Graphic Realizations of Sequences

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Under the direction of
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This talk is an overview of results developed in the paper “Two sufficient conditions for a graphic sequence to have a realization with prescribed clique size” written by Jian-Hua Yin and Jiong-Sheng Li.
Outline

1. Necessary and sufficient conditions for $\pi \in NS_n$ to be graphic

2. Sufficient conditions for $\pi \in GS_n$ to be $A_{r+1}$-graphic

3. Sufficient conditions for $\pi \in GS_n$ to be almost $A_{r+1}$-graphic

4. Applications
Simple Graphs and Degree Sequences
Simple Graphs and Degree Sequences

When is $\pi$ graphic? When is $\pi_{A_{r+1}}$-graphic? When is $\pi$ almost $A_{r+1}$-graphic? Applications
When is $\pi$ graphic? When is $\pi A_{r+1}$-graphic? When is $\pi$ almost $A_{r+1}$-graphic?

Simple Graphs and Degree Sequences

(4,3,3,3,2,2,2,1,0)
Sequences in $NS_n$ and $GS_n$

$NS_n :=$ The family of all decreasing sequences of $n$ whole numbers, $\pi = (d_1, d_2, \ldots, d_n)$ a typical element

$GS_n :=$ All sequences in $NS_n$ which are degree sequences for some simple graph

$\sigma(\pi) :=$ The sum of all $n$ terms of $\pi$. 
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- It is necessary that $\sigma(\pi)$ is even for $\pi$ to belong to $GS_n$. This condition is not sufficient, however. Consider $(4, 1, 1)$.
- It is sufficient that $\sigma(\pi)$ is even and $d_{d_1+1} \geq d_1 - 1$ for $\pi$ to belong to $GS_n$ (Yin, 2005).
When is $\pi$ graphic? When is $\pi \text{ A}_{r+1}$-graphic? When is $\pi$ almost $\text{A}_{r+1}$-graphic? Applications

Sequences in $\mathcal{NS}_n$ and $\mathcal{GS}_n$

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- It is necessary that $\sigma(\pi)$ is even for $\pi$ to belong to $\mathcal{GS}_n$. This condition is not sufficient, however. Consider $(4, 1, 1)$.
- It is sufficient that $\sigma(\pi)$ is even and $d_{d_1+1} \geq d_1 - 1$ for $\pi$ to belong to $\mathcal{GS}_n$ (Yin, 2005). This condition is not necessary, though. Consider $(3, 1, 1, 1)$. 
A necessary and sufficient condition for $\pi$ to be in $GS_n$.

Let the threefold process of

- deleting $d_i$ from $\pi$,
- subtracting 1 from the first $d_i$ terms of the new sequence, and
- reordering the resulting terms to be non-increasing

be referred to as *laying off* the term $d_i$ from $\pi$. 

Theorem (Kleitman and Wang, 1973)

A sequence is graphic if and only if it is graphic after laying off any of its terms.

Note: This generalizes a theorem due to Havel and Hakimi (1955).
When is $\pi$ graphic? When is $\pi$ $A_{r+1}$-graphic? When is $\pi$ almost $A_{r+1}$-graphic?

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A second characterization of sequences that are graphic.

**Theorem (Erdős and Gallai, 1960)**

Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$. Then $\pi$ is graphic if and only if $\sigma(\pi)$ is even and, for all $t \in [n]$,

$$\sum_{i=1}^{t} d_i \leq t(t - 1) + \sum_{i=t+1}^{n} \min(t, d_i).$$

For example, the sequence $(6, 6, 5, 4, 3, 2, 2)$ is not graphic since the inequality above does not hold for $t = 3$. 
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What is meant by $A_{r+1}$-graphic?

- A sequence $\pi \in GS_n$ is $A_{r+1}$-graphic if a realization of $\pi$ exists with an $(r+1)$-clique on $(r+1)$-vertices of maximal degree.
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**Caution:** A sequence can be $A_{r+1}$-graphic and have realizations that don’t even contain a $(r + 1)$-clique. For example, $\pi = (3, 3, 3, 3, 2, 2, 2)$ can be realized as each of the following simple graphs.
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- A sequence $\pi \in GS_n$ is $K_{r+1}$-graphic if a realization of $\pi$ exists with an $(r + 1)$-clique.
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- A sequence $\pi \in GS_n$ is $A_{r+1}$-graphic if a realization of $\pi$ exists with an $(r + 1)$-clique on $(r + 1)$-vertices of maximal degree.

- A sequence $\pi \in GS_n$ is $K_{r+1}$-graphic if a realization of $\pi$ exists with an $(r + 1)$-clique.

- A sequence is $A_{r+1}$-graphic if and only if it is $K_{r+1}$-graphic.
Example

(5, 4, 4, 3, 3, 1, 1, 1)
Example

\[(5, 4, 4, 3, 3, 1, 1, 1)\]
Theorem (Gould, 1999)

Suppose a sequence $\pi = (d_1, d_2, \ldots, d_n)$ has a realization $G$ with a subgraph $H$ on $r$ vertices. Then there exists a realization $G^*$ of $\pi$ with a subgraph $H^*$ isomorphic to $H$ such that the vertices of $H^*$ are the vertices of $G^*$ corresponding to the first $r$ terms of $\pi$. 
Sketch of proof

Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$.
Let $G$ be a realization of $\pi$.
Let $H$ be a subgraph of $G$. 
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$$(v_1, \ldots, \underbrace{v_k}_{\text{position } j}, \ldots, v_m)$$
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(position \( j \))

Note that \( j < k \) thus \( d_j \geq d_k \),

thus \( v_j \) has at least as many
neighbors as \( v_k \).
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A modified laying off procedure

Let \( \pi = (d_1, d_2, \ldots, d_n) \in NS_n \).

\( \pi_0 := (d_1, d_2, \ldots, d_n) \)
A modified laying off procedure

Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$.

$\pi_0 := (d_1, d_2, \ldots, d_n)$

$\pi_1 := (d_2 - 1, d_3 - 1, \ldots, d_{r+1} - 1, d_{r+2}^{(1)}, \ldots, d_n^{(1)})$
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$\pi_i := (d_{i+1} - i, d_{i+2} - i, \ldots, d_{r+1} - i, d_{r+2}^{(i)}, \ldots, d_n^{(i)})$
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**Theorem (Rao, 1979)**

A sequence $\pi \in NS_n$ is $A_{r+1}$-graphic if and only if $\pi_{r+1}$ is graphic.
Two sufficient conditions for \( \pi \in GS_n \) to be \( A_{r+1} \)-graphic

Let us suppose \( \pi = (d_1, d_2, \ldots, d_n) \in GS_n \) where \( n \geq r + 1 \).
Two sufficient conditions for $\pi \in GS_n$ to be $A_{r+1}$-graphic

Let us suppose $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ where $n \geq r + 1$.

**Theorem (Yin, 2005)**

*If $d_{r+1} \geq r$ and either
  * $d_i \geq 2r - i$ for all $i \in [r - 1]$, or
  * $\pi$ has at least $2r + 2$ terms and $d_{2r+2} \geq r - 1$

then $\pi$ is $A_{r+1}$-graphic.*
Two sufficient conditions for \( \pi \in GS_n \) to be \( A_{r+1} \)-graphic

Let us suppose \( \pi = (d_1, d_2, \ldots, d_n) \in GS_n \) where \( n \geq r + 1 \).

**Theorem (Yin, 2005)**

If \( d_{r+1} \geq r \) and either

- \( d_i \geq 2r - i \) for all \( i \in [r - 1] \), or
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then \( \pi \) is \( A_{r+1} \)-graphic.

Note this condition is certainly not necessary. For example, we have already learned that \((5, 4, 4, 3, 3, 1, 1, 1)\) is \( K_4 \)-graphic.
When is $\pi$ graphic? When is $\pi$ $A_{r+1}$-graphic? When is $\pi$ almost $A_{r+1}$-graphic? Applications

$(5, 4, 4, 3, 3, 1, 1, 1)$

Diagram with vertices labeled $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$. Edges connect $v_1$ to $v_2$, $v_3$, and $v_4$; $v_2$ to $v_3$ and $v_4$; $v_3$ to $v_4$; $v_4$ to $v_5$, $v_6$, and $v_7$; $v_5$ to $v_6$; $v_6$ to $v_7$; and $v_7$ to $v_8$.
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Almost $A_{r+1}$-graphic?

We declare $\pi$ to be almost $A_{r+1}$-graphic whenever $\pi$ has a realization that is one edge shy of containing a copy of $K_{r+1}$. 
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Theorem (Yin, 2005)

If $\pi$ is an element of $GS_n$ with at least $r + 1$ terms and either

- $d_{r+1} \geq r - 1$ and $d_i \geq 2r - i$ for all $i \in [r - 1]$, or
- $\pi$ has at least $2r + 2$ terms, $d_{r-1} \geq r$, and $d_{2r+2} \geq r - 1$

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then $\pi$ is almost $A_{r+1}$-graphic.

This condition is not necessary, which can be noted by once again considering the graphic sequence $(5, 4, 4, 3, 3, 1, 1, 1)$. 
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A pair of conjectures and recent theorems which can be proven using the tools described in this talk

**Theorem (Conjectured by Erdös in 1991)**

Let $\pi$ be a sequence in $GS_n$. Then $\pi$ is $A_{r+1}$-graphic if

- $r^2 \leq \frac{2}{3}n$ and
- $\sigma(\pi) \geq (r - 1)(2n - r) + 2$.

**Theorem (Yin, 2005)**

Let $\pi$ be a sequence in $GS_n$. Then $\pi$ is almost $A_{r+1}$-graphic if

- $3r^2 - r - 1 \leq n$ and
- $\sigma(\pi) \geq (r - 1)(2n - r) + 2 - (n - r)$. 
Thank you for attending this presentation!

A special thank you also to Dr. John Caughman and Dr. Joyce O’Halloran for helping me prepare this 501 project.