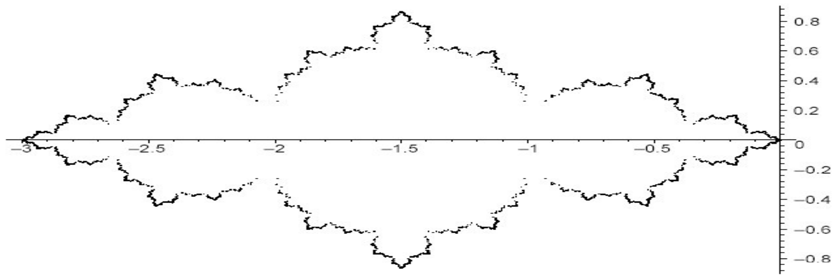


Fig. 2.5. The roots of the reduced independence polynomial for $G = P_3$.

The roots of the reduced independence polynomial for G^{11}, where $G = P_3$ are shown above. This is a polynomial of degree $2^{11} = 2048$. Notice the similarity between the placement of the roots in the plot above and the boundary of the black region in the fractal shown below.

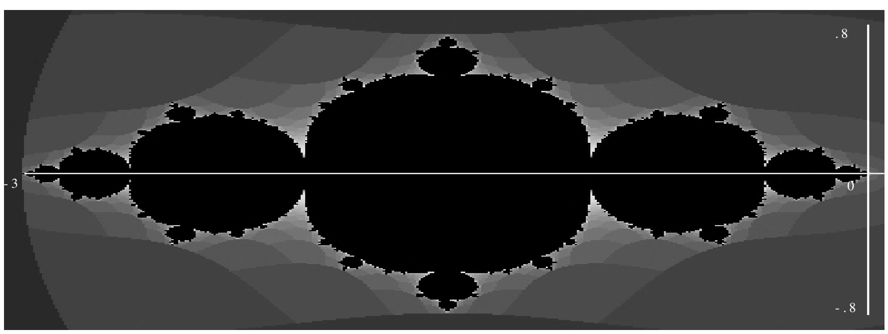


Fig. 2.6. A mystery fractal (see Figure 3.1).

As these figures illustrate, it appears that the roots of the modified independence polynomials are approaching a fractal-like object. Is it indeed a fractal? We will need a bit of background in iteration theory before we can answer this question definitively.

Section 3: Fractals & Iteration of Polynomials

There is a large body of work devoted to the study of iteration of polynomials (see [Barnsley, 1988], [Beardon, 1991], [Devaney, 1992]), which we will only be touching on in this paper. Unless otherwise noted, all definitions and results from this section can be found in [Beardon, 1991].

We will be working in the metric space ($\mathbb{C}$, $|\cdot|$), which is the complex plane combined with the absolute value metric, where $d(z, w) = |z - w|$ for $z, w \in \mathbb{C}$. For the remainder of this section, assume that *f* is a polynomial of degree at least 2.

For a polynomial *f* and a positive integer *k*, we define $f^{\circ k}$ as the map $f \circ f \circ \ldots \circ f_k$ with $f^{\circ(0)}$ as the identity map. Define $f^{\circ(-k)}$ as the map $f^{\circ(-1)} \circ f^{\circ(-1)} \circ \ldots \circ f^{\circ(-1)}{}_k$ where $f^{\circ(-1)}(A) = \{ z \in \mathbb{C} : f(z) \in A \}$ for $A \subseteq \mathbb{C}$.

The **forward orbit** of a point $z_0 \in \mathbb{C}$ with respect to *f* is the set: $\mathcal{O}^+(z_0) = \{f^{\circ k}(z_0)\}_{k=0}^{\infty}$.

For a polynomial *f*, its **filled Julia set** $K(f)$ is the set of all points *z* whose forward orbit $\mathcal{O}^+(z)$ is bounded in ($\mathbb{C}$, $|\cdot|$). The **Julia set** of *f*, $J(f)$ is the boundary $\partial K(f)$. The **Fatou set** $F(f)$ is the complement of $J(f)$ in $\mathbb{C}$.

Example 3.1. For the polynomial $f(x) = x^2 + 3x$, $f^{\circ 0}(x) = x$, $f^{\circ 1}(x) = f(x) = x^2 + 3x$, $f^{\circ 2}(x) = (f \circ f)(x) = x^4 + 6x^3 + 12x^2 + 9x$, etc… Let $z_0 = 1$, then

$$\mathcal{O}^+(1) = \{f^{\circ k}(1)\}_{k=0}^{\infty} = \{f^{\circ 0}(1), f^{\circ 1}(1), f^{\circ 2}(1), \ldots\} = \{1, f(1), f^{\circ 2}(1), f^{\circ 3}(1), \ldots\} = \{1, 4, 28, 868, \ldots\}$$

which is clearly unbounded, and therefore 1 is not in the filled Julia set of *f*.

On the other hand, setting $z_0 = -1$ yields

$$\mathcal{O}^+(-1) = \{f^{\circ k}(-1)\}_{k=0}^{\infty} = \{-1, f(-1), f^{\circ 2}(-1), f^{\circ 3}(-1), \ldots\} = \{-1, -2, -2, -2, \ldots\}$$ which is

bounded and so –1 is in $K(f)$. Figure 3.1 shows the filled Julia set of *f*. $K(f)$ is the region in black, indicating that those values have a bounded forward orbit. All points in gray have unbounded orbits, and are therefore in $F(f)$. ◄

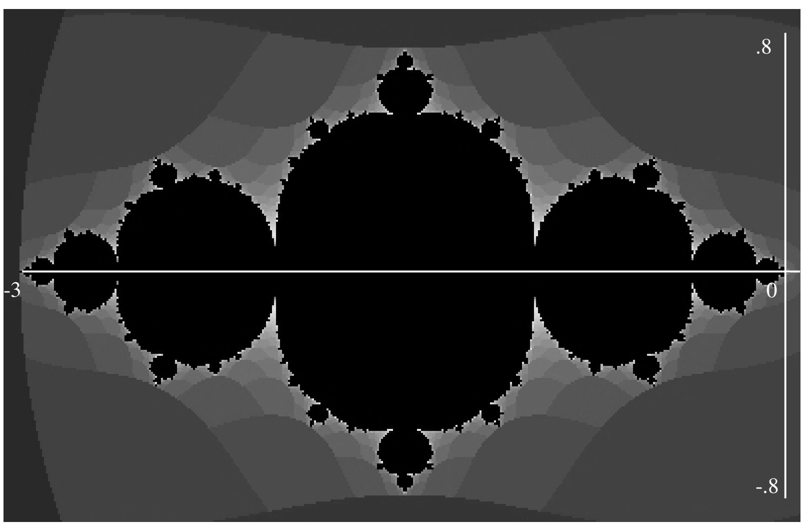


Fig. 3.1. The filled Julia set of *f* from Example 3.1.

Now that we have defined the sets $K(f)$, $F(f)$, and $J(f)$, we can explore some of their properties. If *g* is a map of a set *X* into itself, a subset *A* of *X* is **completely invariant** if $g(A) = A = g^{-1}(A)$. In [Beardon, 1991] it is shown that if *A* is completely invariant then the complement of A, the interior of A, the closure of A, and the boundary of A are all completely invariant as well. It is then shown that $F(f)$ is completely invariant, which leads to the following theorem.

Theorem 3.1. *The sets* $K(f)$, $F(f)$, *and* $J(f)$ *are all completely invariant.*

For any positive integer k, $F(f)$ invariant implies that $F\left(f^{\circ k}\right)=F\left(f^{\circ(k-1)}\right)=\ldots=F(f)$, so $F\left(f^{\circ k}\right)=F(f)$ and similarly $J\left(f^{\circ k}\right)=J(f)$.

There is an alternative definition for the Fatou and Julia sets, based on the notion of equicontinuity. A family of maps $\mathcal{F}$ from a metric space (X,d) to a metric space (X_1,d_1) is **equicontinuous** at the point x_0 in X if, for every positive ε, there is some positive δ such that for every x in X, and for all f in $\mathcal{F}$, $d(x_0,x)<\delta$ implies $d_1(f(x_0),f(x))<\varepsilon$. This definition extends the usual notion of continuity to a family of functions, and implies that all functions f in an equicontinuous family of functions $\mathcal{F}$ map the open ball with center x_0 and radius δ into a ball of radius at most ε.

It is shown in [Beardon, 1991] that an equivalent definition of the Fatou and Julia sets is as follows: For a non-constant polynomial f, $F(f)$ is the maximal open subset of $(\mathbb{C}, |\cdot|)$ on which $\left\{f^{\circ k}\right\}_{k=0}^{\infty}$ is equicontinuous. $J(f)$ is the complement of $F(f)$ in $(\mathbb{C}, |\cdot|)$.

By this new definition it is clear that $F(f)$ is an open subset of ($\mathbb{C}$, | · |), which makes $J(f)$ a closed subset of ($\mathbb{C}$, | · |). It can actually be shown that $J(f)$ is a **perfect** set, that is, a set which is equal its set of accumulation points. So $J(f)$ is closed, bounded, and uncountable.

There is yet another characterization of the sets $F(f)$ and $J(f)$, which Beardon describes as the "central idea in iteration theory." [Beardon, 1991] A point ζ is a **fixed point** of *f* if $f(\zeta)=\zeta$. If we assume that for some choice of z_0, the sequence $\{f^{\circ k}(z_0)\}_{k=0}^{\infty}$ (the forward orbit of z_0) converges to a point *w* then we have (from the continuity of f): $w=\lim_{n\to\infty} f^{\circ n}(z_0)=f\left(\lim_{n\to\infty} f^{\circ n-1}(z_0)\right)=f(w)$. So *w* is a fixed point of *f*, and all forward orbits of points in $\mathbb{C}$, if they converge, converge to a fixed point of *f*.

We can characterize the fixed points of a polynomial *f* by the behavior of the derivative of the function at the point ζ. For all polynomials, the derivative $f'(\zeta)$ is defined for any fixed point ζ, so we define ζ to be:

1. a **super-attracting fixed point** if $|f'(\zeta)|=0$;
2. an **attracting fixed point** if $|f'(\zeta)|<1$;
3. a **repelling fixed point** if $|f'(\zeta)|>1$; and
4. an **indifferent fixed point** if $|f'(\zeta)|=1$, which includes the following cases:
    a. a **rationally indifferent fixed point** if $|f'(\zeta)|$ is a root of unity, and

b. an **irrationally indifferent fixed point** if $|f'(\zeta)| = 1$ but is not a root of unity.

A **critical point** *z* of *f* is a point which has no local inverse, that is, *f* fails to be injective in any neighborhood of *z*. The distinction between super-attracting and attracting fixed points is that if ζ is super-attracting it is also a critical point of *f*, while attracting fixed points are not critical points. For an indifferent fixed point ζ, the best linear approximation to *f* near ζ is a rotation about ζ. Rationally indifferent fixed points can be approximated with a rotation of finite order, while irrationally indifferent fixed points are approximated with a rotation of infinite order.

The following two theorems describe the behavior of points in a neighborhood around ζ when iterating *f*.

Attracting Fixed Point Theorem 3.2. [Devaney, 1992] *Suppose ζ is an attracting (or super-attracting) fixed point for f. Then there is an interval I that contains ζ in its interior and in which the following condition is satisfied: if $x \in I$, then $f^{\circ n}(x) \in I$ for all n and $f^{\circ n}(x) \to \zeta$ as $n \to \infty$.*

Proof. ζ an attracting fixed point $\Rightarrow |f'(\zeta)| < 1,$ so there is a number $\lambda > 0$ such that $|f'(\zeta)| < \lambda < 1.$ We may therefore choose a number $\delta > 0$ so that $|f'(x)| < \lambda$ provided *x* belongs to the interval $I = [\zeta - \delta, \zeta + \delta].$ Now let *p* be any point in *I*. By the Mean Value Theorem, $\dfrac{|f(p) - f(\zeta)|}{|p - \zeta|} < \lambda,$ so that $|f(p) - f(\zeta)| < \lambda|p - \zeta|.$ Since ζ is a fixed point, it follows that $\left|f(p)-\zeta\right|<\lambda\left|p-\zeta\right|$. Since $\lambda<1$, this means that the distance from $f(p)$ to ζ is smaller than the distance from p to ζ. In particular, $f(p)$ also lies in the interval I. Therefore we may apply the same argument to $f(p)$ and $f(\zeta)$, finding $\left|f^{\circ 2}(p)-\zeta\right|=\left|f^{\circ 2}(p)-f^{\circ 2}(\zeta)\right|<\lambda\left|f(p)-f(\zeta)\right|<\lambda^2\left|p-\zeta\right|$. Since $\lambda<1$, $\lambda^2<\lambda$ and the points $f^{\circ 2}(p)$ and ζ are even closer together than $f(p)$ and ζ. We may continue using this argument to find that, for any $n>0$, $\left|f^{\circ n}(p)-\zeta\right|<\lambda^n\left|p-\zeta\right|$. Now $\lambda^n\to 0$ as $n\to\infty$, so $f^{\circ n}(p)\to\zeta$ as $n\to\infty$. ◄

By the same argument, we also have:

Repelling Fixed Point Theorem 3.3. [Devaney, 1992] *Suppose ζ is a repelling fixed point for f. Then there is an interval I that contains ζ in its interior and in which the following condition is satisfied: if $x\in I$ and $x\neq\zeta$, then there is an integer $n>0$, such that $f^{\circ n}(x)\notin I$.*

The figures below illustrate the behavior of a point z which is "close" to ζ when f is repeatedly applied. If z is close to ζ, then we can estimate $\left|f(z)-\zeta\right|=\left|f(z)-f(\zeta)\right|\approx\left|f'(\zeta)\right|\cdot\left|z-\zeta\right|$. If $\left|f'(\zeta)\right|<1$, this implies that $\left|f(z)-\zeta\right|\leq\left|z-\zeta\right|$, so points which are near an attracting fixed point will move closer with repeated applications of f.

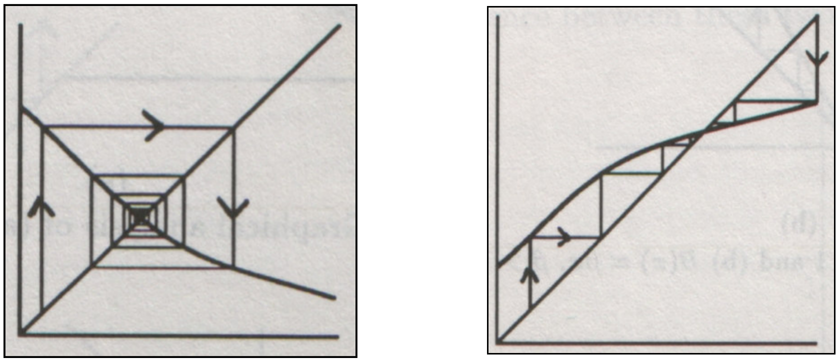


Fig. 3.2. The behavior of points near an attracting fixed point.

Similarly, $|f'(\zeta)| > 1$ implies that $|f(z) - \zeta| \geq |z - \zeta|$, so points near a repelling fixed point will tend to move away when *f* is applied.

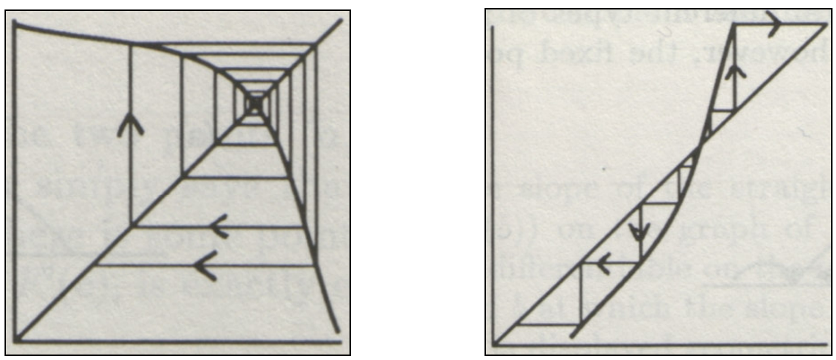


Fig. 3.3. The behavior of points near a repelling fixed point.

Example 3.2. For the function $f(x) = x^2 - 2$, 2 is a fixed point, since $f(2) = 2^2 - 2 = 4 - 2 = 2$. $|f'(2)| = |2 \cdot 2| = 4 > 1$, so 2 is a repelling fixed point of *f*. This implies that points in a neighborhood around 2 should escape that neighborhood, as can be seen by taking points close to 2 and iterating:

$$\{f^{\circ k}(2.001)\}_{k=0}^{\infty} = \{2.001,\ 2.004,\ 2.016,\ 2.064,\ 2.261,\ 3.11,\ 7.699,\ 57.274,\ 3278.344, \ldots\}$$

$\{f^{\circ k}(1.999)\}_{k=0}^{\infty} = \{1.999, 1.996, 1.984, 1.936, 1.749, 1.060, -0.876, -1.233, ...\}$.

On the other hand, the function $f(x) = -2x^2 + 2x$ has a fixed point at ½.

$\left|f'\left(\frac{1}{2}\right)\right| = \left|-4\left(\frac{1}{2}\right) + 2\right| = 0$, so ½ is a super-attracting fixed point of *f*. Taking points close to ½ shows: $\{f^{\circ k}(.45)\}_{k=0}^{\infty} = \{0.45, 0.495, 0.49995, 0.5, ...\}$ and

$\{f^{\circ k}(.55)\}_{k=0}^{\infty} = \{0.55, 0.495, 0.49995, 0.5, ...\}$ ◄

If $|f'(\zeta)| = 1$ then the situation is not as clear cut. In fact, the behavior of points in a neighborhood around an indifferent fixed point varies depending on the function involved, as the following example shows.

Example 3.3. The function $f(x) = -x$ has a fixed point at 0. This is an indifferent fixed point since $|f'(0)| = |-1| = 1$. This fixed point is neither attracting nor repelling, since for all $a \neq 0$,

$\{f^{\circ k}(a)\}_{k=0}^{\infty} = \{a, -a, a, -a, ...\}$.

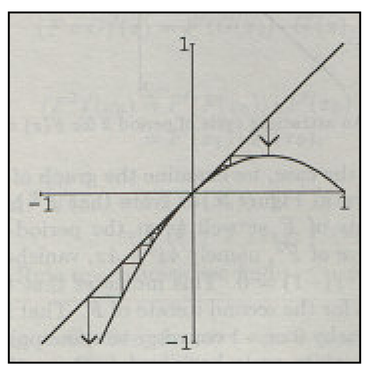


Fig. 3.4. An indifferent fixed point.

The function $f(x) = x - x^2$ also has an indifferent fixed point at 0, since $|f'(0)| = |1 - 2(0)| = 1$. The behavior of this fixed point is shown in the figure at right. Points $a > 0$ in a neighborhood around 0 are attracted to 0: $\{f^{\circ k}(.5)\}_{k=0}^{\infty} = \{.5, .25, .1875, .152344, ...\}$, whereas points $a < 0$ are repelled from 0:

$\{f^{\circ k}(-.5)\}_{k=0}^{\infty} = \{-.5, -.75, -1.3125, -3.0352, -12.2473, ...\}$.

Finally, an indifferent fixed point may attract all orbits in a neighborhood. The function $f(x) = x - x^3$ has an indifferent fixed point at 0 that attracts all $|a| < 1$:

$\{f^{\circ k}(-.5)\}_{k=0}^{\infty} = \{-.5, -.375, -.3223, -.2888, -.2647, ...\}$. Conversely the function $f(x) = x + x^3$ repels all orbits away from 0. These fixed points are called **weakly** attracting (or repelling), since the convergence (divergence) is slow. ◄

A point z_0 is a **periodic point** of *f* if, for some positive integer *k*, $f^{\circ k}(z_0) = z_0$. The smallest such *k* is the **period** of z_0. If *k* =1 then z_0 is a fixed point of *f*. The forward orbit of a periodic point with period *k* is $\{f^{\circ i}(z_0)\}_{i=0}^{\infty} = \{z_0, f(z_0), f^{\circ 2}(z_0), ..., f^{\circ k}(z_0), ...\}$ $= \{z_0, f(z_0), f^{\circ 2}(z_0), ..., f^{\circ k}(z_0) = z_0, f(z_0), f^{\circ 2}(z_0), ..., f^{\circ (k-1)}(z_0), z_0, f(z_0), ...\}$, which is a cycle of length *k*.

As with fixed points, we can characterize periodic points using $\lambda = \left(f^{\circ k}\right)'(z_0)$, the derivative of $f^{\circ k}$ evaluated at z_0, where *k* is the period of the cycle. λ is known as the **multiplier** of the cycle, and is independent of which z_0 is chosen from the cycle (we prove this below). The cycle is:

1. a **super-attracting cycle** if $|\lambda| = 0$;
2. an **attracting cycle** if $0 < |\lambda| < 1$;
3. a **repelling cycle** if $|\lambda| > 1$;
4. a **rationally indifferent cycle** if λ is a root of unity; and
5. an **irrationally indifferent cycle** if $|\lambda| = 1$ but λ is not a root of unity.

It can be shown (see notes below) that:

i. attracting (or super-attracting) cycles lie in $F(f)$;

ii. repelling cycles lie (and are dense) on $J(f)$;

iii. rationally indifferent cycles lie on $J(f)$; and

iv. irrationally indifferent cycles may lie in $J(f)$ or $F(f)$.

<u>Notes</u>:

i. As shown above in Theorem 3.2, $|\lambda| < 1$ and $\left|f^{\circ n}(p) - \zeta\right| < \lambda^n |p - \zeta|$ in a neighborhood around ζ. This shows that $f^{\circ n}$ maps a disc D (centered at ζ) into itself, leading to the conclusion that $\{f^{\circ k}(D)\}_{k=0}^{\infty}$ is equicontinuous, and thus lies in $F(f)$.

ii. **Proof.** Suppose the origin is a repelling fixed point of f. Then, near the origin $f(z) = az + \ldots$, where $|a| > 1$ and consequently as $n \to \infty$, $\left(f^{\circ 1}\right)'(0) = a^n \to \infty$.

Suppose for contradiction that 0 is in $F(f)$. Then $\{f^{\circ n}\}$ is equicontinuous on a neighborhood $\mathcal{N}$ of 0, and so some sequence of the iterates $f^{\circ n}$ converge uniformly on $\mathcal{N}$ to some analytic function *g*. Now, $g(0) = 0$, so $g'(0)$ is finite. On the other hand, the uniform convergence implies that for the given sequence, $g'(0) = \lim\left[\left(f^{\circ n}\right)'(0)\right] = \infty$, which is a contradiction. Therefore the repelling fixed point 0 is in $J(f)$. By conjugation, this implies that any repelling fixed point of *f* is in $J(f)$. If $\{\zeta_1,...,\zeta_k\}$ is any repelling cycle for *f*, then each ζ_i is in $J\left(f^{\circ n}\right)$, and since $J\left(f^{\circ n}\right) = J(f)$, the cycle is in $J(f)$. ◀

iii. See [Beardon, 1991].

iv. See [Beardon, 1991].

The chain rule gives a straightforward way to compute the multiplier. Assume that $\{z_0,..., z_{k-1}\}$ is a cycle in *f*. Computing the derivative for several iterates gives:

$\left(f^{\circ 2}\right)'(z_0) = f'(f(z_0)) \cdot f'(z_0) = f'(z_1) \cdot f'(z_0)$,

$\left(f^{\circ 3}\right)'(z_0) = f'\left(f^{\circ 2}(z_0)\right) \cdot \left(f^{\circ 2}\right)'(z_0) = f'(z_2) \cdot f'(z_1) \cdot f'(z_0)$, and with repeated application of the chain rule we have

$\left(f^{\circ k}\right)'(z_0) = f'\left(f^{\circ k-1}(z_0)\right) \cdot \left(f^{\circ k-1}\right)'(z_0) = f'(z_{k-1}) \cdot f'(z_{k-2}) \cdot ... \cdot f'(z_0)$. So the multiplier is $\lambda = \left(f^{\circ k}\right)'(z_0) = f'(z_{k-1}) \cdot f'(z_{k-2}) \cdot ... \cdot f'(z_0)$, the product of the derivative of *f* at all points on the orbit. Note that this is independent of which z_0 is chosen from the cycle.

Example 3.4. For the polynomial $f(x) = x^2 - 1$, $\{f^{\circ i}(-1)\}_{i=0}^{\infty} = \{-1,0,-1,0,...\}$, and so contains the cycle {-1,0} of length $k = 2$. The multiplier for this cycle is $\lambda = \left(f^{\circ 2}\right)'(-1) = f'(z_1)\cdot f'(z_0) = f'(0)\cdot f'(-1) = 0\cdot(-2) = 0$. This cycle is therefore super-attracting. ◀

Another characterization of the Julia set can be found by using the backward orbit of a point. For of a point $z_0 \in \mathbb{C}$, its **backward orbit** with respect to *f* is the set $O^-(z_0) = \bigcup_{k=0}^{\infty} f^{\circ(-k)}(z_0)$. A polynomial *f* has at most one **exceptional point** whose backwards orbit is finite. In [Beardon, 1991] it is shown that an exceptional point for *f*, if it exists, lies in $F(f)$.

Example 3.5. For $f(x) = x^2$, 0 is an exceptional point since $O^-(0) = \{0\}$. ◀

As we discovered above, repelling forward cycles of *f* lie on $J(f)$. When looking at backward orbits, it is therefore not surprising to find that $J(f)$ attracts backwards orbits of *f*, as outlined in the following theorems.

Theorem 3.4. *For a polynomial f (of degree ≥ 2), a non-empty open set W which meets* $J(f)$, *and for all sufficiently large integers n,* $f^{\circ n}(W) \supset J(f)$. (See [Beardon, 1991] for proof.)

Theorem 3.5. *For a polynomial f of degree at least 2,*

i. *If z is not exceptional, then $J(f)$ is contained in the closure of $O^{-}(z)$.*

ii. *If $z \in J(f)$, then $J(f)$ is the closure of $O^{-}(z)$.*

Proof. [Beardon, 1991] Consider any non-exceptional *z* and any non-empty open set *W* which meets $J(f)$. As *W* meets $J(f)$, Theorem 3.4 implies that *z* lies in some $f^{\circ n}(W)$ and so $O^{-}(z)$ meets *W*, which proves (i). If *z* is in $J(f)$, then the closed, completely invariant set $J(f)$ contains the closure of the backward orbit $O^{-}(z)$, and in conjunction with (i) yields (ii). ◄

In making the connection between the Julia set of a function *f* and the backward orbit of a point, we can actually do better than these two theorems. Using a metric on the finite sets $f^{\circ(-k)}(z_0)$, we can show that they converge to $J(f)$. Note that since the sets $f^{\circ(-k)}(z_0)$ are finite, they are necessarily compact.

The **Hausdorff metric** measures the distance between two compact subsets *A* and *B* of $(\mathbb{C}, |\cdot|)$ as $h(A,B) = \max(d(A,B), d(B,A))$ where $d(A,B) = \max_{a \in A} \min_{b \in B} |a-b|$. To compute this metric, first find the point in *A* which is closest to *B*, and the point in *B* that is farthest from it, and compute the distance between them (see the line in Figure 3.5, below). Next do the opposite, with *B* and *A*. The Hausdorff distance is the maximum of these two values.

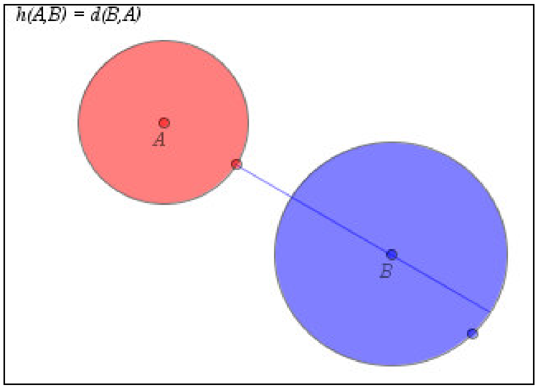


Fig. 3.5. Computing the Hausdorff metric I.

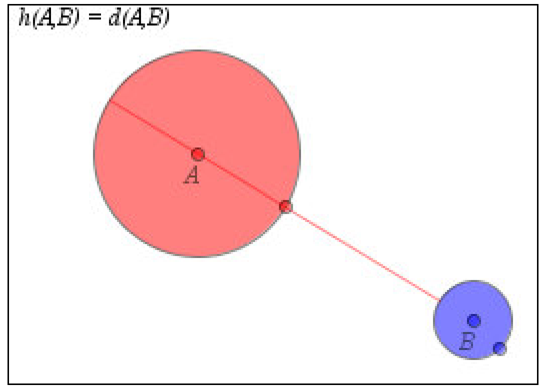


Fig. 3.6. Computing the Hausdorff metric II.

Example 3.6. Let $A = \{1, 2, 3\}$ and $B = \left\{\frac{1}{2}, 6, 7\right\}$. The values of $|a-b|$ are shown in Table 3.7.

Table 3.7. Computing the Hausdorff metric.

$\lvert a-b \rvert$		A		
		1	2	3
	0.5	0.5	1.5	2.5
B	6	5	4	3
	7	6	5	4

$$d(A,B) = \max_{a\in A} \min_{b\in B} |a-b| = \max_{a\in A} |a-.5| = 2.5,$$

$$d(B,A) = \max_{b\in B} \min_{a\in A} |b-a| = \max_{b\in B} |b-1| = 6,$$

$$h(A,B) = \max(d(A,B), d(B,A)) = \max(2.5, 6) = 6.$$ ◄

We need only a few more definitions to round out our discussion of iteration theory. A **Möbius map** is a rational map of the form $\phi(z) = \dfrac{az+b}{cz+d}$, with $ad - bc \neq 0$, for *a*, *b*, *c*, and *d* fixed complex numbers. The condition $ad - bc \neq 0$ ensures that ϕ is 1-1 and therefore invertible. Two polynomials *f* and *g* are **conjugate** if there exists a Möbius map ϕ such that $g = \phi \circ f \circ \phi^{-1}$. For two conjugate functions,

$g^{\circ k} = \left(\phi \circ f \circ \phi^{-1}\right)^{\circ k} = \phi \circ f \circ \phi^{-1} \circ \ldots \circ \phi \circ f \circ \phi^{-1} = \phi \circ f^{\circ k} \circ \phi^{-1}.$

Theorem 3.6. If $g = \phi \circ f \circ \phi^{-1}$ for some Möbius map ϕ, then $F(g) = \phi(F(f))$ and $J(g) = \phi(J(f))$. The sets $J(g)$ and $J(f)$ are then said to be analytically conjugate, as are $F(g)$ and $F(f)$.

A **Siegel disk** is a forward invariant component of $F(f)$ which is analytically conjugate to a Euclidean rotation of the unit disc onto itself. For our discussion here, we only need to know that they are contained in $F(f)$.

With the Hausdorff metric at our disposal, we now have the following theorem.

Theorem 3.7. [Brown, 2003] *Let f be a polynomial, and* z_0 *a point which does not lie in any attracting cycle or Siegel disk of f. Then* $\lim_{k\to\infty} f^{\circ(-k)}(z_0) = J(f)$, *where the limit is taken with respect to the Hausdorff metric on compact subspaces of* $(\mathbb{C}, |\cdot|)$.

Attracting cycles are also contained in $F(f)$, so we have from Theorem 3.7 that for any

$z_0 \in J(f)$, $\lim_{k \to \infty} f^{\circ(-k)}(z_0) = J(f)$.

Section 4: The Independence Fractal of a Graph

We now have the background information needed to tackle our main goal: to associate a fractal with our graph. Once we have found our fractal (and shown it exists for all graphs), we then ask what the structure of this fractal can tell us about the structure of the graph.

We start with the roots of the reduced independence polynomial for powers of G. For each $k \geq 1$, let $\text{Roots}\left(f_{G^k}\right)$ be the set of roots of the reduced independence polynomial for G^k, a lexicographic power of G. $\text{Roots}\left(f_{G^k}\right)$ is a finite and therefore compact subset of $(\mathbb{C}, |\cdot|)$. By definition, $f^{\circ(-1)}(0) = \{ z \in \mathbb{C} : f(z) = 0 \}$, so $\text{Roots}\left(f_G\right) = f^{\circ(-1)}(0)$. We have already shown that lexicographic product is associative, so for $k \geq 2$, $G^k = G^{k-1}[G]$, which along with Corollary 2.3 implies that $f_{G^k} = f_{G^{k-1}} \circ f_G$.

So $\text{Roots}\left(f_{G^k}\right) = \text{Roots}\left(f_{G^{k-1}} \circ f_G\right)$. The roots of the polynomial $f_{G^{k-1}} \circ f_G$ consists of the set of all points mapped by f_G to the roots of $f_{G^{k-1}}$, which is the set $f_G^{\circ(-1)}(\text{Roots}\left(f_{G^{k-1}}\right))$. Therefore $\text{Roots}\left(f_{G^k}\right) = f_G^{\circ(-1)}(\text{Roots}\left(f_{G^{k-1}}\right))$. Since $\text{Roots}\left(f_G\right) = f^{\circ(-1)}(0)$, we have $\text{Roots}\left(f_{G^k}\right) = f^{\circ(-k)}(0)$.

Since $f_G(x) = \sum_{k=1}^{\alpha} i_k x^k$, we have that $f_G(0) = 0$, so $0 \in f_G^{\circ(-1)}(0)$. Applying $f_G^{\circ(-1)}$ to both sides yields $f_G^{\circ(-1)}(0) \subseteq f_G^{\circ(-2)}(0)$, and by repeated applications gives

$f_G^{\circ(-k)}(0) \subseteq f_G^{\circ(-(k+1))}(0)$ for all k. So for any power k of G, Roots$\left(f_{G^k}\right) \subseteq$ Roots$\left(f_{G^{k+1}}\right)$.

That is, roots "stick around" for these polynomials, and once you have found one for f_{G^k}, that root will be a root for all f_{G^m}, $m > k$.

Now, define the **independence fractal** of a graph G as the set $\mathcal{F}(G) = \lim_{k \to \infty} \text{Roots}\left(f_{G^k}\right)$.

The following theorem shows that the independence fractal exists for all graphs G.

Theorem 4.1. *The independence fractal $\mathcal{F}(G)$ of a graph $G \neq K_1$ is precisely the Julia set $J(f_G)$ of its reduced independence polynomial $f_G(x)$. Equivalently, $\mathcal{F}(G)$ is the closure of the union of the reduced independence roots of powers of G^k, k =* 1,2,…,∞.

Proof. If G has independence number 1, then $G = K_n$ for some $n \geq 2$, and $f_G(x) = nx$. Each non-zero point therefore has an unbounded forward orbit, so the Julia set for f_G is $\{0\}$. Now, $G^k = (K_n)^k = K_{n^k}$, since by the definition of lexicographic product, all vertices adjacent in K_n implies that all vertices in $K_n[K_n]$ will be adjacent, and there will be n^k of them. So $f_{G^k}(x) = n^k x$, and the set of roots of this polynomial is $\{0\}$. The union and limiting root set is therefore $\{0\} = J(f)$, and the result holds.

If G has independence number at least 2, then $f_G(x)$ has degree at least 2. Since $f_G(x) = \sum_{k=1}^{\alpha} i_k x^k$, we have that $f_G(0) = 0$ and $f_G'(0) = i_1 = |V(G)| > 1$. Thus 0 is a repelling fixed point of $f_G(x)$ and therefore lies in $J\left(f_G(x)\right)$. In particular, $z_0 = 0$

satisfies the hypothesis of Theorem 3.7. This, along with the fact that Roots $\left(f_{G^k}\right) = f^{\circ(-k)}(0)$ gives $\mathcal{F}(G) = \lim_{k\to\infty}$ Roots $\left(f_{G^k}\right) = \lim_{k\to\infty} f^{\circ(-k)}(0) = J\left(f_G\right)$.

From Theorem 3.5, we know that if $0 \in J(f_G)$, then $J(f_G)$ is the closure of $O^{-}(0)$, so $\mathcal{F}(G) = J(f_G) = \mathrm{CL}\left[O^{-}(z_0)\right] = \mathrm{CL}\left[\bigcup_{k=0}^{\infty} f^{\circ(-k)}(0)\right] = \mathrm{CL}[\bigcup_{k=0}^{\infty}$ Roots $\left(f_{G^k}\right)]$, where CL[] denotes topological closure. So $\mathcal{F}(G)$ is the closure of the union of the reduced independence roots of powers of G^k, which completes the proof. ◀

There are only two graphs which Theorem 4.1 leaves out, the empty graph and $G = K_1$. In the latter case, $f_G(x) = x$ and $f_{G^k}(x) = x$ for all k, so $\mathcal{F}(G) = \{0\}$. The case of the empty graph is considered in [Brown, 2003]. Theorem 4.1 answers the question of whether the limit of the sequence {Roots $\left(f_{G^k}\right)$} exists in general with respect to the Hausdorff metric of compact subsets of $(\mathbb{C}, |\cdot|)$. It does, and the limit is $\mathcal{F}(G)$, the independence fractal of G, which is also the Julia set of f_G.

We can feel comfortable then calling $\mathcal{F}(G)$ the independence *fractal*, since Julia sets are typically fractals (in some sense). By typically, we mean that nearly all Julia sets are fractal-like objects. For instance, for Julia sets generated by the quadratic mapping $z_{n+1} = z_n^2 + c$, most values of c produce a fractal. The resulting object is not a fractal for

$c = 0$ and $c = -2$, and "it does not seem to be known if these two are the only such exceptional values." [Weisstein, 1999]

Now that we have succeeded in establishing our goal of associating a fractal with each graph, we turn our attention to answering a few basic questions about what the structure of the fractal says about the graph. Recall that our overall goal has been to use the fractal to somehow "encode" information about the graph. One obvious attribute of a fractal is whether it is a connected set or disconnected set. As the figures below demonstrate, Julia sets of polynomials are not, in general, connected. The following theorem gives us some guidance as to when we will find a connected or totally disconnected Julia set. A **totally disconnected set** is one whose components (maximally connected subsets) contain just one point.

Theorem 4.2. [Beardon, 1991] *Let f be a polynomial of degree at least two. Its Julia set* $J(f)$ *is connected iff the forward orbit of each of its critical points is bounded in* $(\mathbb{C}, |\cdot|)$. *Its Julia set* $J(f)$ *is totally disconnected if (but not only if) the forward orbit of each of its critical points is unbounded in* $(\mathbb{C}, |\cdot|)$.

Example 4.1. For $f(x) = 2x^2 + 3x$, f has one critical point at

$f'(x) = 4x + 3 = 0 \Rightarrow x = -\frac{3}{4}$, and

$\mathcal{O}^+(-.75) = \{f^{\circ k}(-.75)\}_{k=0}^{\infty} = \{-.75, -1.125, -0.844, -1.107, -0.870, \ldots\}$, which is bounded.

The independence fractal of *f* is therefore connected by Theorem 4.2, as shown in Figure 4.1, below.

For $f(x) = (1+2x)^3 - 1$, *f* has one critical point at $f'(x) = 6(1+2x)^2 = 0 \Rightarrow x = -\frac{1}{2}$, and $\mathcal{O}^+(-.5) = \left\{f^{\circ k}(-.5)\right\}_{k=0}^{\infty} = \{-0.5, -1, -2, -28, -166376, \ldots\}$, which is unbounded. So the independence polynomial of *f* is completely disconnected, as is shown in Figure 4.2.

◄

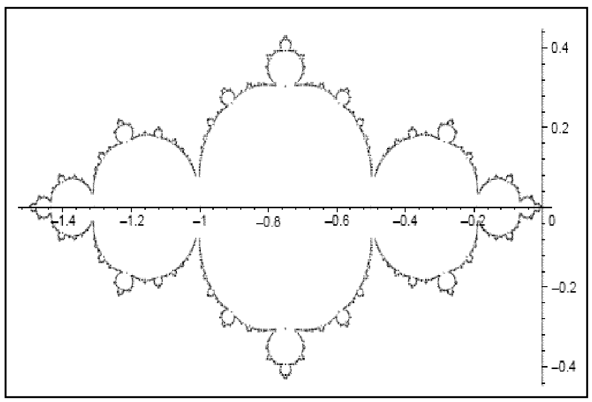


Fig. 4.1. Independence fractal of $f(x) = 2x^2 + 3x$.

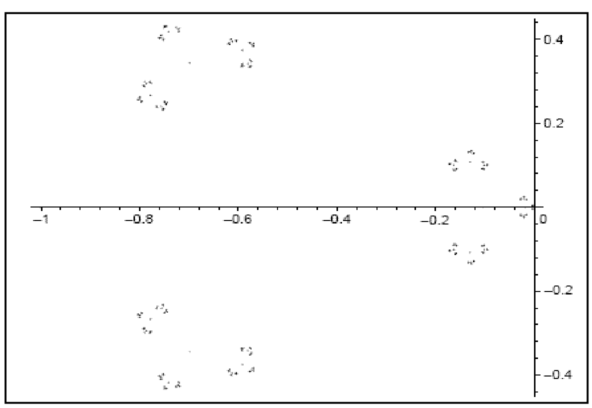


Fig. 4.2. Independence fractal of $f(x) = (1 + 2x)^3 - 1$.

Using Theorem 4.2 we can show that nearly every graph (the only exception being complete graphs) is contained in a graph with the same independence number, whose independence fractal is disconnected.

Theorem 4.3. *Every graph G with independence number at least two is an induced subgraph of a graph H with the same independence number, whose independence fractal is disconnected.*

Proof. Since $f_G(x)$ has degree at least 2, we can write that

$f_G(x) = a_1 x + a_2 x^2 + \ldots + a_\alpha x^\alpha$, where α is the independence number of G, each a_i is a positive number or zero, a_α is at least one, and $a_1 \geq 2$, since a_1 is the number of vertices in the graph. So $|f_G(x)| = |a_1 x + a_2 x^2 + \ldots + a_\alpha x^\alpha| = |a_1 x + A|$, which, along with the fact that $a_1 \geq 2$ leads to $|z| > 1 \Rightarrow |f_G(z)| = |a_1 z + A| > 2|z|$, since $|z| > 1 \Rightarrow A = a_2 z^2 + \ldots + a_\alpha z^\alpha > 0$. This in turn implies that there exists a real number $R > 1$ such that $|z| > R \Rightarrow |f_G(z)| > 2|z|$, so the forward orbit of z is unbounded in ($\mathbb{C}$, $|\cdot|$).

Now, not every critical point of f_G is a root of f_G. Indeed, for a root r of both f_G' and f_G, its multiplicity as a root of f_G is one greater than its multiplicity as a root of f_G'. But $\deg f_G = \deg f_G' + 1$, and so, if every critical point of f_G were a root of f_G then in fact f_G must have only one critical point c, and $f_G(x) = a(x+c)^\alpha$. But we know that $x | f_G(x)$, (since the constant term of our modified independence polynomial is always 0) and so $c = 0$ and $f_G(x) = ax^\alpha$. This could only be the case if $\alpha = 1$, which it is not.

Let c then be a critical point of f_G for which $f_G(c) = w \neq 0,$ and choose a positive integer p large enough that $|p \cdot w| > R$. For the graph $G[K_p],$ we have $f_{G[K_p]}(x) = f_G(px),$ a critical point of which is $\dfrac{c}{p}$. But then $f_{G[K_p]}\left(\dfrac{c}{p}\right) = f_G(c) = w,$ and $\left|f_{G[K_p]}^{\circ k}(w)\right| = \left|f_G^{\circ k}(pw)\right| \to \infty$ as $k \to \infty$. Hence, by Theorem 4.2, the graph $G[K_p],$ which has independence number α, and of which G is an induced subgraph, has a disconnected independence fractal. ◀

Theorem 4.3 goes beyond proving that for nearly every graph G such a graph (disconnected independence fractal, same independence number, G an induced subgraph) exists. It actually finds the graph, and shows that once p becomes sufficiently large, $G[K_p]$ has a disconnected independence fractal for all large p. The following theorem shows that we can extend this result to $K_p[G]$.

Theorem 4.4. *For a graph G and positive integer p,* $f_{K_p[G]}(px) = p \cdot f_G(px)$ $= p \cdot f_{G[K_p]}(x)$. *That is,* $f_{K_p[G]} \circ \phi = \phi \circ f_{G[K_p]},$ *where* ϕ *is the Möbius map* $x \mapsto px$. *Hence,* $\mathcal{F}(K_p[G]) = p \cdot \mathcal{F}(G[K_p])$.

Proof. K_p a complete graph implies that $f_{K_p}(x) = px$. We know from Corollary 2.3 that $f_{G[H]}(x) = f_G(f_H(x)),$ and from Theorem 4.3 $f_{G[K_p]}(x) = f_G(px),$ so $f_{K_p[G]}(px) = f_{K_p}(f_G(px)) = p \cdot f_G(px) = p \cdot f_{G[K_p]}(x)$. For ϕ defined above, ϕ is a

Möbius map, and $\phi \circ f_{G[K_p]} \circ \phi^{\circ(-1)} = f_{K_p[G]}$, so by Theorem 3.6,

$$K(f_{K_p[G]}) = \phi(K(f_{G[K_p]})) = p \cdot K(f_{G[K_p]}).$$ ◄

Theorem 4.4 shows that the independence fractal of $K_p[G]$ is a *p*-scaling of the independence fractal of $G[K_p]$. This implies that if the independence fractal of $G[K_p]$ is disconnected, then the independence fractal of $K_p[G]$ will also be disconnected. Since $K_p[G]$ is the join of *p* copies of *G*, Theorem 4.4 has the following corollary.

Corollary 4.5. *If G is a graph with independence number at least 2, then for all sufficiently large p, the join of p copies of G has a disconnected independence fractal.*

We have shown that for any graph *G* (with connected or disconnected independence fractal), there are many (in fact, infinitely many) graphs with *G* as an induced subgraph and a disconnected independence fractal. Since the graph $K_p[G]$ is connected for *G* connected, it appears that there is no correlation between the connectivity of a graph and the connectivity of its independence polynomial.

Section 5: The Mandelbrot Set & Other Examples

While we have not yet discovered a structural link between G and its independence fractal, we can use some results from iteration theory and fractal geometry, in particular the study of the Mandelbrot set, to completely describe the independence fractals of graphs with independence number up to 2.

For non-empty graphs (graphs with $E(G)$ non-empty), $\alpha > 0$. Graphs with independence number 1 are the complete graphs, and $f_{K_n}(x) = nx$. The only unbounded orbit for this polynomial is $z = 0$, so the Julia set for these graphs is $\{0\}$.

The **Mandelbrot set** $\mathcal{M}$, pictured below in Figure 5.1, is the set of all complex numbers c for which the Julia set of the polynomial $x^2 + c$ is connected. As was shown in Theorem 4.2, $J(x^2 + c)$ is connected only when the critical point $\frac{d}{dx}(x^2 + c) = 2x = 0 \Rightarrow x = 0$ has a bounded forward orbit. For values of c outside the Mandelbrot set, the forward orbit of 0 is unbounded, so $J(x^2 + c)$ is totally disconnected, otherwise known as **fractal dust**. It is shown in [Beardon, 1991] that $\mathcal{M}$ is contained in the disk $|c| < 2$.

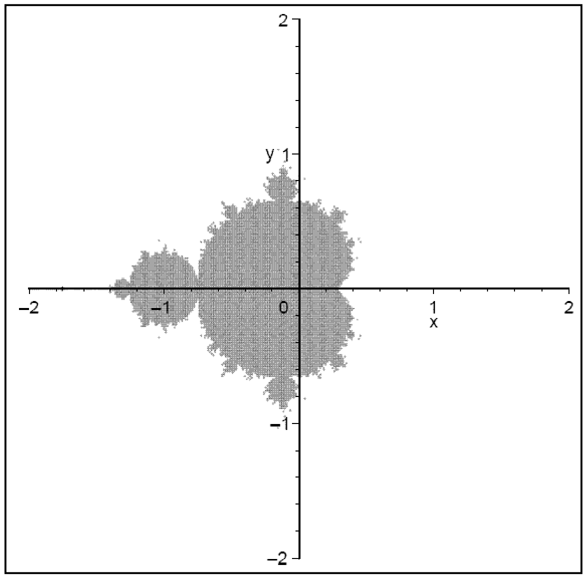


Fig. 5.1. The Mandelbrot set.

For a graph G with independence number 2, and with n vertices and m non-edges (that is $\overline{G}$ has m edges), $f_G(x) = mx^2 + nx$. We can use a Möbius transformation to find polynomials of the form $x^2 + c$ which are conjugate to f_G. Taking $\phi(x) = mx + \frac{n}{2}$, we have $\phi^{\circ(-1)}(x) = \frac{x}{m} - \frac{n}{2m}$ and

$$g_G(x) = \phi \circ f_G \circ \phi^{\circ(-1)} = \left[m\left(mx^2 + nx\right) + \frac{n}{2} \right] \circ \phi^{\circ(-1)} = m^2\left(\frac{x}{m} - \frac{n}{2m}\right)^2 + mn\left(\frac{x}{m} - \frac{n}{2m}\right) + \frac{n}{2}$$

$$= \frac{m^2x^2}{m^2} - \frac{2m^2nx}{2m^2} + \frac{m^2n^2}{4m^2} + \frac{mnx}{m} - \frac{mn^2}{2m} + \frac{n}{2} = x^2 + \frac{n}{2} - \left(\frac{n}{2}\right)^2.$$

So $g_G(x) = x^2 + \frac{n}{2} - \left(\frac{n}{2}\right)^2$, and by Theorem 3.6, $\mathcal{F}(G) = \phi^{\circ(-1)}\left(J(x^2 + c)\right)$, where

$$c = \frac{n}{2} - \left(\frac{n}{2}\right)^2.$$

This result shows that $\mathcal{F}(G) = \phi^{\circ(-1)}\left(J(x^2+c)\right) = \dfrac{J\left(x^2 + \frac{n}{2} - \left(\frac{n}{2}\right)^2\right)}{m} - \dfrac{n}{2m}$, which is a scaling and shifting of $J(x^2+c)$. As shown above, the connectivity of $J(x^2+c)$ (and therefore the connectivity of $\mathcal{F}(G)$) depends entirely on the value c, which in our case is $\frac{n}{2} - \left(\frac{n}{2}\right)^2$, a value dependent only on n, the number of vertices in the graph. The following examples describe the independence fractals of graphs with independence number $\alpha = 2$. G non-empty implies that $n \geq 3$.

Example 5.1. There are two graphs with $\alpha = 2$ and $n = 3$, $K_1 \cup K_2$, the disjoint union of a point and an edge, and P_3, the path on three vertices. The reduced independence polynomials of these two graphs are analytically conjugate to

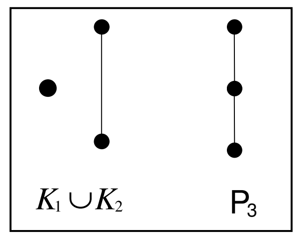


Fig. 5.2. Two graphs with $\alpha = 2$ and $n = 3$.

$g_G(x) = x^2 + \frac{n}{2} - \left(\frac{n}{2}\right)^2 = x^2 + \frac{3}{2} - \left(\frac{3}{2}\right)^2 = x^2 - \frac{3}{4}$. For $g_G(x)$, the forward orbit of 0 is $\mathcal{O}^+(0) = \left\{g_G^{\circ k}(0)\right\}_{k=0}^{\infty} = \{0, -.75, -.1875, -.7148, -.2390, -.6929, \ldots\}$, which is bounded and so $-\frac{3}{4}$ is in the Mandelbrot Set and $J\left(x^2 - \frac{3}{4}\right)$ is connected.

For $G = K_1 \cup K_2$, $f_G(x) = 2x^2 + 3x$ with $m = 2$ and $n = 3$, so

$\phi^{\circ(-1)}(x) = \frac{x}{m} - \frac{n}{2m} = \frac{x}{2} - \frac{3}{4}$. $\mathcal{F}(G) = \phi^{\circ(-1)}\left(J\left(x^2 - \frac{3}{4}\right)\right) = \frac{J\left(x^2 - \frac{3}{4}\right)}{2} - \frac{3}{4}$, which is a

scaling and shifting of $J\left(x^2 - \frac{3}{4}\right)$, and therefore is connected.

For $G = P_3$, $f_G(x) = x^2 + 3x$ with $m = 1$ and $n = 3$, so $\phi^{\circ(-1)}(x) = \frac{x}{m} - \frac{n}{2m} = x - \frac{3}{2}$.

$\mathcal{F}(G) = \phi^{\circ(-1)}\left(J\left(x^2 - \frac{3}{4}\right)\right) = J\left(x^2 - \frac{3}{4}\right) - \frac{3}{2}$, which again is a scaling and shifting of

$J\left(x^2 - \frac{3}{4}\right)$, and therefore is also connected. ◄

We can say a bit more about the independence fractals of graphs with $\alpha = 2$ and $n = 3$.

We show in Theorem 5.1, below, that $J\left(x^2 - \frac{3}{4}\right)$ is contained in a the box

$\left[-\frac{3}{2}, \frac{3}{2}\right] \times \left[-\frac{\sqrt{3}}{2}i, \frac{\sqrt{3}}{2}i\right]$. Since both graphs are conjugate to $g_G(x) = x^2 - \frac{3}{4}$, this result

allows us to find boxes bounding their Julia sets.

Theorem 5.1. *The Julia set* $J\left(x^2 - \frac{3}{4}\right)$ *is contained in the box*

$\left\{z : \left|\mathrm{Re}(z) \le \frac{3}{2}\right| \textit{ and } \left|\mathrm{Im}(z) \le \frac{\sqrt{3}}{2}\right|\right\}$, *where* $z \in \mathbb{C}$, $z = a + bi$, $\mathrm{Re}(z) = a$ *and* $\mathrm{Im}(z) = b$.

Proof. For $g(x)=x^2-\frac{3}{4}$, $|g(z)|^2=\left|z^2-\frac{3}{4}\right|^2=\left|(a+bi)^2-\frac{3}{4}\right|^2=\left|\left(a^2-b^2-\frac{3}{4}\right)+(2ab)i\right|^2$

$$=\left[\left(a^2-b^2-\frac{3}{4}\right)+(2ab)i\right]\left[\left(a^2-b^2-\frac{3}{4}\right)-(2ab)i\right]=\left(a^2-b^2-\frac{3}{4}\right)^2-(2abi)^2$$

$$=a^4-a^2b^2-\frac{3}{4}a^2-a^2b^2+b^4+\frac{3}{4}b^2-\frac{3}{4}a^2+\frac{3}{4}b^2+\frac{9}{16}+4a^2b^2$$

$$=a^4+b^4+2a^2b^2-\frac{3}{2}a^2+\frac{3}{2}b^2+\frac{9}{16}.$$

Since $a^4\geq 0$, if we assume $|b|>\frac{\sqrt{3}}{2}\Rightarrow |g(z)|^2\geq b^4+2a^2b^2-\frac{3}{2}a^2+\frac{3}{2}b^2+\frac{9}{16}$

$$>\left(\frac{\sqrt{3}}{2}\right)^4+2a^2\left(\frac{\sqrt{3}}{2}\right)^2-\frac{3}{2}a^2+\frac{3}{2}\left(\frac{\sqrt{3}}{2}\right)^2+\frac{9}{16}=\frac{9}{16}+\frac{3}{2}a^2-\frac{3}{2}a^2+\frac{9}{8}+\frac{9}{16}=\frac{9}{4}.$$

Similarly, since $b^4,b^2,a^2\geq 0$, if we assume

$$|a|>\frac{3}{2}\Rightarrow |g(z)|^2\geq a^4-\frac{3}{2}a^2+\frac{9}{16}>\left(\frac{3}{2}\right)^4-\frac{3}{2}\left(\frac{3}{2}\right)^2+\frac{9}{16}=\frac{81}{16}-\frac{27}{8}+\frac{9}{16}=\frac{9}{4}.$$

So $|b|>\frac{\sqrt{3}}{2}$ or $|a|>\frac{3}{2}\Rightarrow |g(z)|^2>\frac{9}{4}\Rightarrow |g(z)|>\frac{3}{2}$.

Now, choose a z such that $|b|>\frac{\sqrt{3}}{2}$ or $|a|>\frac{3}{2}$. Then $|g(z)|=\frac{3}{2}+\varepsilon$ for some $\varepsilon>0$, and

$\left|g^{\circ 2}(z)\right|=\left|[g(z)]^2-\frac{3}{4}\right|\geq |g(z)|^2-\frac{3}{4}\geq\left(\frac{3}{2}+\varepsilon\right)^2-\frac{3}{4}=\frac{3}{2}+3\varepsilon+\varepsilon^2>\frac{3}{2}+3\varepsilon.$ Similarly,

$\left|g^{\circ 3}(z)\right|=\left|[g^{\circ 2}(z)]^2-\frac{3}{4}\right|\geq\left(\frac{3}{2}+3\varepsilon\right)^2-\frac{3}{4}>\frac{3}{2}+3^2\varepsilon$, and $\left|g^{\circ(k+1)}(z)\right|\geq\frac{3}{2}+3^k\varepsilon$ for each

$k \geq 1$. This in turn implies that $\left|g^{\circ(k)}(z)\right| \to \infty$ as $k \to \infty$, so z is not in the Julia Set of g.

Therefore, all points in $J\left(x^2 - \frac{3}{4}\right)$ must be contained in the box

$\left\{z : \left|\operatorname{Re}(z) \leq \frac{3}{2}\right| \textit{ and } \left|\operatorname{Im}(z) \leq \frac{\sqrt{3}}{2}\right|\right\}$. ◄

We can in fact show that the bounding box around $J\left(x^2 - \frac{3}{4}\right)$ found in Theorem 5.1 is a tight box (that is, the best box we can find). For $g(x) = x^2 - \frac{3}{4}$,

$g\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - \frac{3}{4} = \frac{9}{4} - \frac{3}{4} = \frac{3}{2}$ and $g'\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right) = 3 > 1$, so $\frac{3}{2}$ is a repelling fixed point and therefore in $J(g)$. Likewise,

$$g\left(-\frac{3}{2}\right) = \left(-\frac{3}{2}\right)^2 - \frac{3}{4} = \frac{3}{2},\ g^{\circ 2}\left(\frac{\sqrt{3}}{2}i\right) = g\left(\left(\frac{\sqrt{3}}{2}i\right)^2 - \frac{3}{4}\right) = g\left(-\frac{3}{4} - \frac{3}{4}\right) = g\left(-\frac{3}{2}\right) = \frac{3}{2},$$

and

$$g^{\circ 3}\left(-\frac{\sqrt{3}}{2}i\right) = g^{\circ 2}\left(\left(-\frac{\sqrt{3}}{2}i\right)^2 - \frac{3}{4}\right) = g^{\circ 2}\left(\frac{3}{4} - \frac{3}{4}\right) = g^{\circ 2}(0) = g\left(0 - \frac{3}{4}\right) = \left(\frac{3}{4}\right)^2 - \frac{3}{4} = \frac{3}{2}.$$

$J(g)$ is completely invariant, so $\pm\frac{3}{2}$ and $\pm\frac{\sqrt{3}}{2}i$ are all in $J(g)$, making

$\left\{z : \left|\operatorname{Re}(z) \leq \frac{3}{2}\right| \textit{ and } \left|\operatorname{Im}(z) \leq \frac{\sqrt{3}}{2}\right|\right\}$ a tight box around $J\left(x^2 - \frac{3}{4}\right)$.

Example 5.1. (cont.) We found that for $G = K_1 \cup K_2$, $\mathcal{F}(G) = \dfrac{J\left(x^2 - \frac{3}{4}\right)}{2} - \dfrac{3}{4}$.

Applying this transformation to the bounding box we found above, we can say that

$$\mathcal{F}(G) = J\left(2x^2 + 3x\right) \subseteq \left[\frac{-\frac{3}{2}}{2} - \frac{3}{4}, \frac{\frac{3}{2}}{2} - \frac{3}{4}\right] \times \left[\frac{-\frac{\sqrt{3}}{2}i}{2} - \frac{3}{4}, \frac{\frac{\sqrt{3}}{2}i}{2} - \frac{3}{4}\right]$$

$$= \left[-\frac{3}{2}, 0\right] \times \left[-\frac{\sqrt{3}}{4}i, \frac{\sqrt{3}}{4}i\right].$$

For $G = P_3$, we apply the transformation:

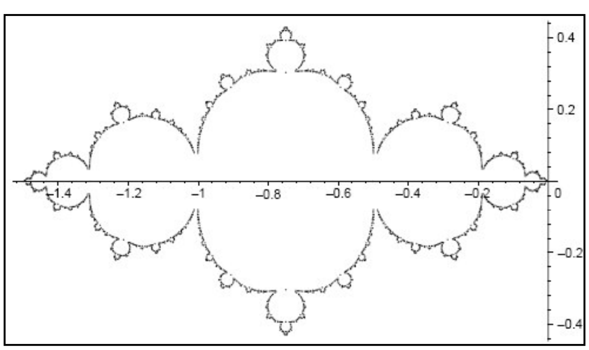
$$\mathcal{F}(G) = J\left(x^2 - \frac{3}{4}\right) - \frac{3}{2} \subseteq \left[-\frac{3}{2} - \frac{3}{2}, -\frac{3}{2} - \frac{3}{2}\right] \times \left[-\frac{\sqrt{3}}{2}i - \frac{3}{2}, -\frac{\sqrt{3}}{2}i - \frac{3}{2}\right]$$

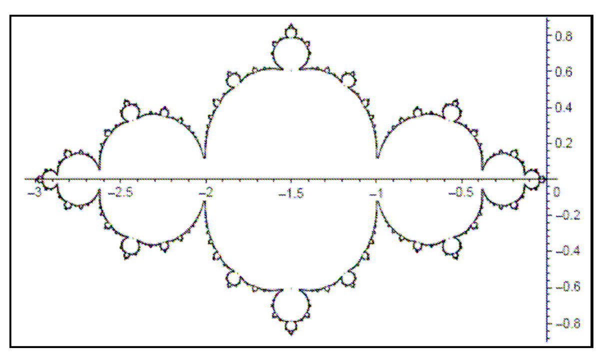
$$= [-3, 0] \times \left[-\frac{\sqrt{3}}{2}i, -\frac{\sqrt{3}}{2}i\right].$$

Figures 5.3 and 5.4 show the plots of $J\left(K_1 \cup K_2\right)$ (on the left) and $J(P_3)$ (on the right).

It is no coincidence that they have a similar look, since each is simply a scaling and

shifting of $J\left(x^2 - \frac{3}{4}\right)$. ◄





Figs. 5.3 and 5.4. The Julia sets of graphs with $\alpha = 2$ and $n = 3$.

Example 5.2. For a graph G with $\alpha = 2$ and $n = 4$, the reduced independence polynomial is analytically conjugate to $g_G(x) = x^2 + \frac{n}{2} - \left(\frac{n}{2}\right)^2 = x^2 - 2$ with $\phi(x) = mx + 2$. In [Devaney, 1992] it is shown that the Julia Set for the polynomial $g_G(x) = x^2 - 2$ is simply the interval $[-2,2]$. Applying $\phi^{\circ(-1)}(x) = \frac{x}{m} - \frac{4}{2m} = \frac{x}{m} - \frac{2}{m}$ to the interval $[-2,2]$ gives $\mathcal{F}(G) = \phi^{\circ(-1)}([-2,2]) = [\phi^{\circ(-1)}(-2), \phi^{\circ(-1)}(-2)] = \left[\frac{-2}{m} - \frac{2}{m}, \frac{2}{m} - \frac{2}{m}\right] = \left[\frac{-4}{m}, 0\right]$.

The graph $G = K_4 - e$, the complete graph on four vertices with one edge removed, has $\alpha = 2$, $n = 4$, and $m = 1$ non-edge. $f_G(x) = x^2 + 4x$ and $\phi(x) = mx + 2 = x + 2$. The independence fractal for this graph is therefore

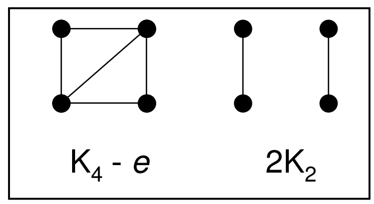


Fig. 5.5. Graphs with $\alpha = 2$ and $n = 4$.

$$\mathcal{F}(G) = \left[\frac{-4}{m}, 0\right] = \left[\frac{-4}{1}, 0\right] = [-4, 0].$$

The graph $G = 2K_2$, the disjoint union of 2 copies of K_2, has $\alpha = 2$, $n = 4$, and $m = 4$ non-edges. $f_G(x) = 4x^2 + 4x$ and $\phi(x) = mx + 2 = 4x + 2$. The independence fractal for this graph is therefore $\mathcal{F}(G) = \left[\frac{-4}{m}, 0\right] = \left[\frac{-4}{4}, 0\right] = [-1, 0]$. ◄

Example 5.3. For graphs G with $\alpha = 2$ and $n \geq 5$, the reduced independence polynomial is analytically conjugate to $g_G(x) = x^2 + c$ where

$c=\frac{n}{2}-\left(\frac{n}{2}\right)^2\leq\frac{5}{2}-\left(\frac{5}{2}\right)^2=\frac{5}{2}-\frac{25}{4}=-\frac{15}{4}<-2.$ Since the Mandelbrot set is contained in the disk $|c|<2,$ our c lies outside the Mandelbrot set, and therefore $J(x^2+c)$ is totally disconnected. This implies that $\mathcal{F}(G)=\phi^{\circ(-1)}\left(J(x^2+c)\right)$ is also totally disconnected.

It is known that for $c<-2,$ $J(x^2+c)$ is contained in the interval $[-q,q]$ where $q=\frac{1}{2}+\sqrt{\frac{1}{4}-c}.$

$$c=\frac{n}{2}-\left(\frac{n}{2}\right)^2\Rightarrow q=\frac{1}{2}+\sqrt{\frac{1}{4}-c}=\frac{1}{2}+\sqrt{\frac{1}{4}-\left(\frac{n}{2}-\left(\frac{n}{2}\right)^2\right)}=\frac{1}{2}+\sqrt{\left(\frac{n}{2}-\frac{1}{2}\right)^2}=\frac{1}{2}+\frac{n}{2}-\frac{1}{2}=\frac{n}{2},$$

so $q=\frac{n}{2}$ and $J(x^2+c)$ is contained in the interval $\left[-\frac{n}{2},\frac{n}{2}\right].$ Applying

$\phi^{\circ(-1)}(x)=\frac{x}{m}-\frac{n}{2m}$ to this interval gives

$\phi^{\circ(-1)}\left(\left[-\frac{n}{2},\frac{n}{2}\right]\right)=\left[\frac{-\frac{n}{2}}{m}-\frac{n}{2m},\frac{\frac{n}{2}}{m}-\frac{n}{2m}\right]=\left[-2\frac{n}{2m},\frac{n}{2m}-\frac{n}{2m}\right]=\left[-\frac{n}{m},0\right].$ The result is that for graphs G with $\alpha=2$ and $n\geq 5$, $\mathcal{F}(G)$ is a dusty subset of the interval $\left[-\frac{n}{m},0\right].$

For any graph G with $\alpha=2$ and $n\geq 5,$ $f_G(x)=ax^2+nx,$ and 0 is a fixed point of $f_G.$

$f_G'(0)=\left(2ax+n\right]_0=n>1,$ so 0 is a repelling fixed point and therefore lies in $J(f_G).$

$$f_G\left(-\frac{n}{m}\right)=a\left(-\frac{n}{m}\right)^2+n\left(-\frac{n}{m}\right)=a\left(\frac{n^2}{m^2}\right)-\left(\frac{n^2}{m}\right)=\frac{a}{m}\left(\frac{n^2}{m}\right)-\left(\frac{n^2}{m}\right)=\left(\frac{a}{m}-1\right)\left(\frac{n^2}{m}-\frac{n^2}{m}\right)=0,$$

$0 \in J(f_G)$, and $J(f_G)$ invariant imply that $-\frac{n}{m} \in J(f_G)$, so the bounding interval of

$\left[-\frac{n}{m}, 0\right]$ is the best we can do.

The graph $G = K_2 \cup K_3$, the disjoint union of K_2 with K_3, has $\alpha = 2$, $n = 5$, and $m = 6$ non-edges. $f_G(x) = 6x^2 + 5x$ and $\mathcal{F}(G)$ is a dusty subset of the interval $\left[-\frac{n}{m}, 0\right] = \left[-\frac{5}{6}, 0\right]$.

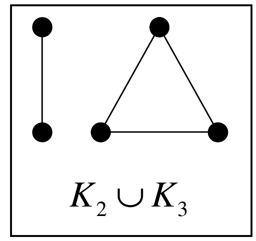


Fig. 5.6. A graph with $\alpha = 2$ and $n \geq 5$.

◄

Using the Mandelbrot set, we have now described the independence fractals for all graphs with independence number 2. Using the results from Examples 5.1-5.3, we can summarize the location of these fractals in the following theorem.

Theorem 5.2. *If G is a non-empty graph with independence number 2 having n vertices and m non-edges, and* $z \in \mathcal{F}(G)$, *then*

(i) $-\frac{n}{m} \leq \mathrm{Re}(z) \leq 0$, *and*

(ii) $\mathrm{Im}(z) = 0$ *unless n* = 3, *in which case* $-\frac{\sqrt{3}}{2m} \leq \mathrm{Im}(z) \leq \frac{\sqrt{3}}{2m}$.

Now that we have a good idea of the behavior of independence fractals for graphs with α = 2, it is natural to consider other families of graphs. The graphs aK_b are the disjoint

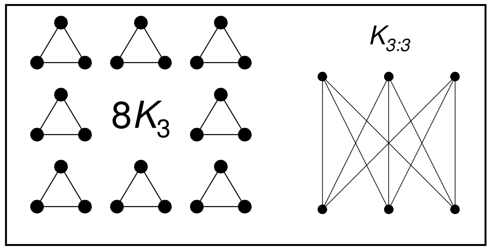


Fig. 5.7. The graphs 8K_3 and $K_{3:3}$.

union of *a* copies of K_b. The graphs $K_{a;b}$ are the family of complete multipartite graphs. Graphs in both families can have arbitrarily high independence numbers. We conclude this section of examples with a theorem which ties these two families together via their independence fractals.

Theorem 5.3. *The independence fractal of* aK_b *is connected if* $b = 2$ *and a is even, and totally disconnected otherwise. Likewise, the independence fractal of* $K_{a;b}$ *is connected if* $b = 2$ *and a is even, and totally disconnected otherwise.*

Proof. First note that $aK_b = \overline{K_a}[K_b]$, $f_{\overline{K_a}}(x) = (1+x)^a - 1$ and $f_{K_b}(x) = bx$. This implies that $f_{aK_b}(x) = f_{\overline{K_a}[K_b]}(x) = f_{\overline{K_a}}\left(f_{K_b}(x)\right) = f_{\overline{K_a}}(bx) = (1+bx)^a - 1$ and

$f'_{aK_b}(x) = ab(1+bx)^{a-1}$, whose only critical point is $z = -\dfrac{1}{b}$. By Theorem 4.2, $\mathcal{F}(G)$ will be connected when the forward orbit of z is bounded, and totally disconnected otherwise.

We are considering only non-empty graphs, so $b \geq 2$. For all $a \geq 1$ and $b \geq 2$,

$$f_{aK_b}\left(-\frac{1}{b}\right) = \left(1 + b\left(-\frac{1}{b}\right)\right)^a - 1 = (1-1)^a - 1 = -1.$$

Case 1: $b = 2$, a even. $f_{aK_2}(x) = (1+2x)^a - 1$, and the forward orbit of $-\frac{1}{b}$ is

$\left\{f_{aK_b}^{\circ k}\left(-\frac{1}{b}\right)\right\}_{k=0}^{\infty} = \left\{f_{aK_2}^{\circ k}\left(-\frac{1}{2}\right)\right\}_{k=0}^{\infty} = \left\{-\frac{1}{2},-1,0,0,\ldots\right\}$, which converges to 0 and is

therefore bounded in $(\mathbb{C}, |\cdot|)$. Thus, $\mathcal{F}(aK_b)$ is connected.

Case 2: $b \geq 3$, a even. $f_{aK_b}\left(-\frac{1}{b}\right) = -1$ and $f_{aK_b}(-1) = (1-b)^a - 1 \geq 2^a - 1 > 1$. Note that

$z > 1 \Rightarrow f_{aK_b}(z) = (1+bz)^a - 1 > (1+2z)^1 - 1 = 2z > z+1$, so the forward orbit of $-\frac{1}{b}$ is

unbounded, and $\mathcal{F}(aK_b)$ is totally disconnected.

Case 3: a ≥ 3, a odd. $f_{aK_b}\left(-\frac{1}{b}\right) = -1$ and $f_{aK_b}(-1) = (1-b)^a - 1 \leq (1-2)^3 - 1 = -2 < -1$.

Note that $z < -1 \Rightarrow f_{aK_b}(z) = (1+bz)^a - 1 < (1+2z)^1 - 1 = 2z = z + z < z - 1$, so the

forward orbit of $-\frac{1}{b}$ is unbounded, and $\mathcal{F}(aK_b)$ is totally disconnected.

Case 4: a = 1. Then $aK_b = K_b$, whose independence fractal we know is {0}, and thus

totally disconnected.

This proves that the independence fractal of aK_b is connected if $b = 2$ and a is even, and

totally disconnected otherwise. Now, $K_{a;b} = K_b\left[\overline{K_a}\right]$ and $aK_b = \overline{K_a}[K_b]$, so by Theorem

4.4, $\mathcal{F}(K_{a;b}) = b \cdot \mathcal{F}(aK_b)$, and $\mathcal{F}(K_{a;b})$ is connected precisely when $\mathcal{F}(aK_b)$ is, which completes the proof. ◄

Section 6: Conclusion

By defining the independence fractal, we set out to create a new piece of identification for each graph. This task is completed, but there are many unanswered questions about this new construction. From our preliminary work here, it is easier to talk about what the fractal *does not* tell us. We showed that a connected graph does not necessarily have a connected independence fractal, and the converse as well. While we were able to use the Mandelbrot set to analyze the independence fractal of graphs with independence number up to 2, all other graphs with $\alpha \geq 2$ await our attention. The Mandelbrot set for cubics is contained in $\mathbb{C} \times \mathbb{C}$, and is not well understood, so we will need to find another guide for many of the remaining graphs.

Some questions which remain for further study include:

- When do two graphs have analytically conjugate independence fractals?
- For which graphs G is $\mathcal{F}(G)$ connected?
- For groups or families of graphs (Cayley graphs, Cyclic graphs, or Polygons, etc.), is there a structure from the graph which is passed to the independence fractal?
- Do structural aspects of a graph (vertex or edge transitivity, for example) show up in independence fractals?
- Is the relationship between a graph and its subgraphs, complement, line graphs, etc., coded in a useful way in the independence fractal?

Section 7: Appendices

Appendix 1 – Derive Output

```
f(x):=x^2+3*x
APPROX(SOLVE(f(x),x))

[[-3,0],
[0,0]]
APPROX(SOLVE(f(f(x)),
x))

[[-1.5,-0.8660254037],
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[0,0]]
APPROX(SOLVE(f(f(f(x))
), x))

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APPROX(SOLVE(f(f(f(f(x
)))), x))

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Section 8: Works Cited

1. Graph Theory. 2005 [cited 2005; Available from: http://en.wikipedia.org/wiki/Graph_theory#History.

2. Barnsley, M., Fractals Everywhere. 1988, San Diego: Academic Press.

3. Beardon, A.F., Iteration of Rational Functions. Graduate Texts in Mathematics. Vol. 132. 1991, New York: Springer-Verlag.

4. Brown, J.I., Dilcher, K., Nowakowski, R.J., Roots of Independence Polynomials of Well Covered Graphs. Journal of Algebraic Combinatorics, 2000. 11: p. 197-210.

5. Brown, J.I., Hickman, C.A., Nowakowski, R.J., The independence fractal of a graph. Journal of Combinatorial Theory, Series B, 2003. 87(2): p. 209-230.

6. Devaney, R., A first course in chaotic dynamical systems: theory and experiment. Addison-Wesley studies in nonlinearity. 1992: Addison-Wesley.

7. Weisstein, E.W. Fractals. MathWorld 1999 2005 [cited 2005 11/20/05]; Available from: http://mathworld.wolfram.com/JuliaSet.html.

8. Weisstein, E.W. Julia Set. MathWorld 1999 2005 [cited 2005 11/20/05]; Available from: http://mathworld.wolfram.com/JuliaSet.html.

9. West, D., Introduction to Graph Theory. 1996, Upper Saddle River, NJ: Prentice-Hall.

10. Wilson, R., An Eulerian Trail Through Konigsberg. Journal of Graph Theory, 1986. 10(3): p. 265-275.