Graphic Realizations of Sequences

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Under the direction of
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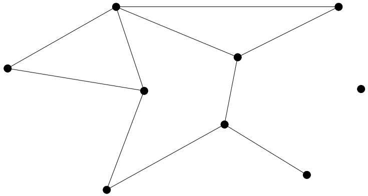
Introduction

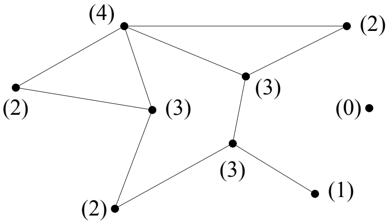
This talk is an overview of results developed in the paper "Two sufficient conditions for a graphic sequence to have a realization with prescribed clique size" written by Jian-Hua Yin and Jiong-Sheng Li.

Outline

Simple Graphs and Degree Sequences



Simple Graphs and Degree Sequences



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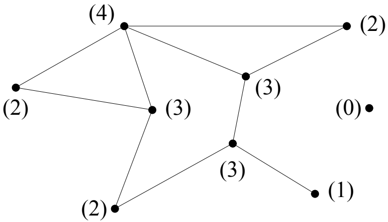

$\longrightarrow$ (4,3,3,3,2,2,2,1,0)

Sequences in NS_n and GS_n

$NS_n :=$ The family of all decreasing sequences of n whole numbers, $\pi = (d_1, d_2, \ldots, d_n)$ a typical element

$GS_n :=$ All sequences in NS_n which are degree sequences for some simple graph

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  This condition is not necessary, though. Consider $(3, 1, 1, 1)$.

A necessary and sufficient condition for π to be in GS_n.

Let the threefold process of

- deleting d_i from π,
- subtracting 1 from the first d_i terms of the new sequence, and
- reordering the resulting terms to be non-increasing

be referred to as *laying off* the term d_i from π.

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Theorem (Kleitman and Wang, 1973)

A sequence is graphic if and only if it is graphic after laying off any of its terms.

Note: This generalizes a theorem due to Havel and Hakimi (1955).

A second characterization of sequences that are graphic.

Theorem (Erdös and Gallai, 1960)

Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$. *Then* π *is graphic if and only if* $\sigma(\pi)$ *is even and, for all* $t \in [n]$,

$$\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{n} \min(t, d_i).$$

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For example, the sequence $(6, 6, 5, 4, 3, 2, 2)$ is not graphic since the inequality above does not hold for $t = 3$.

Outline

1. Necessary and sufficient conditions for $\pi \in NS_n$ to be graphic
2. Sufficient conditions for $\pi \in GS_n$ to be A_{r+1}-graphic
3. Sufficient conditions for $\pi \in GS_n$ to be almost A_{r+1}-graphic
4. Applications

What is meant by A_{r+1}-graphic?

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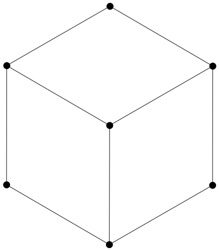
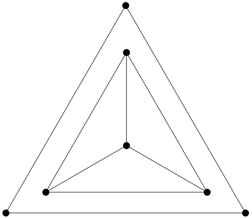
  Caution: A sequence can be A_{r+1}-graphic and have realizations that don't even contain a $(r+1)$-clique. For example, $\pi = (3,3,3,3,2,2,2)$ can be realized as each of the following simple graphs.





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- A sequence $\pi \in GS_n$ is A_{r+1}-graphic if a realization of π exists with an $(r+1)$-clique on $(r+1)$-vertices of maximal degree.
- A sequence $\pi \in GS_n$ is K_{r+1}-graphic if a realization of π exists with an $(r+1)$-clique.

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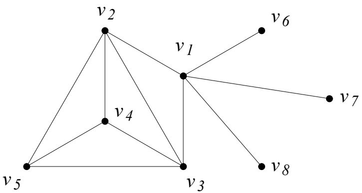
- A sequence is A_{r+1}-graphic if and only if it is K_{r+1}-graphic.

Example

$$(5, 4, 4, 3, 3, 1, 1, 1)$$



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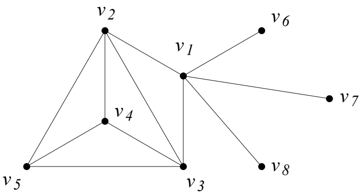

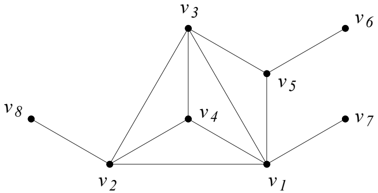

Theorem (Gould, 1999)

Suppose a sequence $\pi = (d_1, d_2, \ldots, d_n)$ has a realization G with a subgraph H on r vertices. Then there exists a realization G^* of π with a subgraph H^* isomorphic to H such that the vertices of H^* are the vertices of G^* corresponding to the first r terms of π.

Sketch of proof

Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$.

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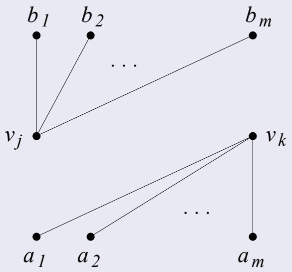

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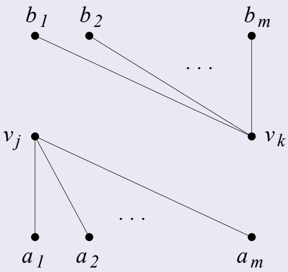

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A modified laying off procedure

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Theorem (Rao, 1979)

A sequence $\pi \in NS_n$ is A_{r+1}-graphic if and only if π_{r+1} is graphic.

Two sufficient conditions for $\pi \in GS_n$ to be A_{r+1}-graphic

Let us suppose $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ where $n \geq r + 1$.

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Theorem (Yin, 2005)

If $d_{r+1} \geq r$ and either

- $d_i \geq 2r - i$ *for all* $i \in [r-1]$, *or*
- π *has at least* $2r + 2$ *terms and* $d_{2r+2} \geq r - 1$

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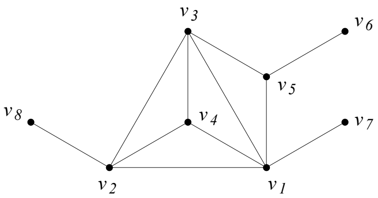
Note this condition is certainly not necessary. For example, we have already learned that $(5, 4, 4, 3, 3, 1, 1, 1)$ is K_4-graphic.

$$(5, 4, 4, 3, 3, 1, 1, 1)$$



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This condition is not necessary, which can be noted by once again considering the graphic sequence $(5,4,4,3,3,1,1,1)$.

Outline

A pair of conjectures and recent theorems which can be proven using the tools described in this talk

Theorem (Conjectured by Erdös in 1991)

Let π be a sequence in GS_n. Then π is A_{r+1}-graphic if

- $r^2 \leq \frac{2}{3}n$ *and*
- $\sigma(\pi) \geq (r-1)(2n-r)+2$.

Theorem (Yin, 2005)

Let π be a sequence in GS_n. Then π is almost A_{r+1}-graphic if

- $3r^2 - r - 1 \leq n$ *and*
- $\sigma(\pi) \geq (r-1)(2n-r)+2-(n-r)$.

Thank You

Thank you for attending this presentation!

A special thank you also to Dr. John Caughman and Dr. Joyce O'Halloran for helping me prepare this 501 project.