“The Friendship Theorem and Projective Planes”

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Friendship Theorem: In a party of n people,

suppose that every pair of people has exactly one common friend.

Then there is a person in the party who knows everyone else.

Introduction

The friendship theorem is a well-known and simply stated theorem from graph theory with many applications outside the field. Several different proofs have been provided over the years, and in this project, we consider some of the recent work surrounding this theorem. First we will present an overview of the history and development of the problem. After introducing all of the necessary notation and preliminaries, we will then give a careful treatment of one particularly nice approach to the problem, a fascinating article by Herbert Wilf that provides a geometric proof using projective planes. His proof begins by assuming the conclusion of the theorem is false, uses this assumption to construct a projective plane out of the “party,” and then produces a contradiction with the incidence matrix of the projective plane – this then proves the theorem true.

As the theorem sounds so “combinatorial,” mathematicians have historically searched for a proof that relies solely on combinatorial arguments. However, much simpler arguments, one of which we will examine here, delve into related branches of mathematics and tie together simple facts from linear algebra and matrix theory. None of Wilf’s arguments stray far beyond a basic undergraduate linear algebra course, and his projective plane arguments are intuitive and simple to comprehend, even for one who has not taken courses in geometry or graph theory.

History and development

The Friendship Theorem traces its roots back to the relatively early days of graph theory. Most authors recognize that the first published proof was given by Erdös, Rényi and Sós in 1966 in a Hungarian journal, although only as an un-named theorem, and since then many different proofs have been given by other authors. The theorem was originally presented in very un-friendly language:

Theorem (Erdös): If G_n is a graph in which any two points are connected by a path of length 2 and which does not contain any cycle of length 4, then $n = 2k+1$ and G_n consists of k triangles which have one common vertex.

Translated into the then-developing language of graph theory, we have something which much more closely resembles Wilf's statement of the friendship theorem:

Theorem (Huneke): If G is a graph in which any two distinct vertices have exactly one common neighbor, then G has a vertex joined to all others.

A flurry of activity surrounding the theorem occurred in the late sixties and early seventies. Since then the production of entirely new proofs has slowed, but an increasing number of applications and extensions have surfaced relating to block designs and coding theory. Historically, true combinatorial proofs have been hard to come by, especially simple ones, and most proofs rely on algebraic techniques.

Wilf gave a proof in 1969 (which we will examine in this paper) with roots in linear algebra and projective geometry. He computes the eigenvalues of the incidence matrix of the graph, and uses this to produce a contradiction. This will become a common way to prove the Friendship Theorem, although Wilf's is unique in that it starts by delving into geometry.

By 1972, Judith Longyear and T. D. Parsons had developed a proof based on counting neighbors, walks and cycles in regular graphs. Their paper also incorporates an extension into set theory. Both Longyear and Wilf reference an unpublished proof given by G. Higman in lecture form at a 1969 conference on combinatorics, but no known printed record of this exists.

More recently, J. M. Hammersley provided a proof at a 1983 conference that avoided using eigenvalues but involved admittedly complicated numerical techniques. Hammersley also extends the friendship theorem into what he calls the "love problem." Friendship is usually taken to be irreflexive (one cannot be friends with oneself), but love, as he points out, can be narcissistic and hence a reflexive relation. Hammersley's work is beyond the scope of this paper, but it is an interesting variation with many unsolved problems.

In 1999, Aigner and Ziegler immortalized the Friendship Theorem in *Proofs from THE BOOK*, covering what were (in Erdös' opinion) the greatest theorems of all time. In his 2001 undergraduate textbook *Introduction to Graph Theory*, D. B. West includes a proof similar to Longyear and Parsons' that counts common neighbors of vertices and cycles.

Craig Huneke claims he first heard of the theorem in 1975 while in graduate school. He constructed a graph-theoretic proof based on counting walks of prime length *p*, but did not publish it until nearly two decades later. After consulting with a colleague and refining his results, Huneke published the proof in the *American Mathematical Monthly* in 2002, his goal being "one proof which is more combinatorial, and another proof which … in some sense combines the combinatorics with the linear algebra" (193).

A number of authors insinuate or directly mention that previous proofs have been complicated or hard to understand. Each author then posits that their own proof is either elementary or easy to understand, seemingly with the goal of one-upping their peers. In the abstract for Longyear and Parsons' paper, they claim prior proofs have relied on "sophisticated mathematics" and that their paper gives "an elementary graph theoretic proof." Wilf states that he gives "a proof which is quite elementary, though no wholly elementary proof is known."

Preliminaries: Definitions and concepts

In our discussion of the Friendship Theorem, we will be examining relationships between people. To give this a mathematical treatment, we need to consider the people of our party and their friendship as commonly-used geometric objects. We will assume for simplicity that whenever we refer to a graph, it is a simple graph, with no loops or multiple edges.

Projective planes and the three plane axioms

We will define a geometric structure **P** on our party, which we will then show is a projective plane. A geometric structure **P** is composed of points, lines, and an incidence relation between them.

- The "points" p of **P** are the people of the party.
- The "line" $l(x)$ of **P** is the set of all friends of x.
- The incidence relation "$p \in l(x)$" is that a point p belongs to line $l(x)$ if p knows x.

We imagine then a set of people represented by points, with lines connecting those who know each other. Any arbitrary graph could then represent visually some collection of relationships, although only certain families of graphs will be shown to satisfy the friendship theorem.

A geometric structure **P** is a projective plane if it satisfies the three projective plane axioms. As a projective plane is an abstract concept, and the "lines" need not be represented by what we would usually think of as lines, nor the points, as we shall see in our examples.

P1: Given any two distinct points, there is exactly one line incident with both of them.

P2: Given any two distinct lines, there is exactly one point incident with both of them.

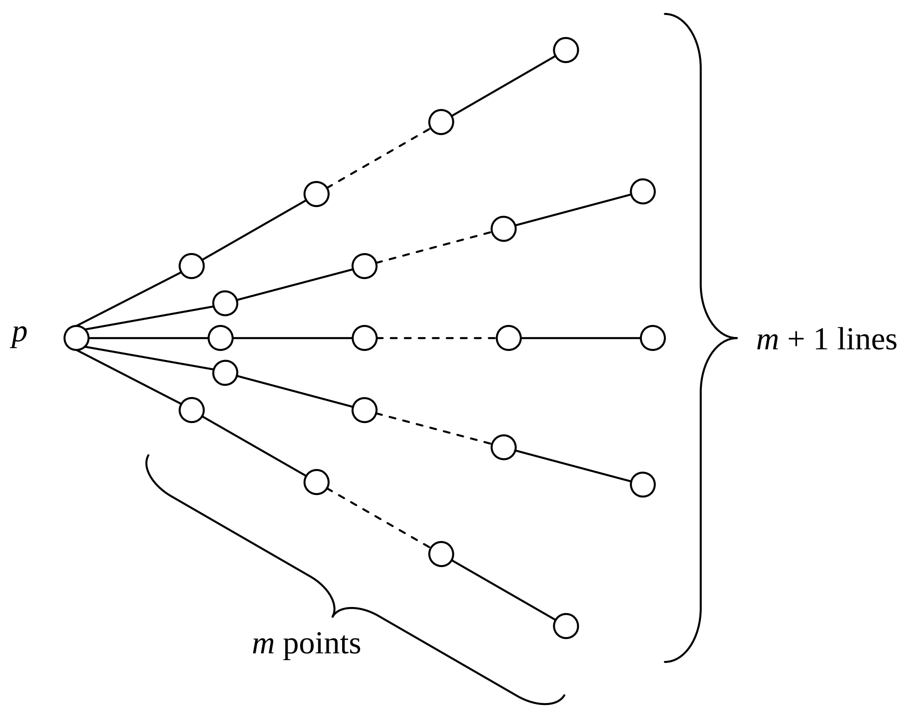
P3: There are four points, no three of which are collinear.

Definitions associated with a projective plane

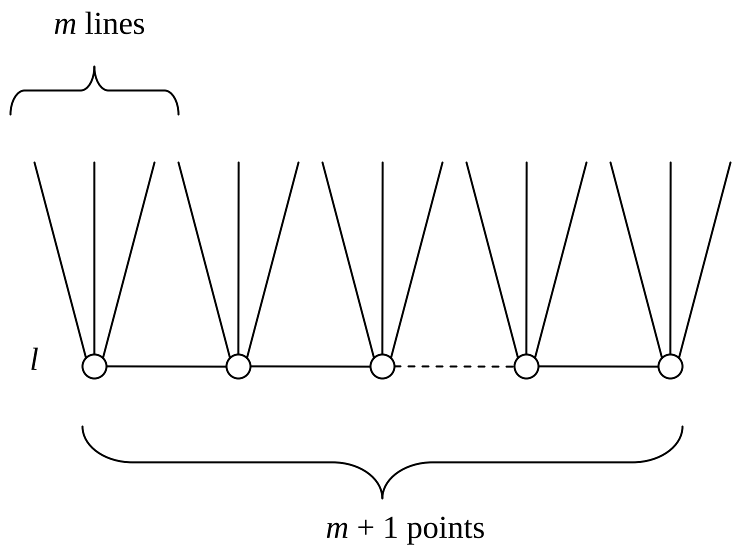
If **P** is a finite projective plane of order m, we define that (Hall 205-7):

- each line of **P** contains $m + 1$ points,
- each point of **P** is on $m + 1$ lines, and
- There are $m^2 + m + 1$ points and $m^2 + m + 1$ lines for some integer $m > 1$.

The first two properties are definitions based on the order of the projective plane, and the third can be derived from the first two: Consider an arbitrary point p of our projective plane **P**. By the second property, there are $m + 1$ lines incident to p. By the first property, since each line contains $m + 1$ points, and each of these lines are already incident with the point p, each of the lines also contains m other points. So we have $m + 1$ lines with m points on each one, and the point p, which gives us $m(m + 1) + 1 = m^2 + m + 1$ points.



Now consider an arbitrary line l of our projective plane **P**. By the first property, there are $m + 1$ points incident to l. By the second property, since each point is incident to $m + 1$ lines, and each of these points is already incident to the line l, each of the points is also incident to m other lines. So we have $m + 1$ points each incident to m lines, as well as our original line l, which gives us $m(m + 1) + 1 = m^2 + m + 1$ lines.



Here, m can be infinite (as is the case with the real projective plane) or finite. For simplicity and space, we will restrict our discussion to finite projective planes.

Incidence matrix of a projective plane – definition & properties

The incidence matrix A of a projective plane **P** is a matrix representation of which points lie on which lines. We identify the i^{th} point with the i^{th} row of A, and the j^{th} line with the j^{th} column of A. Then for a point i and a line j, if point i lies on line j, put a "1" in the ij^{th} position of A; otherwise put a zero. Or, in our context, for a person i and a set of friends of another person j, if person i is friends with person j, put a "1" in the ij^{th} position of A; otherwise put a zero.

Some facts from linear algebra

Later in our discussion it will be useful to know the eigenvalues of any *n*-square matrix of the form $J + mI$. Recall that the multiplicities of an *n*-square matrix must sum to *n*. We will show that the desired eigenvalues are

$m + n$ once, with eigenvector **x** for which $x_0 = x_1 = x_2 \ldots$, and

m with multiplicity $n - 1$ for the complementary eigenvector $x_0 + x_1 + \ldots = 0$

Recall that the eigenvalues of a matrix A are numbers λ such that $Ax = \lambda x$ has a nonzero solution vector, and each such solution is an eigenvector associated with the corresponding value of λ. We will use the fact that if A is an *n*-square triangular matrix, then the eigenvalues of A are the entries on the main diagonal of A (Anton 340).

Let $A = J + mI$, so

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \begin{pmatrix} m & 0 & 0 & \cdots \\ 0 & m & 0 & \cdots \\ 0 & 0 & m & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} m+1 & 1 & 1 & \cdots \\ 1 & m+1 & 1 & \cdots \\ 1 & 1 & m+1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Consider the equation $Ax = \lambda x$, and let $x = \mathbf{1}$, the all-ones vector. Then

$$Ax = \begin{pmatrix} m+1 & 1 & 1 & \cdots \\ 1 & m+1 & 1 & \cdots \\ 1 & 1 & m+1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} = \begin{pmatrix} (m+1)+1+1+\ldots \\ 1+(m+1)+1+\ldots \\ 1+1+(m+1)+\ldots \\ \vdots \end{pmatrix} = \begin{pmatrix} m+n \\ m+n \\ m+n \\ \vdots \end{pmatrix} = (m+n) \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} = \lambda x,$$

so $m + n$ is an eigenvalue with eigenvector **1**, and $x_0 = x_1 = x_2 \ldots$. Since A is symmetric, the eigenvalues of A are all real numbers, and eigenvectors from different eigenspaces are orthogonal (Anton 358). We know $\lambda = m + n$ is an eigenvalue of A with eigenvector **1**.

Let θ be another eigenvalue of A with eigenvector w. Then $\mathbf{1} \cdot w = 0$, since the eigenvectors are orthogonal, and

$$\mathbf{1}w = Aw = (J + mI)w = Jw + mw = \mathbf{0} + mw\,,$$

since

$$Jw = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \end{pmatrix} = \overbrace{\begin{pmatrix} 1\cdot w_1 + 1\cdot w_2 + 1\cdot w_3 + \dots \\ 1\cdot w_1 + 1\cdot w_2 + 1\cdot w_3 + \dots \\ 1\cdot w_1 + 1\cdot w_2 + 1\cdot w_3 + \dots \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}}^{\mathbf{1}\cdot w = 0} = \mathbf{0}\,.$$

Thus $Aw = mw$, and m is also an eigenvalue of A. Since $\mathbf{1}\cdot w = 0$, the eigenvector w must satisfy $w_0 + w_1 + \ldots = 0$, as desired.

For the multiplicities, we use properties of the determinant under elementary row operations.

$$A = \begin{pmatrix} m+1 & 1 & 1 & \cdots \\ 1 & m+1 & 1 & \cdots \\ 1 & 1 & m+1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Add the second row of A to the first, the third row to the first, and so on through all n rows of A – call this new matrix B, and note that $\det A = \det B$:

$$B = \begin{pmatrix} m+n & m+n & m+n & \cdots \\ 1 & m+1 & 1 & \cdots \\ 1 & 1 & m+1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Now multiply the first row of B by $\frac{1}{m+n}$, and call this new matrix C. Note that $\det B = \frac{1}{m+n} \det C$.

$$C = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 1 & m+1 & 1 & \cdots \\ 1 & 1 & m+1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Add the negative of the first row of C to the second, the third, and all successive rows, and call this new matrix D. Note that $\frac{1}{m+n}\det C = \frac{1}{m+n}\det D$.

$$D = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 0 & m & 0 & \cdots \\ 0 & 0 & m & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Multiplying the first row by $m + n$ then yields a new matrix E, for which $\det A = \det E$:

$$E = \begin{pmatrix} m+n & m+n & m+n & \cdots \\ 0 & m & 0 & \cdots \\ 0 & 0 & m & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Now we have a symmetric, n-square upper-triangular matrix. The determinant of this new matrix E is equal to the determinant of our original matrix A, so they have the same eigenvalues, which are the entries on the main diagonal. Thus we have eigenvalue $m + n$ with multiplicity one, and eigenvalue m with multiplicity $n - 1$.

Example: Projective plane of order 2: The Fano plane

The smallest finite projective plane is of order $m = 2$, and is known as the Fano plane. There are seven points and seven lines, each line contains exactly three points, and each point is incident to exactly three lines. In the diagram below, we can consider the straight lines and the inscribed circle to be the lines of the Fano plane and the numbered circles to be the points. However, because of the duality of projective planes, we could also consider the straight lines and circle to be the “points” and the numbered circles to be the “lines,” and the result would still be a projective plane.

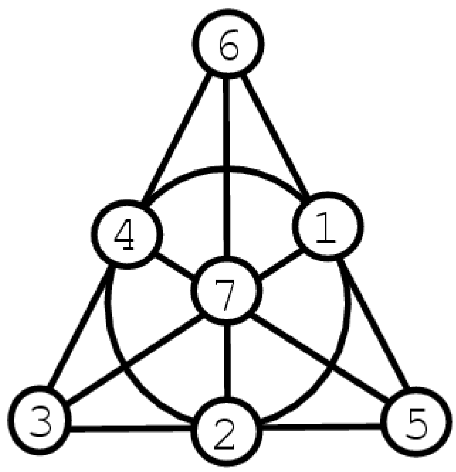


Image from PlanetMath.org

If we consider the points of the Fano plane to be the circles numbered {1,2,3,4,5,6,7}, the lines are given by the segments connecting points {1,2,4}, {2,3,5}, {3,4,6}, {4,5,7}, {5,6,1}, {6,7,2}, and {7,1,3}.

Verifying the projective plane axioms

It is easy to verify the three projective plane axioms by examination. By our definition of the Fano plane, and examination of the diagram above, given any two distinct points, there is exactly one line incident with both of them. Similarly, given any two distinct lines, there is exactly one point incident with both of them. For the third projective plane axiom we can find numerous examples of four points such that no line is incident with more than two of them – for example, the set of points {3,5,6,7} satisfies this axiom.

Incidence matrix of the Fano plane

We defined above that the incidence matrix *A* of a projective plane is a matrix representation of which points lie on which lines. Identifying the i^{th} point with the i^{th} row of *A*, and the j^{th} line with the j^{th} column of *A*, the incidence matrix *A* of the Fano plane would then be given by

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Note that each row and column has three 1's, corresponding to the dual facts that each point lies on exactly three lines and each line contains exactly three points. Note too that any pair of rows or columns has a single 1 in common, corresponding to the facts, respectively, that there is a single line incident with two unique points and two unique lines are incident to a unique point.

Example: Projective plane of order 3

The next largest finite projective plane is of order $m = 3$. There are 13 points and 13 lines, each line contains exactly four points, and each point is incident to exactly four lines. In the diagram below, we can consider the straight and curved segments to be the lines and the circles to be the points. The three projective plane axioms can still be verified by examination, though not quite as clearly due to the increased complexity of the graph.

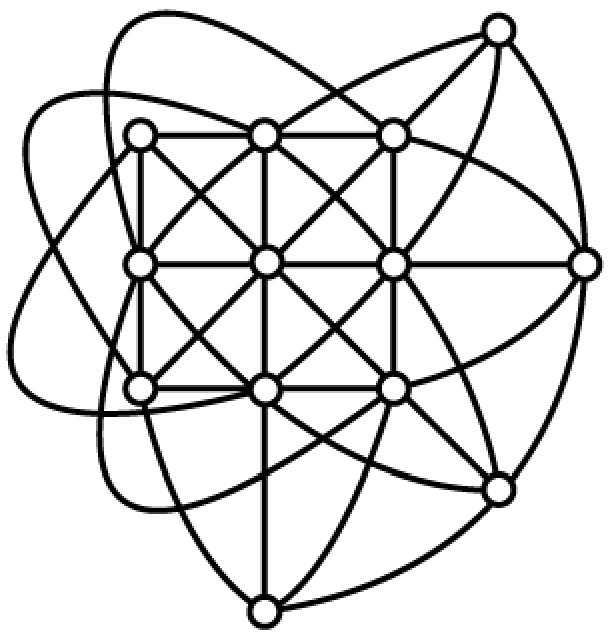


Image from A. McRae's Finite Geometry Problem Page

Incidence matrix of the projective plane of order 3

The incidence matrix of the projective plane of order 3 would be a 13x13 matrix. Each row and column would have four 1's, corresponding to the dual facts that each point lies on exactly four lines and each line contains exactly four points. In addition, any pair of rows or columns would have a single 1 in common, corresponding to the facts, respectively, that there is a single line incident with two unique points and two unique lines are incident to a unique point.

Wilf's proof of the Friendship Theorem

The friendship theorem is a well-known theorem from graph theory with many applications outside the field. Many proofs have been provided, and this article by Herbert Wilf provides a geometric one using projective planes. It begins by assuming the conclusion of the theorem is false, uses this assumption to construct a projective plane out of the "party," and then produces a contradiction with eigenvalues of the incidence matrix of the projective plane – this then proves the theorem true.

Preliminary hypotheses

The Friendship Theorem states: In a party of n people, suppose that every pair of people has exactly one common friend. Then there is a person in the party who knows everyone. Assume the conclusion is false, so there is *not* a person in the party who knows everyone else. Before we go on, we need two hypotheses that guarantee the existence of friends and non-friends given an arbitrary person.

H1: If x and y are different, they have a unique common friend $F(x,y)$.

H2: For every x there is a $y \neq x$ such that y does not know x.

The relation of "knowing" is assumed irreflexive, so the statement "x does not know x" is a correct statement, and symmetric, so if x does not know y then y does not know x (and similarly for "knows").

The party as a projective plane

We defined a geometric structure above on our party, and wish to show the party can be represented as a projective plane. To show this structure is a projective plane, we need to show it satisfies the three projective plane axioms:

P1: Given any two distinct points, there is exactly one line incident with both of them.

By our H1 assumption, since different points x and y in **P** have a *unique* common friend $F(x,y)$, we can say that every pair of points lies on a *unique* line; this is the line consisting of all friends of $F(x,y)$, so $(x,y) \in l(F(x,y))$. This proves P1.

P2: Given any two distinct lines, there is exactly one point incident with both of them.

Also by our H1 assumption, every pair of lines in **P** has exactly one point in common, since $l(x) \cap l(y) = \{F(x,y)\}$. The line $l(x)$ consists of all friends of x, the line $l(y)$ consists of all friends of y, so they must intersect in the unique common friend of x and y defined by H1. This proves P2.

P3: There are four points, no three of which are collinear.

To show **P** is a projective plane, we need only prove *P3*, which Wilf calls a lemma:

Lemma 1: There is a set of four points of **P**, no three of which lie on a line.

Proof: First we need to show that **P** does in fact contain four points.

- Suppose **P** has only one point x. Then H2 is contradicted, for there is no point y such that y does not know x.
- Suppose **P** has only two points x and y. Then H1 is contradicted, for there is no common friend $F(x,y)$.
- Suppose **P** has only three points x, y, and z. By H1, x and y have a common friend, the only option being z, so $z = F(x,y)$. By H1, y and z have a common friend, the only option being x, so $x = F(y,z)$. By H1, x and z have a common friend, the only option being y, so $y = F(x,z)$. But then all three points x, y, and z are friends with each other, contradicting H2.

Thus **P** must contain at least four points.

Choose four distinct points of **P**. If no three lie on a line, we are done. Otherwise, some three have a common friend *a*, as shown in Fig. 1. Here solid lines denote "knowing."

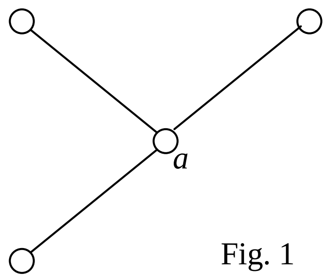


Fig. 1

By *H2*, there is a *b* such that *b* does not know *a*, as shown in Fig. 2. Here dashed lines denote "not knowing."

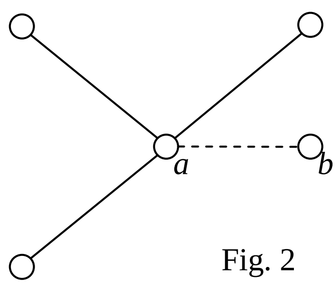


Fig. 2

By *H1*, *a* and *b* must have a common friend $F(a,b)$, and also by *H1*, *a* and $F(a,b)$ must have a common friend $F(a,F(a,b))$, as shown in Fig. 3.

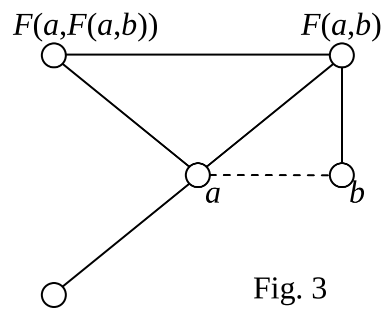


Fig. 3

Pick another friend of a different from $F(a,b)$ and $F(a,F(a,b))$, and label it z. Here, z cannot know $F(a,F(a,b))$ since $F(a,b)$ and a would then have two common friends – z and $F(a,b)$. We now have the picture in Fig. 4. We claim that the points $\{F(a,b), a, b, z\}$ satisfy the lemma; that is, no three of them lie on a line (or, in our context, no three of them are mutual friends).

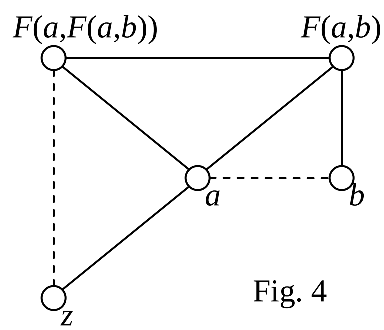


Fig. 4

Consider all possible combinations of three points out of these four.

Suppose $\{F(a,b),a,z\}$ have a common friend. It must be $F(a,F(a,b))$ since we already defined above that a does not know b. But we also defined that z does not know $F(a,F(a,b))$, and if this were true, z and $F(a,b)$ would have two common friends – a and $F(a,F(a,b))$. This is a contradiction, so these three cannot be friends. (See Fig. 5.)

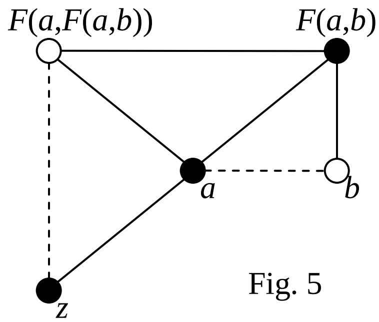


Fig. 5

Suppose $\{F(a,b),a,b\}$ have a common friend. A logical choice would be $F(a,b)$ since this is already the unique common friend of a and b. However, a does not know b, so these three cannot be friends. (See Fig. 6.)

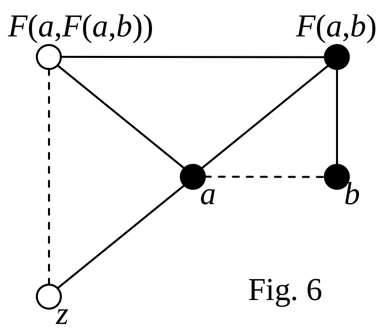


Fig. 6

Suppose $\{F(a,b),z,b\}$ have a common friend. Since a is already the common friend of z and $F(a,b)$, and a does not know b, we have a contradiction, so these three cannot be friends. (See Fig. 7.)

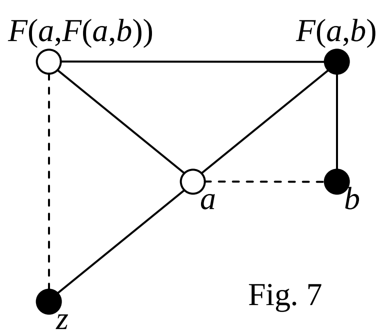


Fig. 7

Suppose $\{a,z,b\}$ have a common friend. It must be $F(a,b)$ since this is already the unique common friend of a and b. But z cannot know $F(a,b)$, for otherwise $F(a,b)$ and a would have two common friends z and $F(a,F(a,b))$, a contradiction, so these three cannot be friends. (See Fig. 8.)

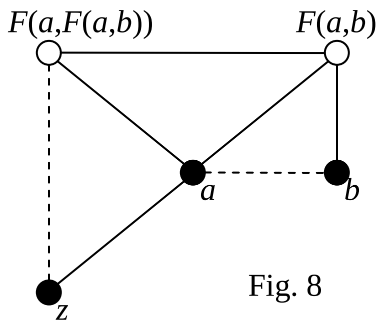


Fig. 8

Since no three of the points can be friends, no three of them can lie on a line, and we have proved *P3*. Wilf then calls this a lemma:

Lemma 2: The structure **P** is a finite projective plane.

Examining the incidence matrix of the party

Let A denote the incidence matrix of our projective plane **P**. From our definitions above, and our knowledge of incidence matrices, we know that A must be $(m^2 + m + 1)$-square since there are $m^2 + m + 1$ points and lines (corresponding to "people" and "friends" in this problem). We also know that since each line contains $m + 1$ points and each point is on $m + 1$ lines, each row and column of A should have $m + 1$ entries labeled "1." Wilf tells us that A has the following properties:

A1: A is symmetric

A2: $\mathrm{Trace}(A) = 0$

A3: $A^2 = Q$, where $q_{ij} = \begin{cases} m + 1 & \text{if } i = j, \\ 1 & \text{if } i \neq j. \end{cases}$

Consider A1: Any incidence matrix is symmetric – if a point i lies on line j (necessitating a "1" in the ij^{th} position of A), then line j contains point i (necessitating a "1" in the ji^{th} position of A).

Consider A2: Our incidence matrix has zero trace, because all its diagonal elements are zero. The ii^{th} element of A corresponds to asking "Is person i friends with person i?" Since we defined "friendship" as irreflexive, the answer is, "no," necessitating a zero in all diagonal places.

Consider A3: When we square A to get Q, the ij^{th} entry is found by taking the dot product of the i^{th} row of A and the j^{th} column of A. Or, since A is symmetric, the i^{th} row of A and the j^{th} row of A. This is equivalent to the number of points any two lines i and j have in common, which is $m + 1$ if $i = j$ (the same line has $m + 1$ points on it, and hence $m + 1$ points in common with itself), and 1 if $i \neq j$ (by the definition of a projective plane, two lines meet in one unique point).

Note that A^2 is also ($m^2 + m + 1$)-square, and $A^2 = \begin{pmatrix} m+1 & 1 & 1 & \cdots \\ 1 & m+1 & 1 & \cdots \\ 1 & 1 & m+1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = J + mI$,

where I is the identity matrix and J has ones in every entry.

Finding a contradiction

We will deduce a contradiction from these three properties of A. Wilf seeks to prove the following lemma:

Lemma 3: There exists no projective plane of order $m > 1$ whose incidence matrix is symmetric with trace zero.

If we can prove this lemma, we will have shown that no such projective plane can exist based on our original assumption that the second half of the friendship theorem was false; therefore, it must be true. We do so by constructing a contradiction from A1, A2, and A3.

Proof: Based on information we discovered from A3, we can calculate the eigenvalues of A^2. We know that A^2 is an $(m^2 + m + 1)$-square matrix of the form $J + mI$. We showed earlier that the eigenvalues of an n-square matrix in this form are

$m + n$ once, and

m with multiplicity $n - 1$.

In A3, we have an $(m^2 + m + 1)$-square matrix of the form $J + mI$, the eigenvalues of which are

$m + (m^2 + m + 1) = m^2 + 2m + 1$ a simple eigenvalue with multiplicity 1, and

m with multiplicity $(m^2 + m + 1) - 1 = m^2 + m$.

Since A^2 is $(m^2 + m + 1)$-square, all the multiplicities must sum to $m^2 + m + 1$; note that here they do indeed obey this rule.

We know that if an eigenvalue of A^2 is λ, a corresponding eigenvalue of A is $\sqrt{\lambda}$.

Hence the eigenvalues of A are:

$\sqrt{m^2 + 2m + 1} = \sqrt{(m+1)^2} = m + 1$ (a simple eigenvalue, with multiplicity 1)

$+\sqrt{m}$ (with multiplicity μ_1)

$-\sqrt{m}$ (with multiplicity μ_2)

Since A is $(m^2 + m + 1)$-square, all the multiplicities must sum to $m^2 + m + 1$, and since the eigenvalue $m + 1$ has multiplicity 1, we must have

$\mu_1 + \mu_2 = m^2 + m$.

We also note that all multiplicities (including μ_1 and μ_2) must be integers, since the multiplicity of an eigenvalue represents the number of times it appears as a zero of the characteristic polynomial.

Previously, we were able to find eigenvalues of A by manipulating A to produce an upper-triangular matrix with the desired eigenvalues along the main diagonal, the number of times they appear on the main diagonal equaling their multiplicity. We also showed earlier that the trace of A is zero. Combining these two facts, we get

$$\mu_1\left(\sqrt{m}\right)+\mu_2\left(-\sqrt{m}\right)+1(m+1)=0$$

$$(\mu_1-\mu_2)\sqrt{m}+m+1=0$$

Using the fact that, from above, $\mu_1 = m^2 + m - \mu_2$, we substitute this in to get

$$\left(\left(m^2+m-\mu_2\right)-\mu_2\right)\sqrt{m}+m+1=0$$

$$\left(m^2+m-2\mu_2\right)\sqrt{m}+m+1=0$$

$$m^2+m-2\mu_2+\frac{m}{\sqrt{m}}+\frac{1}{\sqrt{m}}=0$$

$$m^2+m+\sqrt{m}+\frac{1}{\sqrt{m}}=2\mu_2$$

$$\frac{1}{2}\left(m^2+m+\sqrt{m}+\frac{1}{\sqrt{m}}\right)=\mu_2$$

We will show that the expression on the left cannot be an integer. We know that both m and μ_2 are integers, because m is the order of our projective plane and μ_2 is the multiplicity of one of its eigenvalues. Let $t=\sqrt{m}+\frac{1}{\sqrt{m}}$ in the above equation.

Substituting this in and solving for t, we get $t = 2\mu_2 - m^2 - m$, so t must be an integer since both m and μ_2 are integers. If t is an integer, clearly t^2 is also an integer. However,

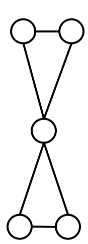
$$t^2 = \left(\sqrt{m} + \frac{1}{\sqrt{m}}\right)\left(\sqrt{m} + \frac{1}{\sqrt{m}}\right) = m + 1 + 1 + \frac{1}{m} = m + 2 + \frac{1}{m}$$

$$t^2 = m + 2 + \frac{1}{m}$$

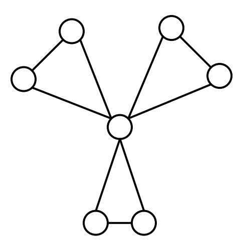
and since we are only considering $m > 1$, the term $\frac{1}{m}$ means that t^2 cannot be an integer. Thus t is also not an integer, and since μ_2 is defined in terms of t, μ_2 cannot be an integer. This is a contradiction, which proves the lemma. Since we have proved the lemma, this contradicts our original assumption that the second half of the friendship theorem is false, so the theorem must be true:

Friendship Theorem: In a party of n people, suppose that every pair of people has exactly one common friend. Then there is a person in the party who knows everyone else.

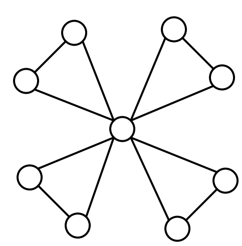
Translating this into graph theory, graphs satisfying the friendship theorem are variations on the "windmill graph." Some examples are given below.



Two couples, one common friend



Three couples, one common friend

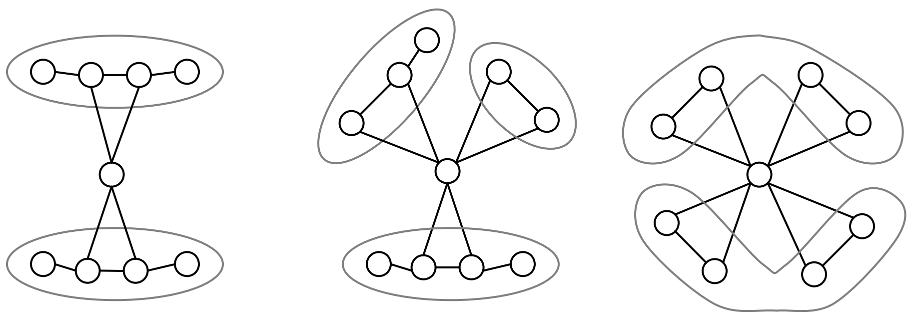


Four couples, one common friend

$\therefore$

Recent work

Numerous authors have tried to find generalizations of or limitations on the friendship theorem, in addition to the sought-after "clean" proofs. The friendship theorem considers, in Wilf's terminology, a party where each couple has exactly one common friend. A common variation is to consider *groups* of people in a party, rather than just couples, and to count how many people *in those groups* have exactly one common friend. Some examples are given below.



2 groups, 1 couple per group with 1 common friend

3 groups, 1 couple per group with 1 common friend

2 groups, 2 couples per group with 1 common friend

Katherine Heinrich examines just such a scenario in her 1990 paper, "The graphs determined by an adjacency property." Her goal in the paper is to determine all graphs G of order at least $k + 1$ with the property that for any k-subset S of $V(G)$, there is a unique vertex x, $x \in V(G) - S$, which has exactly two neighbors in S. As Heinrich notes, the case $k = 2$ is described by the friendship theorem, and she considers only the cases when $k \geq 3$.

The paper hypothesizes that a graph G satisfying this property has exactly $k + 1$ vertices, is regular of degree 2, and is thus a vertex-disjoint union of cycles. To do this, Heinrich first proves that for the base case $k = 3$, G must be a cycle of length four. She does so by defining a vertex set and counting possible edges on that set, limiting them to those satisfying the property in the hypothesis. Heinrich's next result, that the graphs must be regular of degree 2, follows naturally from the property. She then proves that the specified graphs must also have a vertex of degree at least k by considering whether G has a vertex of more than 2 but less than k; this produces a contradiction.

Heinrich's final theorem in the paper postulates that there are *no* graphs satisfying the given property. She proves this by exhaustively considering all possible cases involving the desired unique vertex, its neighbors, and the neighbors of its neighbors. In all of the cases a graph is constructed which contradicts one of the properties proved thus far, and since these are all the possible cases, no graphs exist which satisfy her variation.

Heinrich's work is a useful application of Erdös' original ideas, although because it is necessary to consider so many cases it cannot be considered a clean combinatorial proof. Results specifying graphs that *do* exist, rather than ones that *don't*, are often more widely applicable, but her results are nonetheless an interesting extension of the friendship theorem.

Conclusion

The proof of the friendship theorem has undergone many changes over the years as mathematicians search for a simple, combinatorial proof. They have consistently been foiled in this respect, as most simple proofs rely on findings from linear algebra, graph theory, or both, and the truly combinatorial arguments are often a long and circuitous route to the truth. However, various proofs of this important theorem have the distinction of drawing together many branches of mathematics, and related facts from a variety of fields. Tidbits of information about symmetric matrices and real eigenvalues mesh, while results from modern geometry tie together graph theory and linear algebra to give clean results. Future mathematicians may find a path that does not stray from one field, but for now numerous avenues exist to arrive at the neat conclusion we have found here.

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