GRAPHIC REALIZATIONS OF SEQUENCES

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Abstract. This paper is essentially a discussion of results found in the paper "Two sufficient conditions for a graphic sequence to have a realization with prescribed clique size" written by Jian-Hua Yin and Jiong-Sheng Li. We first define what is meant by a graphic sequence, then offer a few necessary and sufficient conditions for a sequence to be graphic. Next we establish some sufficient conditions for sequences to have graphic realizations with prescribed clique sizes, as well as sufficient conditions for realizations that are one edge shy of containing a clique of predetermined size. Finally we use the theorems developed over the course of this paper to prove a number of recent results.

Contents

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1. Background

A **simple graph**, often denoted by G, is a finite set of vertices V and edges E such that each edge $e \in E$ has two distinct endpoints in V and no two edges in E have the same pair of endpoints. We note that these restrictions mean that simple graphs never have loops (an edge that begins and ends at the same vertex) or multiple edges (pairs of edges that connect the same pair of vertices). Often in the literature, a particular drawing with (labeled or unlabeled) vertices and edges intended to represent a simple graph G is given the name G as well.

We say that an edge $e \in E$ **connects** two vertices in V if these vertices are the endpoints of e. Alternatively, we say that $v, w \in V$ are **adjacent** whenever there exists an edge in E with endpoints v and w. It is convenient to refer to $v, w \in V$ as **neighbors** whenever they are adjacent, and to use the name vw for the edge connecting vertices v and w. We denote by N_v the set of all neighbors of v in V. Below is a simple graph G with labeled vertices and unlabeled edges. Note that, in this graph, v_5 is adjacent to v_3 and $N_{v_1} = \{v_2, v_3, v_4\}$.

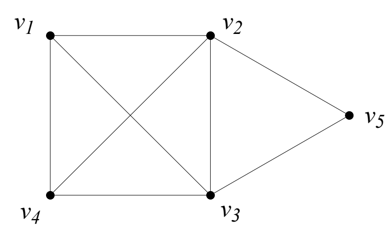


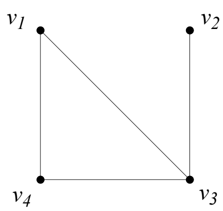
Figure 1. Simple Graph G.

A **complete** graph is a simple graph such that each pair of vertices in V are endpoints of a particular edge in E. More simply put, every pair of vertices in a complete graph are adjacent. Complete graphs with n vertices are often denoted K_n. If the vertex v_5 were deleted from the graph G depicted above, along with all edges in the graph with v_5 for an endpoint, the resulting graph $G - v_5$ would be the complete graph on four vertices, K_4.

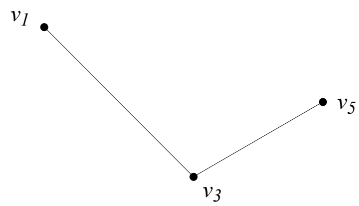
A graph H is said to be a **subgraph** of a graph G if $V_H \subseteq V_G$, and $E_H \subseteq E_G$. If V' is a subset of V, then the subgraph of G **induced** by V' is the maximal subgraph of G with vertex set V'. We note that maximal, in this context, is meant to indicate that every edge in E with both endpoints in V' is an edge of the subgraph induced by V'.

A pair of simple graphs G and H are said to be **isomorphic** if there exists a bijective map $\phi : V_G \to V_H$ such that for all $v, w \in V_G$, v is adjacent to w if and only if $\phi(v)$ is adjacent to $\phi(w)$. For example, if a particular simple graph has three vertices that are pairwise adjacent and a second simple graph has no such set of vertices, then the two are **distinct** (not isomorphic).

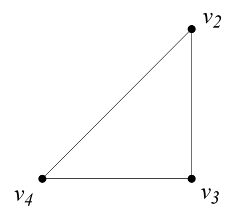
An ***n*-clique** of a graph G is a complete n-vertex subgraph of G. We note that every n-clique of a graph G is induced by some particular set of n vertices, for a complete subgraph is clearly maximal. In Figure 2 we offer three different subgraphs of the simple graph G illustrated in Figure 1. Graph H_1 is a subgraph of G that is not maximal, H_2 is the subgraph induced by the vertices $\{v_1, v_3, v_5\}$, and H_3 is a 3-clique of G that is not isomorphic to H_2.



(a) Subgraph H_1.



(b) Subgraph H_2.

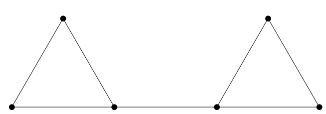


(c) Subgraph H_3.

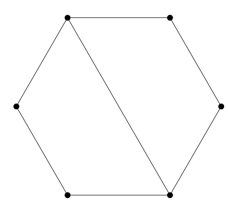
FIGURE 2.

Let $\mathbb{W}$ represent the set of whole numbers $\{0, 1, 2, \dots\}$. Given a simple graph G, we define $d : V \to \mathbb{W}$ to be the map which assigns to each $v \in V$ the number of edges in E that have v as an endpoint. We refer to the value $d(v)$ as the **degree** of v in G. Note that $d(v_3) = 2$, when v_3 is considered as a vertex of the graph H_3 given in Figure 2(c).

The **degree sequence** of a simple graph G is the set of degrees of all vertices in V written in non-increasing order. It is clear that each simple graph has exactly one degree sequence, but that the converse need not hold. Below, we offer two distinct simple graphs G_1 and G_2 which both have the same degree sequence $(3, 3, 2, 2, 2, 2)$.



(a) Simple graph G_1.



(b) Simple graph G_2.

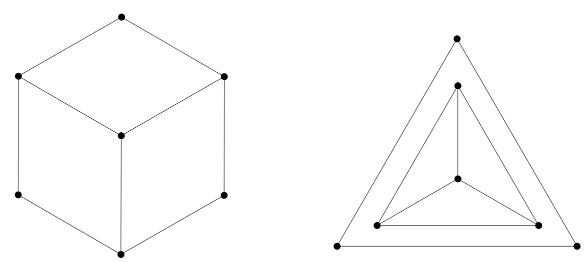
FIGURE 3.

We say that a sequence is **graphic** if it is a degree sequence for some simple graph G. A graphic sequence is said to be **realized** by G if it is the degree sequence of G. For example, the sequence $(3, 3, 2, 2, 2, 2)$ is graphic since it is the degree sequence of the simple graph G_2 depicted above. Note that the sequence $(4, 1, 1)$ is certainly not graphic, since no simple graph on three vertices can contain a degree four vertex.

Let the set of all non-increasing sequences of whole numbers consisting of exactly n terms be denoted NS_n, and let the set of the first n positive whole numbers be denoted $[n]$. Then, provided $d_i \geq d_j \geq 0$ whenever i and j are whole numbers satisfying $1 \leq i \leq j \leq n$, a typical member of NS_n is given by $\pi = (d_1, d_2, \dots, d_n)$. Let those elements of NS_n which are graphic be collectively denoted GS_n. It will be convenient for the discussion to follow to define the map $\sigma : NS_n \to \mathbb{W}$ such that $\sigma(\pi)$ is equal to $\sum_{i=1}^{n} d_i$.

A sequence π belonging to GS_n is called **potentially $\mathbf{K_{r+1}}$-graphic** if there exists a simple graph G which realizes π and which contains K_{r+1} as a subgraph. A sequence π belonging to GS_n is called **potentially $\mathbf{A_{r+1}}$-graphic** if there exists a simple graph which realizes π and whose $r+1$ vertices of highest degree induce an $r+1$ clique.

Consider, for example, $\pi = (3, 3, 3, 3, 2, 2, 2) \in NS_7$. Note that π is a graphic sequence (hence an element of GS_7) since it is the degree sequence of the simple graph G depicted in Figure 4(a). Note that G contains no three vertices which are pairwise adjacent, hence the largest complete subgraph of G is a 2-clique. Despite this relatively small clique size induced by the vertices in G of largest degree, π is potentially A_4-graphic. This can be easily verified, for π is also realized by the simple graph G^*, illustrated in Figure 4(b), and the four vertices of largest degree in G^* induce a 4-clique.



(a) Simple graph G. (b) Simple graph G^*.

FIGURE 4.

It is clear that any sequence that is potentially A_{r+1}-graphic is also potentially K_{r+1} graphic. The final goal of this section is to show that the converse holds as well. We arrive at this conclusion by proving an even stronger result, due to Gould [4], given below.

Theorem 1.1. *Let $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ and let G be a realization of π containing a subgraph H with r vertices. Then there exists a realization G^* of π containing a subgraph H^* isomorphic to H where the vertices of H^* are the vertices of G^* whose degrees correspond to the first r terms of π.*

It is possible that this result is intuitively clear to the reader, but let us attempt to sow a seed of doubt before diving into a proof of Theorem 1.1. The simple graph G illustrated in Figure 4(a), is a realization of the sequence $(3, 3, 3, 3, 2, 2, 2)$. Note that one subgraph of

G is a square (often called a 4-cycle in the literature). Further notice that every subgraph of G that is a square fails to have for a vertex set the four vertices of G of largest degree. However, the simple graph G^* given in Figure 4(b) is also a realization of the same sequence $(3, 3, 3, 3, 2, 2, 2)$. Since the four vertices of G^* of largest degree induce a 4-clique, G^* has our desired subgraph on its vertices of largest degree.

Let us look at a second example. Notice that the sequence $(5, 4, 4, 3, 3, 1, 1, 1)$ is realized by the graph G_1 illustrated in Figure 5, where vertices are labeled so that $d(v_i) = d_i$ for all $i \in [8]$.

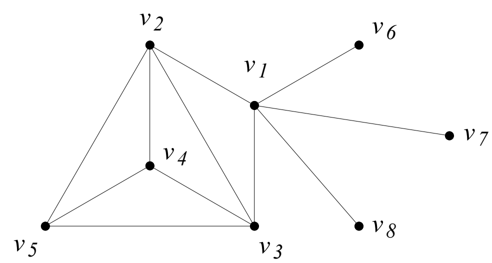


FIGURE 5. Simple Graph G_1.

It is easy to see that G_1 contains a copy of K_4, and that this K_4 subgraph has vertices $\{v_2, v_3, v_4, v_5\}$. It is also easy to see that G_1 does *not* contain a subgraph K_4 on the vertices in G_1 of highest degree. The theorem above implies that a different graphic representation of $(5, 4, 4, 3, 3, 1, 1, 1)$ exists, call it G_2, with a K_4 subgraph whose vertices are $\{v_1, v_2, v_3, v_4\}$. Indeed, this is the case, as shown in Figure 6.

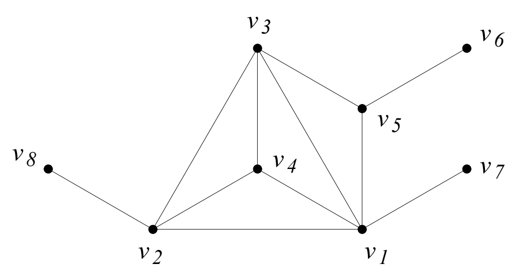


FIGURE 6. Simple Graph G_2.

We now proceed with our proof of Theorem 1.1.

Proof. Let $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ and let G be a realization of π containing a particular subgraph H with r vertices. Let the vertices of G be labeled such that for all $i \in [n]$, vertex $v_i \in V$ has degree d_i. List the vertices of H so that the indices of these vertices are strictly increasing, and let this list be denoted L.

Inspect L. If $L = (v_1, v_2, \ldots, v_r)$, we are finished. Otherwise, L contains at least one vertex whose index and position in L do not match. Let v_k be the left-most vertex in L whose index k and position, say j, do not match. Note that vertex v_k occupies a position in L that is strictly less than k. In short, $j < k$, hence $d(v_j) \geq d(v_k)$, and we see that v_j has at least as many neighbors in G as v_k. If all of the neighbors of v_k in H are also neighbors of v_j, then we can form a new list L' by replacing v_k with v_j, noting that this new set of vertices induces a subgraph of G containing H. Otherwise, there must be some non-empty set of neighbors of v_k in H, call them $\{a_i\}$ for $i \in [m]$, that are not adjacent to v_j. We depict the situation in Figure 7.

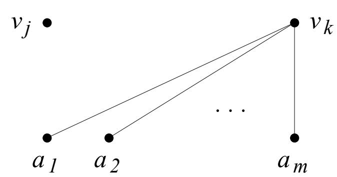


FIGURE 7.

Now, it may or may not be the case that edges exist in G that are not drawn above. In other words, v_j and v_k may well be neighbors in G, and any pair a_i and a_j, for $i, j \in [m]$ may be adjacent as well. The critical observation is that the edges drawn most certainly do exist in G, and no edges exist in G which connect v_j to any a_i for $i \in [m]$.

Recall that v_j has at least as many neighbors in G as v_k. Since v_k has m neighbors (besides, potentially v_j) that v_j doesn't, it must be true that v_j has at least m neighbors (besides, potentially v_k) that v_k doesn't. Let a particular set of m neighbors of v_j that are not adjacent to (nor equal to) v_k be denoted $\{b_i\}$ for $i \in [m]$. Hence the drawing illustrated in Figure 8 is a known subgraph of G.

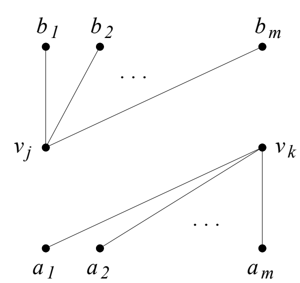


FIGURE 8.

Let each of the edges illustrated in Figure 8 be deleted, and replaced with those edges depicted in Figure 9.

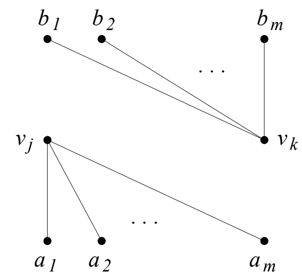


FIGURE 9.

Note that the edge switch just performed has no effect on the degree of any vertex in G, and that every new edge introduced certainly did not previously exist in G. In short, we have created a new simple graph G^* with precisely the same degree sequence that G had.

Note that v_j is now adjacent in G^* to every vertex in $H - v_k$ that v_k was initially adjacent to. Furthermore, we have not changed the adjacency or non-adjacency of any other vertex in $H - v_k$ to any other vertex in $H - v_k$. Hence, we may now delete v_k from our list L, replacing it with v_j, and our revised list of vertices induces a subgraph of G^* containing a subgraph isomorphic to H.

Now, the algorithm described above can be repeated until the (finite) list L becomes equal to $(v_1, v_2, \ldots, v_r)$. When this happens, we will have succeeded in producing a graph whose r vertices with degrees corresponding to the first r terms of π induce a subgraph containing a subgraph H^* isomorphic to H, as desired.

□

One obvious consequence of the preceding theorem is that a sequence that is potentially K_{r+1}-graphic is necessarily potentially A_{r+1}-graphic, a theorem independently established by Rao [12]. Since the converse of this statement is clear, we have completely verified the following result.

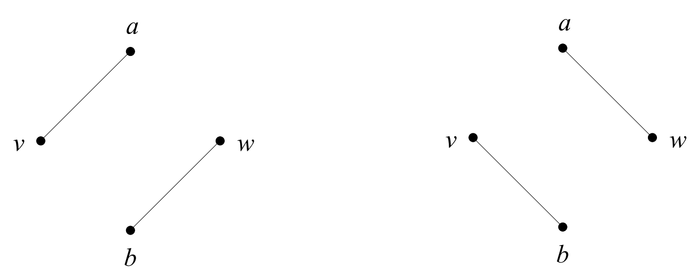
Corollary 1.1. *A sequence $\pi \in GS_n$ is potentially A_{r+1}-graphic if and only if it is potentially K_{r+1}-graphic.*

2. SOME NECESSARY AND SUFFICIENT CONDITIONS FOR $\pi \in NS_n$ TO BE GRAPHIC

Let G be an arbitrary simple graph. Since each edge in E has two distinct endpoints, it is clear that summing the degrees of all vertices in V counts every edge exactly twice. Concisely, $\sum_{v \in V} d(v) = 2|E|$, where $|E|$ is taken to mean the number of edges in E. As an immediate consequence, every degree sequence necessarily has an even sum. Thus, a necessary condition for a sequence $\pi \in NS_n$ to belong to GS_n is that $\sigma(\pi)$ is even.

Before we present a necessary and sufficient condition for a sequence $\pi \in NS_n$ to belong to GS_n, we build up a few pieces of needed machinery. Let $\pi = (d_1, d_2, \ldots, d_n)$ be an arbitrary sequence in NS_n, and let $1 \leq k \leq n$. If we delete d_k from π (so that it is a sequence of length $n-1$), subtract 1 from the left-most d_k terms remaining, then reorder the resulting terms to be non-increasing, we form the sequence π'_k. We will often write this new sequence as $(d'_1, d'_2, \ldots, d'_{n-1})$. This sequence is called the **residual sequence** obtained by **laying off** d_k from π.

Suppose that a graph G contains at least four vertices, and that two of these vertices have the property that each has a neighbor the other does not. For convenience, we assign the names v and w to the vertices of G with this property. Let a be a neighbor of v that is not adjacent to w and b be a neighbor of w that is not adjacent to v. We thus find that G has the subgraph H depicted below in Figure 10(a).



(a) Subgraph H. (b) Subgraph H^*.

Figure 10.

Note that the edges vw and ab may very well belong to E, but for our purposes this is irrelevant. The main observation to make at this point is that the edges vb and wa certainly do *not* belong to E. Now, let edges va and wb be deleted and replaced by edges vb and wa. For convenience, let this new modified version of the graph G be designated G^*. It is clear that G^* is a simple graph containing the subgraph H^* illustrated above in Figure 10(b).

The critical observation is that none of the vertices in this subgraph H^* have a degree any different than they started out having in H. Indeed, no vertex in G^* has a degree any different than it had as a vertex in G, since our edge switch could only possibly have impacted the degrees of the vertices v, w, a, and/or b. In particular, G and G^* have identical degree sequences. The modification we have just performed to transform G into G^*, which we henceforth call a **2-switch**, will prove useful in the discussion to follow.

Our first theorem of this section is a result due to Kleitman and Wang [6] (a generalization of an algorithm due to Havel and Hakimi). The observation is that a sequence $\pi \in NS_n$ and any possible residual sequence formed by laying off a term of π are either both graphic or both not graphic.

Theorem 2.1. *Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ and let $k \in [n]$. Then $\pi \in GS_n$ if and only if $\pi'_k \in GS_{n-1}$.*

Proof. First, suppose that π'_k is a graphic sequence. Let a graph G be drawn realizing π'_k. Now, each vertex of G has some particular degree, and d_k of these degrees are 1 smaller than they were prior to the laying off of d_k. Let the corresponding d_k vertices of G be connected to a new vertex, say w. Clearly w has degree d_k. In fact, it is clear that the degrees of the vertices of this new graph are, when arranged from greatest to least, precisely equal to π. Since π has a graphic realization, we conclude that $\pi \in GS_n$.

Next, suppose that $\pi = (d_1, d_2, \ldots d_n)$ is a graphic sequence. Let a graph G be drawn realizing π such that for all $i \in [n]$, vertex $v_i \in V$ has degree d_i. Recall that N_{v_k} is the set of all d_k neighbors of vertex v_k. Collect d_k vertices of $V - \{v_k\}$ with maximal degree sum, denoting this set M_{v_k}. Now, if $N_{v_k} = M_{v_k}$ then deleting vertex v_k from G results in a realization of π'_k directly. Suppose instead that $N_{v_k} \neq M_{v_k}$. Then since these sets have equal cardinalities, there must exist some $v_a \in N_{v_k}$ and $v_j \in M_{v_k}$ such that neither v_j nor v_a belongs to $N_{v_k} \cap M_{v_k}$. Now, v_a is a neighbor of v_k, and v_j is not. Furthermore, v_j has at least as many neighbors as v_a by construction of M_{v_k}. Therefore, v_j must have a neighbor v_b which is not a neighbor of v_a (since v_a is known to have a neighbor v_k which is not a neighbor of v_j). In short, the vertex set $\{v_a, v_b, v_j, v_k\}$ induces the subgraph of G depicted in Figure 11 below.

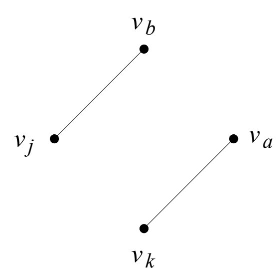


FIGURE 11.

We have satisfied the necessary conditions for employing our 2-switch. By removing edges v_jv_b and v_av_k and adding edges v_jv_k and v_av_b, we produce the graph G^* with exactly the same degree sequence that G has. Furthermore, if we reconstruct N_{v_k} and M_{v_k} for this new graph G^*, we find that the cardinality of their intersection is exactly 1 larger than it was before (v_j has been included by our introduction of edge v_jv_k). Thus, by repeating the procedure described above as many times as needed (a necessarily finite number of times), we will eventually produce a graph whose vertex v_k is adjacent to d_k vertices of highest degree other than v_k. Once this has been accomplished, we need only delete vertex v_k to produce a realization of π'_k, hence $\pi'_k \in GS_{n-1}$, as desired.

□

Our next theorem, a result of Erdös and Gallai [2], provides another necessary and sufficient condition for a sequence in NS_n to be graphic. The proof we offer is closely modeled after a proof given by Choudum [15].

Theorem 2.2. *Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ such that $\sigma(\pi)$ is even. Then $\pi \in GS_n$ if and only if for all $t \in [n]$,*

$$\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{n} \min(t, d_i).$$

Before we begin our proof, we note that some versions of this theorem, including the version cited by Li et al. in [1], place $t \in [n-1]$ rather than $[n]$. This is a matter of taste, since the claim holds trivially for $t = n$. For brevity, let the family of sequences consisting of exactly n terms satisfying the inequality above for all $t \in [n]$ be collectively referred to as EG_n in honor of Erdös and Galliai. Then, for $\pi \in NS_n$ with an even sum, we wish to establish that $\pi \in GS_n$ if and only if $\pi \in EG_n$.

Proof. First, suppose that $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$. Let a graph G be drawn realizing π, such that for all $i \in [n]$, $v_i \in V$ has degree d_i. Let $V_t := \{v_i\}_{i\in[t]}$. Now the sum $\sum_{i=1}^{t} d_i$ can be thought of as an enumeration of edges in E with at least one endpoint in V_t. Note that those edges in E with both endpoints in V_t will be counted twice by this sum. Since no more than $\binom{k}{2}$ edges in E can have endpoints exclusively in V_t, edges with both endpoints in V_t can contribute no more than $2\binom{t}{2} = t(t-1)$ to $\sum_{i=1}^{t} d_i$. On the other hand, those edges in E with exactly one endpoint in V_t will be counted exactly once by the sum $\sum_{i=1}^{t} d_i$. Note that, for each i such that $t+1 \leq i \leq n$, the number $\min(t, d_i)$ is the largest possible number of neighbors that d_i has in V_t. Consequently, the sum $\sum_{i=t+1}^{n} \min(t, d_i)$ is at least as large as the number of the number of edges in E with exactly one endpoint in V_t. Hence, we find that $\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{n} \min(t, d_i)$, hence $\pi \in EG_n$ as desired.

Next, we prove that $\pi \in EG_n$ implies $\pi \in GS_n$ by means of induction on the sum $\sigma(\pi)$. First, suppose that $\sigma(\pi) = 0$. Then both $\sum_{i=1}^{t} d_i$ and $\sum_{i=t+1}^{n} \min(t, d_i)$ are necessarily equal to zero as well (for each d_i is itself equal to zero). Furthermore, $t(t-1)$ is a strictly increasing function on the domain $(\frac{1}{2}, \infty)$. Since $t(t-1) = 0$ for $t = 1$, we learn that $t(t-1) \geq 0$ for all $t \in [n]$. Hence the desired inequality holds for all $\pi \in NS_n$ whose terms have sum 0. Furthermore, the sequence π is clearly graphic, since it can be realized as n distinct vertices with no edges. Hence, every $\pi \in EG_n$ such that $\sigma(\pi) = 0$ is also an element of GS_n, as desired.

Next, suppose that $\sigma(\pi) = 2$. One way that this could occur is if $d_1 = 2$ and all other terms equal zero. However, we see that our inequality fails for $t = 1$, in that the left side is equal to 2 while the right side is equal to zero. The only remaining possibility is that $d_1 = d_2 = 1$ and all remaining terms equal zero. If we set $t = 1$, we see that $\sum_{i=1}^{1} d_i = \sum_{i=2}^{n} \min(1, d_i) = 1$, hence our inequality holds. If $t \geq 2$, we see that $\sum_{i=1}^{t} d_i = 2$. Since $t(t-1) = 2$ when $t = 2$, and $t(t-1)$ is strictly increasing over the domain of interest, it follows that $t(t-1) \geq \sum_{i=1}^{t} d_i$, and our inequality holds once more. Thus, the only sequence $\pi \in NS_n$ such that $\sigma(\pi) = 2$ and $\pi \in EG_n$ has precisely two terms equal to 1 and the remaining $n-2$ terms equal to zero. This is clearly graphic as well, realized by any simple graph with n vertices and 1 edge.

Having established a sufficient base case, we now set the stage for our inductive step. Suppose, for our induction hypothesis, that every sequence in NS_n with even sum $s-2$ belonging to EG_n has a graphic realization. (Note that this automatically guarantees s is even.) Let $\pi = (d_1, d_2, \ldots, d_n)$ be an arbitrary sequence in NS_n such that $\sigma(\pi) = s$ and $\pi \in EG_n$. Since membership in neither EG_n nor membership in GS_n is in any way impacted by any string of zero terms at the tail of π, we may assume without loss of generality that $d_n \geq 1$.

Now π is a sequence that begins with a string of terms equal to d_1 (though the string may be quite short). Let d_k be the final term in this string of terms equal to d_1, or the term d_{n-1}, whichever comes first. By our choice of d_k, we guarantee ourselves that $d_k - 1 \geq d_{k+1}$, or $d_k - 1 = d_{n-1} - 1 \geq d_n - 1$. In either case, we see that

$$\pi^* = (d_1, \ldots, d_{k-1}, d_k - 1, d_{k+1}, \ldots, d_{n-1}, d_n - 1)$$

is a sequence belonging to NS_n such that $\sigma(\pi^*) = s - 2$. As a notational convenience in the work to follow, we rename these terms $(e_1, e_2, \ldots, e_n)$ so that

$$e_i = \begin{cases} d_i, & \text{if } i \neq k, n; \\ d_i - 1, & \text{otherwise.} \end{cases}$$

Before going any further, we require one additional tool. In particular, we will frequently need the inequality

$$\min(a, b) - 1 \leq \min(a, b - 1) \qquad \forall a, b \in \mathbb{R}. \tag{1}$$

This is easily verified. Suppose first that $a \leq b$, thus $\min(a, b) - 1 = a - 1$. Since $a - 1$ is obviously not greater than a, and $a - 1 \leq b - 1$ by hypothesis, the inequality holds. Next, suppose instead that $b < a$. It follows that $\min(a, b) - 1 = b - 1$ which is very clearly not larger than a or $b - 1$, and the inequality holds once more. We conclude that $\min(a, b) - 1 \leq \min(a, b - 1)$ holds for any $a, b \in \mathbb{R}$, as claimed. Having settled this, we now proceed to verify that $\pi^* \in EG_n$ by considering five different cases which together represent every possible scenario.

(1) Suppose that $k \leq t \leq n$. By hypothesis, $\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{n} \min(t, d_i)$, hence
$$\left(\sum_{i=1}^{t} d_i\right) - 1 \leq t(t-1) + \left(\sum_{i=t+1}^{n} \min(t, d_i)\right) - 1$$
Since $k \leq t$, one of the terms of $\sum_{i=1}^{t} d_i$ is d_k, and we may declare that the 1 that has been subtracted from the left side has the lone effect of changing d_k to $d_k - 1 = e_k$. Thus, the left side may be rewritten as $\sum_{i=1}^{t} e_i$. Now, we declare that the one subtracted from the right side has the effect of changing the final term of $\sum_{i=t+1}^{n} \min(t, d_i)$ from $\min(t, d_n)$ to $\min(t, d_n) - 1$. Since $\min(t, d_n) - 1 \leq \min(t, d_n - 1)$ by (1), we have that
$$\sum_{i=t+1}^{n} \min(t, d_i) - 1 \leq \sum_{i=t+1}^{n-1} \min(t, d_i) + \min(t, d_n - 1) = \sum_{i=t+1}^{n} \min(t, e_i).$$
Piecing these results together we find that $\sum_{i=1}^{t} e_i \leq t(t-1) + \sum_{i=t+1}^{n} \min(e_i, t)$, as desired.

(2) Suppose that $t \in [k-1]$ and $d_t \leq t - 1$. We note that for this case (and each remaining case), $1 \leq t < k$, hence d_t and all terms of π with smaller index than t precede d_k, and are therefore equal to $d_k = d_t$. It follows that
$$\sum_{i=1}^{t} d_i = \sum_{i=1}^{t} e_i = t d_t.$$
Now, by hypothesis, $d_t \leq t - 1$, hence $\sum_{i=1}^{t} e_i \leq t(t-1)$. Also, since d_k, d_n, and t are all equal to at least one, the expression $\sum_{i=t+1}^{n} \min(t, d_i)$ has at least two terms each at least equal to 1. Thus $\sum_{i=t+1}^{n} \min(t, d_i) - 2$ is non-negative. Note that, by once more employing (1), we have
$$0 \leq \sum_{i=t+1}^{n} \min(t, d_i) - 2 \leq \sum_{i=t+1}^{n} \min(t, e_i).$$
Adding this (necessarily non-negative) sum to the right hand side of $\sum_{i=1}^{t} e_i \leq t(t-1)$, derived a few sentences ago, we have $\sum_{i=1}^{t} e_i \leq t(t-1) + \sum_{i=t+1}^{n} \min(t, e_i)$, as desired.

(3) Suppose that $t \in [k-1]$ and $d_t = t$. We will need to verify, for reasons soon to be clear, that $\sum_{i=t+2}^{n} d_i$ is at least equal to 2. It is clear that this inequality holds if $\sum_{i=t+2}^{n} d_i$ has at least two terms, since each such term is greater than or equal to d_n which is itself greater than or equal to 1 by assumption. Now, if $\sum_{i=t+2}^{n} d_i$ consists of only one term, then $d_{t+2} = d_n$, and in particular, $t = n - 2$. Since $n - 1 = t + 1 \leq k$ by hypothesis while $k \leq n - 1$ by our construction of d_k, we find that $k = n - 1$. Hence the sequence π consists of $k = n - 1$ copies of $t = n - 2$,

followed by a single term d_n. In other words, $\sigma(\pi) = (n-1)(n-2) + d_n$. Since both $\sigma(\pi)$ and $(n-1)(n-2)$ are even, it follows that d_n is as well. Finally, since $d_n \geq 1$, we conclude that d_n is, in fact, an even integer greater than or equal to 2. Thus, we can be quite certain that, regardless of how many terms $\sum_{i=t+2}^{n} d_i$ has, it is irrefutably greater than or equal to 2.

Now then, $\sum_{i=1}^{t} e_i = \sum_{i=1}^{t} d_i = td_t = t^2 = t^2 - t + t = t(t-1) + d_{t+1}$. By adding $\sum_{i=t+2}^{n} d_i - 2$ to the right hand side of this equality, shown in the previous paragraph to be non-negative, we derive the inequality

$$\sum_{i=1}^{t} e_i \leq t(t-1) + d_{t+1} + \sum_{i=t+2}^{n} d_i - 2. \tag{2}$$

Furthermore, since $d_i \leq d_t = t$ for $i > t$, it follows that

$$\begin{aligned} t(t-1) + d_{t+1} + \sum_{i=t+2}^{n} d_i - 2 &= t(t-1) + \sum_{i=t+1}^{n} \min(t, d_i) - 2 \\ &\leq t(t-1) + \sum_{i=t+1}^{n} \min(t, e_i). \end{aligned} \tag{3}$$

Combining inequalities (2) and (3) yields $\sum_{i=1}^{t} e_i \leq t(t-1) + \sum_{i=t+1}^{n} \min(t, e_i)$, as desired.

(4) Suppose that $t \in [k-1]$, $d_t \geq t+1$, and $d_n \geq t+1$. Now,

$$\sum_{i=1}^{t} e_i = \sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{n} \min(t, d_i)$$

by hypothesis. Furthermore, each term in the sum $\sum_{i=t+1}^{n} \min(t, d_i)$ is equal to t due to our assertion that $d_n \geq t+1$. In fact, the sum would remain unchanged even if we reduce both d_k and d_n by 1, since both $d_k - 1$ and $d_n - 1$ are still greater than or equal to t. In short, $\sum_{i=t+1}^{n} \min(t, d_i) = \sum_{i=t+1}^{n} \min(t, e_i)$, and we immediately have the conclusion $\sum_{i=1}^{t} e_i \leq t(t-1) + \sum_{i=t+1}^{n} \min(t, e_i)$, as desired.

(5) Finally, suppose that $t \in [k-1]$, $d_t \geq t+1$, and $d_n \leq t$. Since $d_n \leq t$, there exists some particular term in π that is the smallest indexed term to be less than or equal to t; let this term be denoted d_r. Note that $\min(d_i, t) = t$ for all $i \in [r-1]$ whereas $\min(d_i, t) = d_i$ for all $i \geq r$.

Now, we claim that there does not exist a $t \in [k-1]$ such that

$$\sum_{i=1}^{t} d_i = t(t-1) + \sum_{i=t+1}^{n} \min(t, d_i).$$

For, suppose there does exist such a t. Then

$$\begin{aligned} td_t = \sum_{i=1}^{t} d_i &= t(t-1) + \sum_{i=t+1}^{r-1} \min(t, d_i) + \sum_{i=r}^{n} \min(t, d_i) \\ &= t(t-1) + t(r-1-t) + \sum_{i=r}^{n} d_i \\ &= t(r-2) + \sum_{i=r}^{n} d_i. \end{aligned} \tag{4}$$

Multiplying both sides of (4) by $\frac{t+1}{t}$ yields

$$\begin{aligned} (t+1)d_t &= (t+1)(r-2) + \frac{t+1}{t} \sum_{i=r}^{n} d_i \\ &> (t+1)(r-2-t+t) + \frac{t+1}{t} \sum_{i=r}^{n} d_i - \frac{1}{t} \sum_{i=r}^{n} d_i \\ &= t(t+1) + (t+1)(r-1-(t+1)) + \sum_{i=r}^{n} d_i \\ &= t(t+1) + \sum_{i=t+2}^{r-1} (t+1) + \sum_{i=r}^{n} d_i. \end{aligned} \tag{5}$$

Now, consider the expression $\min(t+1, d_i)$. Since d_i is defined to be strictly greater than t for all $i < r$, we see that $\min(t+1, d_i) = t+1$, for $i \in [r-1]$. Furthermore, d_i is known to be less than or equal to t for all $i \geq r$, hence $\min(t+1, d_i) = d_i$ for $r \leq i \leq n$. Finally, we note that $(t+1)d_k = \sum_{i=1}^{t+1} d_i$ since $t+1 \in [k]$. Thus, we may rewrite (5) as

$$\sum_{i=1}^{t+1} d_i > (t+1)(t) + \sum_{i=t+2}^{n} \min(t+1, d_i),$$

which is a direct contradiction of the hypothesis that $\pi \in EG_n$. For our trouble, we may now conclude that, for all t under consideration in this case, the following strict inequality holds:

$$\sum_{i=1}^{t} d_i < t(t-1) + \sum_{i=t+1}^{n} \min(t, d_i).$$

Recall that $d_k = d_t$ is strictly larger than t, hence $\min(t, d_k) = \min(t, d_k - 1) = \min(t, e_k)$. Finally, subtracting one from the right side of the inequality above and recalling that $\min(t, d_n) - 1 \leq \min(t, d_n - 1)$ yields

$$\sum_{i=1}^{t} e_i = \sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{n} \min(t, d_i) - 1$$
$$\leq t(t-1) + \sum_{i=t+1}^{n} \min(t, e_i),$$

as desired. Since this case was the only remaining case to check, we conclude that $\pi^* \in EG_n$.

Since $\pi^* \in EG_n$, we conclude by our induction hypothesis that π^* is graphic. Let a graph G be drawn realizing π^*, such that for all $i \in [n]$, $v_i \in V$ has degree e_i. In particular, we focus our attention on the vertices labeled v_k and v_n. If these two vertices are not adjacent, then we may clearly add an edge connecting them, immediately producing a graph that realizes π. Suppose instead that they are adjacent. We note that G has a maximum degree e_1 of no greater than $n-1$, and that the vertex v_k (having degree $e_k = d_k - 1 \leq d_1 - 1 = e_1 - 1$) may therefore have a degree of no greater than $n-2$. In short, there exists some vertex in $G - v_k$ that is not adjacent to v_k, say v_a. Now since v_a is not equal to v_k or v_n, $e_a = d_a \geq d_n > d_n - 1 = e_n$, hence the vertex v_a has more neighbors that vertex v_n. Consequently, v_a has a neighbor, say v_b, that is not adjacent to v_n. Similar to a situation we have seen before in this paper, we learn that $\{v_a, v_b, v_n, v_k\}$ induces the subgraph of G depicted in Figure 12 below.

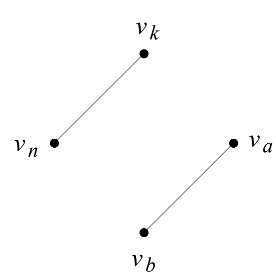


FIGURE 12.

Since the edges $v_n v_b$ and $v_k v_a$ are certainly not edges belonging to G, we may perform a 2-switch. Our new graph G^* still faithfully realizes π^*, but certainly does not have an edge connecting v_k to v_n. By adding this edge, we have produced a graph which realizes π. Hence, $\pi \in GS_n$ and our proof is complete.

□

Before moving on, let us consider the sequence $(6, 6, 5, 4, 3, 2, 2)$. Since the sum of the first three terms of this sequence is 17 whereas $3(3-1) + \sum_{i=4}^{7} \min(3, d_i)$ is only 16, this sequence is not graphic by Theorem 2.2. It is interesting to note that laying off the first three terms of $(6,6,5,4,3,2,2)$ produces a sequence that is no longer strictly whole numbers. Perhaps, given a sequence $\pi \in NS_n$, a failure to satisfy the inequality given in Theorem 2.2 for some $t \in [n]$ corresponds exactly to an instance of laying off the first t terms of π and producing a sequence containing at least one term that is not a whole number. To our knowledge, this conjecture is open.

Our final result of this section is a sufficient condition for a sequence $\pi \in NS_n$ to belong to GS_n which will prove most useful in the sequel. Both the theorem and its proof are from Yin et al. [1].

Theorem 2.3. *If $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, such that $\sigma(\pi)$ is even and $d_{d_1+1} \geq d_1 - 1$, then π is graphic.*

We note that the statement of the theorem above differs slightly from that given in [1]. In particular, we do not adopt the convention of replacing d_1 with r, even though this will lead to subscripts with subscripts. We intend to use r as a variable in a different, yet similar, context later in our discussion, and wish to avoid unnecessary confusion.

Before we offer a proof of Theorem 2.3, consider the sequence $(5,5,5,4,4,4,3,2,1,1)$. We could certainly create residuals of this ten term sequence by laying off one term after another, until we arrived at a sequence we could easily identify as being graphic or not. Alternatively, we could verify that this sequence satisfies the inequality of Theorem 2.2 for $1 \leq t \leq 10$. However, both of these plans seem rather tedious. Indeed, it is easy to see that a sequence of any great length would quickly make both of theses approaches untenable. Thus it is rather remarkable that observing an even number of odd entries along with a sixth term of at least four allows us to conclude that this sequence is certainly graphic by Theorem 2.3.

On the other hand, we note that Theorem 2.3 is a sufficiency condition for a sequence to be graphic, not a necessary one. Indeed, the sequence $(3,1,1,1)$ is graphic as shown in Figure 13, even though the fourth term of this sequence is not at least 2. Thus we note that this rather powerful tool for determining if a given sequence in NS_n is graphic, though quite useful and easy to apply, does not characterize all graphic sequences.

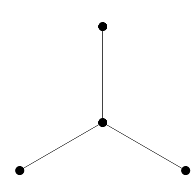


Figure 13.

Proof. We have that $\pi \in NS_n$ and that π has an even sum. If we can show that, for all $t \in [n]$, the inequality

$$\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{n} \min(t, d_i)$$

holds, then $\pi \in GS_n$ by Theorem 2.2. In order to demonstrate that π indeed satisfies the inequality above, we consider four separate cases.

(1) Suppose that $1 \leq t \leq d_1 - 1$. Clearly the sum $\sum_{i=1}^{t} d_i$ consists of t terms, each no larger than d_1. As a result, $\sum_{i=1}^{t} d_i \leq td_1$. Note that $td_1 = t(t-1) + t(d_1 + 1 - t)$. Hence

$$\begin{aligned}\sum_{i=1}^{t} d_i &\leq t(t-1) + t(d_1 + 1 - t) \\ &= t(t-1) + \sum_{i=t+1}^{d_1+1} t.\end{aligned}$$

By hypothesis, $d_{d_1+1} \geq d_1 - 1 \geq t$. Consequently $d_i \geq t$, hence $\min(t, d_i) = t$, for all $i \in [d_1 + 1]$. This gives us the inequality

$$\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{d_1+1} \min(t, d_i).$$

Finally, since d_{d_1+1} is well-defined by hypothesis, it is clear that $d_1 + 1 \leq n$. Thus we certainly do not decrease the sum on the right by adding over all i such that $t + 1 \leq i \leq n$. Thus

$$\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{n} \min(t, d_i),$$

as desired.

(2) Suppose that $d_1 + 1 \leq t \leq n$. For identical reasons to those given above,

$$\sum_{i=1}^{t} d_i \leq t(t-1) + t(d_1 + 1 - t).$$

By hypothesis, $d_1 + 1 - t \leq 0$. Clearly the inequality above will still hold if we replace the non-positive term $t(d_1 + 1 - t)$ with one that is non-negative. In short, we once more have the desired inequality

$$\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{n} \min(t, d_i).$$

(3) Suppose that $t = d_1$ and $d_{d_1} = d_1 - 1$. Since $d_{d_1} \geq d_{d_1+1} \geq d_1 - 1$, this is the least value possible for d_{d_1} to take on. Note that the largest value that d_i can take on for each $i \in [d_1 - 1]$ is d_1. Hence, $\sum_{i=1}^{t} d_i = \sum_{i=1}^{d_1} d_i$ is not greater than $d_1(d_1 - 1) + d_1 - 1$. Since $d_{t+1} = d_{d_1+1} \geq d_1 - 1$ by hypothesis and $t = d_1 > d_1 - 1$ by inspection, we find that $\min(t, d_{t+1}) \geq d_1 - 1$. Putting these thoughts together, we have the desired inequality

$$\begin{aligned}\sum_{i=1}^{t} d_i \leq d_1(d_1 - 1) + d_1 - 1 &= t(t-1) + d_1 - 1 \\ &\leq t(t-1) + \min(t, d_{t+1}) \\ &\leq t(t-1) + \sum_{i=t+1}^{n} \min(t, d_i).\end{aligned}$$

(4) Finally suppose that $t = d_1$ and $d_{d_1} = d_1$. Clearly, this is the largest (and only other) value possible for d_{d_1} to take on. Now, let us assume for the moment that $d_{d_1+1} = d_1 - 1$ and $d_{d_1+2} = 0$. Then π is a sequence consisting of d_1 terms each equal to d_1, a single term equal to $d_1 - 1$, and every remaining term equal to 0. Thus, $\sigma(\pi) = d_1(d_1) + d_1 - 1 = (d_1 + 1)(d_1) - 1$. Since $(d_1 + 1)(d_1)$ is the product of two consecutive integers, it is even. Consequently $\sigma(\pi)$ is odd, contradicting our hypothesis. We conclude that our assumption was false, and either $d_{d_1+1} \neq d_1 - 1$ or $d_{d_1+2} \neq 0$. By hypothesis, these assertions are equivalent to saying $d_{d_1+1} = d_1$ or $d_{d_1+2} \geq 1$.

If the first of the two assertions holds, then $\min(d_1, d_{d_1+1}) = d_1$. If the second holds, then $\min(d_1, d_{d_1+1}) + \min(d_1, d_{d_1+2}) \geq (d_1 - 1) + 1 = d_1$. Either way, we learn that $\sum_{i=d_1+1}^{n} \min(d_1, d_i)$ is greater than or equal to d_1. Therefore,

$$\begin{aligned}\sum_{i=1}^{t} d_i = \sum_{i=1}^{d_1} d_i = d_1(d_1) &= d_1(d_1 - 1) + d_1 \\ &\leq d_1(d_1 - 1) + \sum_{i=d_1+1}^{n} \min(d_1, d_i) \\ &= t(t-1) + \sum_{i=t+1}^{n} \min(t, d_i),\end{aligned}$$

as desired. Now, in each of the exhaustive cases above our sequence $\pi \in NS_n$ with even sum satisfied the inequality $\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{n} \min(t, d_i)$ for all $t \in [n]$. Therefore, we conclude that $\pi \in GS_n$ by Theorem 2.2, completing our proof.

$\square$

3. Some sufficient conditions for $\pi \in NS_n$ to be potentially A_{r+1}-graphic

We will find frequent need, in the discussion to follow, of a modified version of the laying off procedure described earlier. Let us fix positive whole numbers n and r such that $n \geq r+1$. Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ be a sequence with $d_1 \leq n-1$ and $d_{r+1} \geq r$. Our modified laying off procedure will produce a family of sequences, $\{\pi_i\}$ for $0 \leq i \leq r+1$.

We begin by defining $\pi_0 := \pi$. Next, we define

$$\pi_1 := (d_2 - 1, d_3 - 1, \ldots, d_{r+1} - 1, d_{r+2}^{(1)}, \ldots, d_n^{(1)}),$$

where $(d_{r+2}^{(1)}, \ldots, d_n^{(1)})$ is simply the sequence $(d_{r+2} - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ reordered so as to be non-increasing. Note that our modified laying off procedure consisted of deleting the first term d_1, subtracting 1 from each of the next d_1 terms, then reorganizing the last $n-r-1$ terms of our sequence to be non-increasing. This is, in fact, very much our general plan for producing π_i, and we state this explicitly.

For $i \in [r+1]$, and given the sequence

$$\pi_{i-1} = (d_i - (i-1), d_{i+1} - (i-1), \ldots, d_{r+1} - (i-1), d_{r+2}^{(i-1)}, \ldots, d_n^{(i-1)}),$$

we define

$$\pi_i := (d_{i+1} - i, d_{i+2} - i, \ldots, d_{r+1} - i, d_{r+2}^{(i)}, \ldots, d_n^{(i)}),$$

where $(d_{r+2}^{(i)}, \ldots, d_n^{(i)})$ is simply a reordering of the last $n-r-1$ terms so that they are non-increasing.

We note that laying off $d_i - i + 1$ from π_{i-1} to generate π_i causes $d_i - i + 1$ terms to each get 1 smaller, starting with the term indexed with an $i+1$. Hence, the largest indexed term to have 1 subtracted from it, when forming π_i from π_{i-1}, will be indexed by the integer $i + d_i - i + 1 = d_i + 1$.

Before we consider a theorem that depends on the family of sequences $\{\pi_i\}$ for $0 \leq i \leq r+1$ just defined, we consider a concrete example. Let $\pi = (5, 4, 4, 3, 3, 1, 1, 1) \in NS_8$. Let us fix $r+1 = 4$, which is certainly less than or equal to 8. The algorithm described above produces the following family of sequences:

$$\begin{aligned}
\pi_0 &= (5, 4, 4, 3, 3, 1, 1, 1) \\
\pi_1 &= (3, 3, 2, 2^{(1)}, 1^{(1)}, 1^{(1)}, 0^{(1)}) \\
\pi_2 &= (2, 1, 1^{(2)}, 1^{(2)}, 1^{(2)}, 0^{(2)}) \\
\pi_3 &= (0, 1^{(3)}, 1^{(3)}, 0^{(3)}, 0^{(3)}) \\
\pi_4 &= (1^{(4)}, 1^{(4)}, 0^{(4)}, 0^{(4)})
\end{aligned}$$

The modified laying off procedure described above consists of removing the leading term $d_i - i + 1$ twice (once when it is deleted and a second time when 1 is subtracted from each of the next $d_i - i + 1$ terms). In short, an even number, namely $2(d_i - i + 1)$, is removed from the sum total of π_{i-1} in order to generate π_i. Concisely, $\sigma(\pi_{i-1}) - \sigma(\pi_i) = 2(d_i - i + 1)$. Consequently, $\sigma(\pi)$ is even if and only if $\sigma(\pi_i)$ is even for all $i \in [r + 1]$.

Our next theorem asserts that the sequence π_{r+1} tells us a great deal about whether π is potentially A_{r+1}-graphic or not. In particular, we shall see that π is potentially A_{r+1}-graphic if and only if π_{r+1} is graphic. Note that, from the concrete family of sequences we constructed above, this implies $(5, 4, 4, 3, 3, 1, 1, 1)$ is potentially A_4-graphic since the sequence $(1, 1, 0, 0)$ is quite clearly graphic. The theorem is due to Rao [12], but the proof that follows is our own.

Theorem 3.1. *Fix whole numbers n and r such that $n \geq r + 1$. Let $\pi = (d_1, d_2, \dots, d_n) \in NS_n$ with $d_{r+1} \geq r$. Then π is potentially A_{r+1}-graphic if and only if π_{r+1} is graphic.*

Proof. First, suppose that $\pi = (d_1, d_2, \dots, d_n)$ is potentially A_{r+1}-graphic. Let G be a graph that realizes π such that for all $i \in [n]$, $v_i \in V$ has degree d_i, and so that the $r + 1$ vertices of highest degree induce an $(r + 1)$-clique. We proceed with the same style proof used in verifying Theorem 2.1.

Denote by N_1 the d_1 neighbors of vertex v_1. Denote by M_1 the set of vertices $\{v_i\}$ for $2 \leq i \leq d_{d_1+1}$. If $N_1 = M_1$ then deleting vertex v_1 from G results in a realization of π_1. If instead $N_1 \neq M_1$, then since these sets have equal cardinalities, there must exist some $v_a \in N_1$ and $v_b \in M_1$ such that v_a does not belong to M_1 and v_b does not belong to N_1. Note that neither v_b nor v_a is a vertex belonging to the induced $r+1$-clique in G, as all of these vertices belong to $M_1 \cap N_1$, by construction. Now, v_a is a neighbor of v_1, and v_b is not. Furthermore, v_b has at least as many neighbors as v_a by construction of M_1. Thus, since v_a has a neighbor that v_b does not have, it follows that v_b has a neighbor v_c that is not adjacent to v_a. In short, our graph contains the subgraph illustrated in Figure 14.

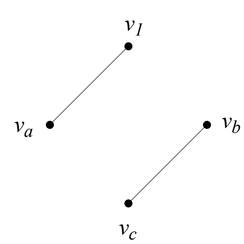


FIGURE 14.

Note that edges v_1v_b and v_av_c are known not to belong to G. Thus we may perform a 2-switch, creating a new graph G^* with the same degree sequence that G has. If we reform sets N_1 and M_1 for this new graph, we find that their intersection is larger by exactly one element (for v_b has been included). Furthermore, we note that no vertex belonging to the induced $r+1$-clique has been added to or removed from this clique in our process. Therefore, we have a new graph G^* which is an A_{r+1} graphic realization of π. It is clear that this procedure can be repeated until the intersection of N_1 and M_1 is equal to N_1. Finally, deleting vertex v_1 from the resulting graph produces a realization of π_1.

We have seen in the preceding paragraph that whenever π is potentially A_{r+1} graphic, there exists an A_{r+1} graphic realization of π such that removing the vertex of highest degree produces a graphic realization of π_1. We note that π_1 is A_r graphic by virtue of Corollary 1.1. Consequently, we need merely follow the steps outlined above to produce a realization (necessarily A_{r-1} graphic) of π_2. Indeed, by repeating the procedure, we will certainly produce a graphic realization of π_{r+1}, as was desired to show.

Next, suppose that π_{r+1} is graphic. Let a simple graph G be given that realizes π_{r+1}. To this graph add a new vertex. Some set of terms in π_{r+1} each got one smaller in the process of forming π_{r+1} from π_r, and these terms are the degrees of a specific set of vertices in G. Connect each of these vertices to the new vertex. It is clear that the resulting graph is a realization of π_r. It is equally clear that this procedure can be repeated until a graphic realization of π is produced. All that remains is a verification that the resulting $r+1$ vertices with largest degrees in this realization induce an $(r+1)$-clique. But this is clear as well; by the manner in which our laying off process was defined, each vertex added to π_i to ultimately realize π_{i-1} is always adjacent to every vertex previously added. Hence, we see that if π_{r+1} is graphic, it does indeed follow that π is potentially A_{r+1}-graphic.

□

In order to derive further sufficient conditions for $\pi \in NS_n$ to be potentially A_{r+1}-graphic, we will need to develop some rather technical machinery. First, we define a specific number associated with each sequence in the family $\{\pi_i\}$ for $0 \leq i \leq r+1$. Recall that a typical element of this family is given by $\pi_i = (d_{i+1}-i, d_{i+2}-i, \ldots, d_{r+1}-i, d_{r+2}^{(i)}, \ldots, d_n^{(i)})$. In particular, this sequence has the $(n-r-1)$ term tail $(d_{r+2}^{(i)}, \ldots, d_n^{(i)})$. Let $t_i \in [n-r-1]$ be the unique whole number that exactly counts the number of elements of this tail that are within 1 (in size) of $d_{r+2}^{(i)}$. Put another way, $t_i := \max\{j | d_{r+2}^{(i)} - d_{r+1+j}^{(i)} \leq 1\}$.

Earlier we constructed a concrete family of sequences from $\pi = (5,4,4,3,3,1,1,1)$ with $r+1 = 4$. By the definition just given of t_i for $0 \leq i \leq r+1$, we have the following:

$$\begin{aligned}
&\pi_0 = (5,4,4,3,3,1,1,1) && t_0 = 1\\
&\pi_1 = (3,3,2,2^{(1)},1^{(1)},1^{(1)},0^{(1)}) && t_1 = 3\\
&\pi_2 = (2,1,1^{(2)},1^{(2)},1^{(2)},0^{(2)}) && t_2 = 4\\
&\pi_3 = (0,1^{(3)},1^{(3)},0^{(3)},0^{(3)}) && t_3 = 4\\
&\pi_4 = (1^{(4)},1^{(4)},0^{(4)},0^{(4)}) && t_4 = 4
\end{aligned}$$

Next, we consider the following lemma, due to Yin et al. [1]. Essentially, this lemma gives us a clearer picture of what the final $n-r-1$ terms of π_i look like for each $i \in [r+1]$.

Lemma 3.1. *Let n and r be fixed positive whole numbers such that $n \geq r+1$, and let $\pi = (d_1, d_2, \ldots d_n) \in NS_n$ such that $d_{r+1} \geq r$, $\sigma(\pi)$ is even and $n-2 \geq d_1 \geq \cdots \geq d_r \geq d_{r+1} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n$. Let t_i be as defined above. Then:*

1. *$t_{r+1} \geq t_r \geq \cdots \geq t_0 \geq d_1 + 1 - r$.*
2. *For each $i \geq 1$, $d^{(i)}_{r+1+k} = d^{(i-1)}_{r+1+k}$ for $k > t_i$. Consequently, $d^{(r+1)}_{r+1+k} = d_{r+1+k}$ for $k > t_{r+1}$.*
3. *$\sigma(\pi_{i-1}) - \sigma(\pi_i) = 2(d_i - i + 1)$ for $i \in [r+1]$. Consequently, $\sum_{i=1}^{r+1} d_i = r(r+1) + \sum_{r+2}^{n} d_i - \sigma(\pi_{r+1})$.*

Proof. We consider each of the proposed statements in turn.

(1) Recall that t_0 is defined so that d_{r+1+t_0} is the term of π with largest index within 1 of d_{r+1}. Since, by hypothesis, $d_{r+1} = d_{d_1+2}$, we see immediately that the index $r+1+t_0$ is not less than the index d_1+2. It follows that $r+1+t_0 \geq d_1+2$, hence $t_0 \geq d_1 + 1 - r$.

Let $i \in [r+1]$ and consider $\pi_i = (d_{i+1} - i, d_{i+2} - i, \ldots, d_{r+1} - i, d^{(i)}_{r+2}, \ldots, d^{(i)}_n)$. In particular, note that the tail of this sequence is non-increasing, and that the first t_i elements of this tail are all within 1 of $d^{(i)}_{r+2}$ and strictly larger than $d^{(i)}_{r+1+t_i+1}$ by definition of t_i.

Now, laying off the leading term of π_i (to form π_{i+1}) reduces the first several terms of π_i by 1. Suppose this laying off results in all of $(d^{(i)}_{r+2}, d^{(i)}_{r+3}, \ldots, d^{(i)}_{r+1+t_i})$ being reduced by 1. Since $d_{r+1+t_i} > d_{r+1+t_i+1}$ by definition of t_i, we see that reordering the tail of π_i to form π_{i+1} does not involve reordering any of the first t_i such terms. Furthermore, these first t_i terms are just as within 1 of $d_r + 2$ as they started out being. In short, t_{i+1} is certainly not less than t_i. Suppose instead that the laying off process results instead in none of the terms of $(d^{(i)}_{r+2}, d^{(i)}_{r+3}, \ldots, d^{(i)}_{r+1+t_i})$ being reduced by 1. Clearly no reordering of the tail of the resulting sequence is then required to form π_{i+1}, and we once more find that t_{i+1} is not less than t_i.

The only other possibility is that the laying off process under consideration will reduce only part of $(d^{(i)}_{r+2}, d^{(i)}_{r+3}, \ldots, d^{(i)}_{r+1+t_i})$ by 1. Let us denote by $d^{(i)}_a$, for $r+2 < a < r+1+t_i$, the term of π_i with largest index that is reduced by 1 in our laying off procedure to form π_{i+1}. Since $d^{(i)}_{r+2} - d^{(i)}_{r+1+t_i} \leq 1$, we see immediately that $d^{(i)}_{r+1+t_i} \geq d^{(i)}_{r+2} - 1$, hence we have the non-increasing sequence

$$d^{(i)}_{a+1}, \ldots, d^{(i)}_{r+1+t_i}, d^{(i)}_{r+2} - 1, \ldots, d^{(i)}_a - 1.$$

Now $d_a^{(i)} \geq d_{r+2}^{(i)} - 1$, since $r+2 < a < r+1+t_i$. But $d_{r+2}^{(i)} - 1 > d_{r+1+t_i+1}^{(i)}$ by definition of t_i, hence $d_a^{(i)} - 1 \geq d_{r+1+t_i+1}^{(i)}$. Therefore

$$\underbrace{d_{a+1}^{(i)}, \ldots, d_{r+1+t_i}^{(i)}, d_{r+2}^{(i)} - 1, \ldots, d_a^{(i)} - 1}_{t_i \text{ terms}}, d_{r+1+t_i+1}^{(i)}, \ldots, d_n^{(i)}$$

is non-increasing and is therefore equal to $(d_{r+2}^{(i+1)}, \ldots, d_n^{(i+1)})$. In particular, $d_{a+1}^{(i)} = d_{r+2}^{(i+1)}$ and $d_a^{(i)} - 1 = d_{r+1+t_i}^{(i+1)}$. Note that $d_a^{(i)}$ is either equal to $d_{a+1}^{(i)}$ or one larger by virtue of our index a being strictly between $r+2$ and $r+1+t_i$. In either case, $d_a^{(i)} - 1$ is certainly within 1 of $d_{a+1}^{(i)}$. Thus $d_{r+2}^{(i+1)} - d_{r+1+t_i}^{(i+1)} \leq 1$, and we find once more that t_{i+1} is not less than t_i. Since we have now examined every possible case, we conclude that $t_{i+1} \geq t_i$ for all $i \in [r+1]$. Along with the fact that $t_0 \geq d_1 + 1 - r$, which we derived earlier, we have the desired combined inequality

$$t_{r+1} \geq t_r \geq, \ldots, t_1 \geq t_0 \geq d_1 + 1 - r.$$

(2) Recall that the sequence π_i is formed by deleting the leading term of π_{i-1}, subtracting 1 from each of the remaining terms up through the term indexed by the integer $d_i + 1$, and reordering the last $n - r - 1$ terms to be non-increasing. If it turns out that $d_i + 1 < r + 1 + t_{i-1}$ for each $i \in [r+1]$, then by the last case considered in the preceding proof, none of the terms $d_{r+1+t_{i-1}+1}^{(i-1)}, \ldots, d_n^{(i-1)}$ will be reordered (or reindexed) in this modified laying off process. In short, we would have $d_{r+1+k}^{(i-1)} = d_{r+1+k}^{(i)}$ for all (relevant) $k \geq t_{i-1} + 1$. In fact, since $t_i + 1 \geq t_{i-1} + 1$ for all $i \in [r+1]$, we need only insist that $k > t_i$, as desired. Hence, all that remains is to show that the inequality $d_i + 1 < r + 1 + t_{i-1}$ does, in fact hold for all $i \in [r+1]$.

First, we note that the inequality in question can be rewritten $d_i + 1 - r \leq t_{i-1}$. Thus, if $i = 1$, the inequality to be verified is $d_1 + 1 - r \leq t_0$, which was shown true in the preceding proof. Suppose, for an induction hypothesis, that $d_i + 1 - r \leq t_{i-1}$ for some particular $i \in [r]$. Since $d_{i+1} \leq d_i$ by hypothesis and $t_{i-1} \leq t_i$ by the preceding proof, we have $d_{i+1} + 1 - r \leq d_i + 1 - r \leq t_{i-1} \leq t_i$. Since we have succeeded in completing our inductive step, we conclude the inequality holds for all $i \in [r+1]$, as desired.

Now, we have verified that $d_{r+1+k}^{(i-1)} = d_{r+1+k}^{(i)}$ for all $k > t_i$. Since t_{r+1} is greater than or equal to t_i for all $i \in [r+1]$, setting $k > t_{r+1}$ makes $d_{r+1+k}^{(i-1)} = d_{r+1+k}^{(i)}$ true for all $i \in [r+1]$. Stringing these r distinct equalities together gives us $d_{r+1+k} = d_{r+1+k}^{(r+1)}$ for $k > t_{r+1}$, as was to be proven.

(3) We have shown earlier that $\sigma(\pi_{i-1}) - \sigma(\pi_i) = 2(d_i - i + 1)$ for $i \in [r+1]$. Writing this equality out for each $i \in [r+1]$ and summing the results together yields

$$\sigma(\pi_0) - \sigma(\pi_{r+1}) = 2(\sum_{i=1}^{r+1} d_i - \sum_{i=1}^{r+1}(i-1)).$$

We note that $\sigma(\pi_0) = \sum_{i=1}^{r+1} d_i + \sum_{r+2}^{n} d_i$, and that $\sum_{i=1}^{r+1}(i-1) = \frac{1}{2}r(r+1)$. Subtracting $\sum_{i=1}^{r+1} d_i$ from both sides of the equation above and adding $r(r+1)$ yields the desired

$$\sum_{i=1}^{r+1} d_i = r(r+1) + \sum_{r+2}^{n} d_i - \sigma(\pi_{r+1}).$$

□

The following lemma is another piece of technical machinery which will prove useful in the sequel. The inequality is of some interest, but of critical importance is how much we learn about the structure of our given sequence if and when the bound is realized.

Lemma 3.2. *Let n and r be fixed positive whole numbers such that $n \geq r+1$, and let $\pi = (d_1, d_2, \ldots d_n) \in NS_n$ such that $d_{r+1} \geq r$, $\sigma(\pi)$ is even and $n-2 \geq d_1 \geq \cdots \geq d_r \geq d_{r+1} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n$. If $t_{r+1} \leq d_{r+2}^{(r+1)}$, then*

$$\sum_{i=1}^{r+1} d_i \leq r(r+1) + d_{r+2}^{(r+1)}(d_{r+1} - d_{r+2}^{(r+1)} + 1) - 1,$$

with equality if and only if

(a) $d_{r+3}^{(r+1)} = \cdots = d_{r+1+t_{r+1}}^{(r+1)} = d_{r+2}^{(r+1)} - 1$
(b) $d_{r+1} = \cdots = d_{r+1+t_{r+1}}$, *and*
(c) $t_{r+1} = d_{r+2}^{(r+1)}$.

Proof. By Lemma 3.1(2),

$$\begin{aligned}\pi_{r+1} &= (d_{r+2}^{(r+1)}, d_{r+3}^{(r+1)}, \ldots, d_{r+1+t_{r+1}}^{(r+1)}, d_{r+1+t_{r+1}+1}^{(r+1)}, \ldots, d_n^{(r+1)}) \\ &= (d_{r+2}^{(r+1)}, d_{r+3}^{(r+1)}, \ldots, d_{r+1+t_{r+1}}^{(r+1)}, d_{r+1+t_{r+1}+1}, \ldots, d_n).\end{aligned}$$

Note that the first t_{r+1} terms of π_{r+1} are all at least $d_{r+2}^{(r+1)} - 1$ by definition of t_{r+1}. Hence,

$$\sigma(\pi_{r+1}) \geq d_{r+2}^{(r+1)} + (t_{r+1} - 1)(d_{r+2}^{(r+1)} - 1) + \sum_{i=r+2+t_{r+1}}^{n} d_i, \tag{6}$$

with equality if and only if $d_{r+3}^{(r+1)} = \cdots = d_{r+1+t_{r+1}}^{(r+1)} = d_{r+2}^{(r+1)} - 1$. Meanwhile, consider the last $n-r-1$ terms of π, namely $(d_{r+2}, \ldots, d_{r+1+t_{r+1}}, d_{r+1+t_{r+1}+1}, \ldots, d_n)$. Since each of the first t_{r+1} terms of this sequence are no larger than d_{r+1} due to the fact that $\pi \in NS_n$, we have

$$\sum_{i=r+2}^{n} d_i \leq t_{r+1} d_{r+1} + \sum_{i=r+2+t_{r+1}}^{n} d_i, \tag{7}$$

with equality if and only if $d_{r+1} = \cdots = d_{r+1+t_{r+1}}$. Combining inequalities (6) and (7) with our result from Lemma 3.1(3) (which involves adding and subtracting the sum $\sum_{i=r+2+t_{r+1}}^{n} d_i$) yields

$$\begin{aligned} \sum_{i=1}^{r+1} d_i &= r(r+1) + \sum_{i=r+2}^{n} d_i - \sigma(\pi_{r+1}) \\ &\leq r(r+1) + t_{r+1} d_{r+1} - \left(d_{r+2}^{(r+1)} + (t_{r+1} - 1)(d_{r+2}^{(r+1)} - 1) \right) \\ &= r(r+1) + t_{r+1}(d_{r+1} - d_{r+2}^{(r+1)} + 1) - 1 \qquad (8) \\ &\leq r(r+1) + d_{r+2}^{(r+1)}(d_{r+1} - d_{r+2}^{(r+1)} + 1) - 1. \qquad (9) \end{aligned}$$

Note that the inequality between lines (8) and (9) is justified by hypothesis, and the bound is realized if and only if $t_{r+1} = d_{r+2}^{(r+1)}$. It is therefore now clear that the inequality

$$\sum_{i=1}^{r+1} d_i \leq r(r+1) + d_{r+2}^{(r+1)}(d_{r+1} - d_{r+2}^{(r+1)} + 1) - 1$$

depends on three others, whose conditions for equality are known. Hence, as desired, equality holds in the inequality displayed above if and only if

(*a*) $d_{r+3}^{(r+1)} = \cdots = d_{r+1+t_{r+1}}^{(r+1)} = d_{r+2}^{(r+1)} - 1$
(*b*) $d_{r+1} = \cdots = d_{r+1+t_{r+1}}$, and
(*c*) $t_{r+1} = d_{r+2}^{(r+1)}$.

□

We are finally in a position to be able to prove a rather powerful sufficient condition for $\pi \in NS_n$ to be potentially A_{r+1}-graphic. Both the theorem and the proof are due to Yin et al. [1].

Theorem 3.2. *Let n and r be fixed positive whole numbers such that $n \geq r+1$, and let $\pi = (d_1, d_2, \ldots d_n) \in NS_n$ such that $d_{r+1} \geq r$, $\sigma(\pi)$ is even and $n-2 \geq d_1 \geq \cdots \geq d_r \geq d_{r+1} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n$. If $d_i \geq 2r - i$ for each $i \in [r-1]$, then π is potentially A_{r+1}-graphic.*

Proof. Recall that π is potentially A_{r+1}-graphic if and only if π_{r+1} is graphic, as verified in Theorem 3.1. Hence, we need only focus our attention on proving that

$$\pi_{r+1} = (d_{r+2}^{(r+1)}, d_{r+3}^{(r+1)}, \ldots, d_{r+1+t_{r+1}}^{(r+1)}, d_{r+1+t_{r+1}+1}, \ldots, d_n)$$

is graphic. To do so, we consider three separate cases, enumerated below.

(1) Suppose $t_{r+1} \geq d^{(r+1)}_{r+2} + 1$. It follows immediately that $d_{d^{(r+1)}_{r+2}+1} \geq d_{t_{r+1}}$. Since t_{r+1} is clearly less than $r+1+t_{r+1}$, we also have the inequality $d_{t_{r+1}} \geq d_{r+1+t_{r+1}}$. Furthermore, $d_{r+1+t_{r+1}} \geq d^{(r+1)}_{r+2} - 1$ by definition of t_{r+1}. Stringing these inequalities together yields $d_{d^{(r+1)}_{r+2}+1} \geq d^{(r+1)}_{r+2} - 1$, thus π_{r+1} is graphic by Theorem 2.3.

(2) Suppose $t_{r+1} \leq d^{(r+1)}_{r+2}$ and $d_{r+1} \geq 2r-1$. Note that since $\pi \in NS_n$, the first $r+1$ terms of the sequence $(d_1, d_2, \ldots, d_n)$ are each greater than or equal to d_{r+1}. Thus we have the inequality $\sum_{i=1}^{r+1} d_i \geq (r+1)d_{r+1}$, with equality holding if and only if $d_1 = \cdots = d_{r+1}$. We combine this result with the one derived in Lemma 3.2, obtaining $(r+1)d_{r+1} \leq r(r+1) + d^{(r+1)}_{r+2}(d_{r+1} - d^{(r+1)}_{r+2} + 1) - 1$, hence

$$0 \leq (r - d_{r+1})(r+1) + d^{(r+1)}_{r+2}(d_{r+1} - d^{(r+1)}_{r+2} + 1) - 1.$$

Now, for all $x \in \mathbb{R}$, let $f(x) = x(d_{r+1} - x + 1)$. Note that f attains its maximum value when $x = \frac{1+d_{r+1}}{2}$, and is strictly decreasing on the domain $(\frac{1+d_{r+1}}{2}, \infty)$. Now, by assumption, $d^{(r+1)}_{r+2} \geq t_{r+1}$. Furthermore, $t_{r+1} \geq d_1 + 1 - r$ by Lemma 3.1(1). Since $d_1 \geq d_{r+1}$, we clearly have $d_1 + 1 - r \geq d_{r+1} + 1 - r$. Finally, since by hypothesis $\frac{1+d_{r+1}}{2} \geq r$, we have $d_{r+1} + 1 - r \geq \frac{1+d_{r+1}}{2}$. Combining this set of inequalities, we have

$$d^{(r+1)}_{r+2} \geq t_{r+1} \geq d_1 + 1 - r \geq d_{r+1} + 1 - r \geq \frac{1 + d_{r+1}}{2}.$$

In particular, we see that $d^{(r+1)}_{r+2} \geq d_{r+1} + 1 - r$, and that both of these values belong to the domain $(\frac{1+d_{r+1}}{2}, \infty)$. Therefore, $f(d^{(r+1)}_{r+2}) \leq f(d_{r+1} + 1 - r)$, and we have

$$\begin{aligned}
0 &\leq (r - d_{r+1})(r+1) + d^{(r+1)}_{r+2}(d_{r+1} - d^{(r+1)}_{r+2} + 1) - 1 \\
&= (r - d_{r+1})(r+1) + f(d^{(r+1)}_{r+2}) - 1 \\
&\leq (r - d_{r+1})(r+1) + f(d_{r+1} + 1 - r) - 1 \\
&= (r - d_{r+1})(r+1) + (d_{r+1} + 1 - r)r - 1 \\
&= 2r - d_{r+1} - 1 \\
&\leq 0.
\end{aligned}$$

Thus $f(d^{(r+1)}_{r+2}) = f(d_{r+1} + 1 - r)$, and we learn that $d^{(r+1)}_{r+2} = d_{r+1} + 1 - r$. Since $2r - d_{r+1} - 1 = 0$, we learn that $d^{(r+1)}_{r+2} = r$. Indeed, since our many inequalities have proven to be equalities, we have $d^{(r+1)}_{r+3} = \cdots = d^{(r+1)}_{r+1+t_{r+1}} = d^{(r+1)}_{r+2} - 1 = r - 1$

and $t_{r+1} = d_{r+2}^{(r+1)} = r$, which together reveal that

$$\pi_{r+1} = (r, \overbrace{r-1, \ldots, r-1}^{r-1 \text{ terms}}, d_{2r+2}, \ldots, d_n).$$

Note that the term d_{2r+2}, being the term immediately to the right of $d_{r+1+t_{r+1}}$, is $r-2$ or smaller by definition of t_{r+1}. Also note that $r+(r-1)^2$ is the sum of an odd and even integer, hence odd. Since $\sigma(\pi_{r+1})$ is even, it must be the case that $d_{2r+2} \geq 1$. Thus, if we lay off the first term of π_{r+1} we have

$$(\overbrace{r-2, \ldots, r-2}^{r-1 \text{ terms}}, d'_{2r+2}, \ldots, d'_n)$$

where the tail $(d'_{2r+2}, \ldots, d'_n)$ is to be interpreted as a reordering of the last $n-2r-1$ terms so that they are non-increasing. This sequence is certainly a non-increasing sequence of whole numbers, has an even sum, and has the property that $d_{(r-2)+1} \geq (r-2)-1$, thus by Theorem 2.3 it is graphic. By Theorem 2.1, it follows that π_{r+1} is graphic as well, as desired.

(3) Suppose $t_{r+1} \leq d_{r+2}^{(r+1)}$ and $d_{r+1} \leq 2r-2$. Since d_{r+1} is at least r by hypothesis, we may rephrase the condition $d_{r+1} \leq 2r-2$ as $d_{r+1} = r+x$ for some x such that $0 \leq x \leq r-2$. Consider the sum $\sum_{i=1}^{r+1} d_i = \sum_{i=1}^{r-x-1} d_i + \sum_{i=r-x}^{r+1} d_i$. Since $d_i \geq 2r-i$ for all $i \in [r-1]$, we find that

$$\sum_{i=1}^{r-x-1} d_i \ \geq \ \sum_{i=1}^{r-x-1} (2r-i) = \frac{(3r+x)(r-x-1)}{2}.$$

It is clear that each of the $x+2$ terms in the sum $\sum_{i=r-x}^{r+1} d_i$ is at least as large as d_{r+1}. Combining this result with the previous one, we find that

$$\sum_{i=1}^{r+1} d_i \geq \frac{(3r+x)(r-x-1)}{2} + (x+2)d_{r+1}.$$

Combining this inequality with the one obtained in Lemma 3.2 yields the rather cumbersome inequality

$$0 \leq r(r+1) + f(d_{r+2}^{(r+1)}) - 1 - \frac{(3r+x)(r-x-1)}{2} - (x+2)d_{r+1}.$$

We have already seen that $d_{r+2}^{(r+1)} \geq t_{r+1} \geq d_1 + 1 - r$ in the preceding case analysis. Since, by hypothesis, $d_1 \geq 2r-1$, we immediately have $d_1+1-r \geq r$. Since $d_{r+1} < 2r-1$, we clearly see that $r > \frac{1+d_{r+1}}{2}$. Combining this set of inequalities, we have

$$d_{r+2}^{(r+1)} \geq t_{r+1} \geq d_1 + 1 - r \geq r > \frac{1+d_{r+1}}{2}.$$

In particular, $d_{r+2}^{(r+1)} \geq r$, and both of these integers belong to the domain $(\frac{1+d_{r+1}}{2}, \infty)$. Thus, $f(d_{r+2}^{(r+1)}) \leq f(r)$. Using this inequality (along with $d_{r+1} = r + x$) we have

$$\begin{aligned}
0 &\leq r(r+1) + f(d_{r+2}^{(r+1)}) - 1 - \frac{(3r+x)(r-x-1)}{2} - (x+2)d_{r+1} \\
&\leq r(r+1) + f(r) - 1 - \frac{(3r+x)(r-x-1)}{2} - (x+2)d_{r+1} \\
&= r(r+1) + r(d_{r+1} - r + 1) - 1 - \frac{(3r+x)(r-x-1)}{2} - (x+2)d_{r+1} \\
&= r(r+1) + r(x+1) - 1 - \frac{(3r+x)(r-x-1)}{2} - (x+2)(r+x) \\
&= -\frac{1}{2}\left(x - (r-2)\right)\left(x - (r-1)\right).
\end{aligned}$$

Since the roots of the final quadratic expression above are consecutive integers, and the leading coefficient is negative, it cannot take on a strictly positive value for any integer x. Thus we conclude that $-\frac{1}{2}\left(x-(r-2)\right)\left(x-(r-1)\right) \leq 0$ (with equality possible only for $x = r-2$). Since equality is not only possible but certain, we have that $f(d_{r+2}^{(r+1)}) = f(r)$, hence $d_{r+2}^{(r+1)} = r$. Since all our inequalities have again proven to be equalities, we once more have $d_{r+3}^{(r+1)} = \cdots = d_{r+1+t_{r+1}}^{(r+1)} = d_{r+2}^{(r+1)} - 1 = r - 1$, thus

$$\pi_{r+1} = (r, \overbrace{r-1, \ldots, r-1}^{r-1 \text{ terms}}, d_{2r+2}, \ldots, d_n).$$

Hence, for identical reasons to those given in the preceding case, we conclude that π_{r+1} is graphic. Since we have now investigated all possible cases, we conclude that π is potentially A_{r+1}-graphic, as desired.

□

Before moving on to the next section, we offer one final sufficient condition for $\pi \in NS_n$ to be potentially A_{r+1}-graphic, also due to Yin et al. [1]. The reader will notice that the conditions of the following theorem are quite similar in many respects to those given in Theorem 3.2. There are two main differences, however. One is that we impose stricter requirements on how much longer our sequence must be than the desired clique size of our hoped for graphic realization. The other is that we need only verify that a single term of our sequence is in some sense "large enough", rather than checking to see that each of our first several terms are, as needed in Theorem 3.2.

Theorem 3.3. *Let n and r be fixed positive whole numbers such that $n \geq 2r+2$, and let $\pi = (d_1, d_2, \ldots d_n) \in NS_n$ such that $d_{r+1} \geq r$, $\sigma(\pi)$ is even and $n-2 \geq d_1 \geq \cdots \geq d_r \geq d_{r+1} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n$. If $d_{2r+2} \geq r-1$, then π is potentially A_{r+1}-graphic.*

Proof. As we did in the previous proof, we consider three separate cases, enumerated below.

(1) Suppose that $t_{r+1} \geq d_{r+2}^{(r+1)}+1$. For reasons identical to those given in the preceding proof, we have that π_{r+1} is consequently graphic, thus π is potentially A_{r+1}-graphic by Theorem 3.1.

(2) Suppose that $t_{r+1} \leq d_{r+2}^{(r+1)}$ and $d_{r+1} \geq 2r-1$. Since $d_i \geq d_{r+1} \geq 2r-1 \geq 2r-i$ for all $i \in [r-1]$, π satisfies all necessary conditions for employing Theorem 3.2, hence π is potentially A_{r+1}-graphic in this case as well.

(3) Suppose that $t_{r+1} \leq d_{r+2}^{(r+1)}$ and $d_{r+1} \leq 2r-2$. In our analysis of case 2 in our proof of Theorem 3.2, we derived the inequality

$$0 \leq (r-d_{r+1})(r+1) + f(d_{r+2}^{(r+1)}) - 1$$

where $f(x) = x(d_{r+1} - x + 1)$. The conditions necessary for asserting the above inequality are all met in this case as well. Now, recall that f is a strictly decreasing function on the domain $(\frac{1+d_{r+1}}{2}, \infty)$. Further note that since $d_{r+1} \leq 2r-2$, we have that $\frac{1+d_{r+1}}{2} \leq r - \frac{1}{2} < r$. Now

$$\begin{aligned}(r-d_{r+1})(r+1) + f(r+1) - 1 &= (r-d_{r+1})(r+1) + (r+1)(d_{r+1}-r) - 1 \\ &= -1 \\ &< 0.\end{aligned}$$

Therefore, $d_{r+2}^{(r+1)}$ must not be greater than or equal to $r+1$. But this means that $d_{r+2}^{(r+1)} \leq r$, and since $t_{r+1} \leq d_{r+2}^{(r+1)}$ by hypothesis, we find that $t_{r+1} \leq r$. Consequently, $r+1+t_{r+1}+1 \leq 2r+2$, thus $d_{r+1+t_{r+1}+1} \geq d_{2r+2} \geq r-1$. But $d_{r+1+t_{r+1}+1} \leq d_{r+2}^{(r+1)} - 2 = r-2$ by definition of t_{r+1}. We have reached a contradiction, hence we find that no π can satisfy the conditions of this third case. Having now considered every possibility, we conclude that π is A_{r+1}-graphic, as desired.

□

Consider, for example the sequence $(5, 5, 4, 4, 4, 4, 4, 3, 3, 3, 3, 2)$. We could convince ourselves fairly readily, by an application of Theorem 2.3, that this sequence is graphic. However, it is not immediately clear that a graph which realizes this sequence will necessarily contain a clique of any given size. Note that there are an even number of odd terms, that the fifth term through seventh terms are equal and at least four, and that the tenth term is at least three. By the preceding theorem, we deduce that there exists a graph that realizes our sequence with a K_5 subgraph.

In the next section, we limit our attention to sequences in GS_n. The two theorems stated and proved are both due to Yin et al. [1], and are the key results of this paper.

4. Two sufficient conditions for $\pi \in GS_n$ to be potentially A_{r+1}-graphic

Our first theorem of this section is quite similar to Theorem 3.2, though here we are interested in sequences already known to be graphic. The other key difference is that, in the theorem to follow, we are not forcing our sequence $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ to satisfy the rather stringent condition

$$n - 2 \geq d_1 \geq \cdots \geq d_r \geq d_{r+1} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n,$$

which was a necessary assumption in Theorem 3.2.

Theorem 4.1. *Let n and r be fixed positive whole numbers such that $n \geq r+1$, and let $\pi = (d_1, d_2, \ldots d_n) \in GS_n$ such that $d_{r+1} \geq r$. If $d_i \geq 2r - i$ for all $i \in [r-1]$, then π is potentially A_{r+1}-graphic.*

Proof. We will prove this theorem by means of induction on r. If $r = 1$, then by hypothesis $d_2 \geq 1$. In other words, regardless of our choice of n satisfying $n \geq r+1$, any graph realizing such a sequence $\pi \in GS_n$ necessarily contains an edge. Since this is a copy of K_2, we immediately conclude that π is indeed potentially A_2-graphic.

Now, suppose that the theorem holds for a particular (positive) integer value $r-1$. In other words, we assume that for every $n-1$ that is at least $(r-1)+1$ and every $\pi \in GS_{n-1}$ such that $d_{(r-1)+1} \geq r-1$, if $d_i \geq 2(r-1) - i$ for all $i \in [(r-1)-1]$, then π is potentially $A_{(r-1)+1}$-graphic. We wish to verify that the theorem holds for the integer r as well, thus we let π be an arbitrary sequence of length $n \geq r+1$ which satisfies each of the conditions of this theorem. We break up our argument into a pair of cases.

(1) Suppose that $d_1 = n-1$ or that there exists some integer t such that $r+1 \leq t \leq d_1+1$ and $d_t > d_{t+1}$. Laying off d_1 from π produces the sequence $\pi'_1 = (d'_1, d'_2, \ldots, d'_r, \ldots d'_{n-1})$. Clearly $n-1 \geq (r-1)+1$, since by assumption $n \geq r+1$. It is also clear that π'_1 is graphic, by Theorem 2.1. Thus, we have already verified the first two conditions necessary for employing our induction hypothesis. Before verifying the third condition, we note that d_1 of the terms of π'_1 are one less than they were as terms of π. Note that there does not exist a term to the right of $d_{d_1}+1$ that is greater than $d_{r+1}-1$. Thus, we find that the sequence $(d'_1, \ldots, d'_r)$ is equal to the sequence $(d_2 - 1, \ldots, d_{r+1} - 1)$, for all reordering that occurs in our laying off process will necessary occur to the right of d'_r. In particular, we have

$$d'_i = d_{i+1} - 1 \quad \text{for all } i \in [r]. \tag{10}$$

Equality (10) yields $d'_r = d_{r+1} - 1$, whereas $d_{r+1} - 1 \geq r - 1$ by hypothesis. Consequently, we learn that $d'_{(r-1)+1} = d'_r \geq r-1$, which is the third of our four induction hypothesis conditions. Finally, note that $d_{i+1} \geq 2r - (i+1)$ for each $i \in [r-2]$, by hypothesis. Incorporating this fact with (10), we have $d_i' \geq 2r - (i+1) - 1 = 2(r-1) - i$ for all $i \in [(r-1)-1]$, and we have therefore met every condition necessary to assert that π_1' is potentially $A_{(r-1)+1}$-graphic.

Let G be a simple graph realizing π_1' whose r vertices of highest degree induce the subgraph K_r. It is clear that adding a new vertex to G and connecting this vertex to each of the vertices whose degrees were reduced by 1 in passing from π to π_1' will necessarily involve connecting our new vertex to each of the r vertices in G with highest degree. In short, by adding our new vertex, we have formed a graph G^* whose $r+1$ vertices of highest degree induce a K_{r+1} subgraph, and this graph is a realization of π. Hence π is potentially A_{r+1}-graphic, as claimed.

(2) Suppose that $d_1 \neq n-1$ and $d_{r+1} = \cdots = d_{d_1+2}$. Since π is assumed to be graphic, it must be the case that $d_1 \leq n-2$. In short, we have $n-2 \geq d_1 \geq \cdots \geq d_r \geq d_{r+1} = \cdots = d_{d_1+2} \geq \cdots \geq d_n$. Consequently, by Theorem 3.2, π is potentially A_{r+1}-graphic. Since we have considered every possible case, we find that we have successfully completed our inductive step. The proof of the theorem is thus complete.

□

While Theorem 4.1 provides a relatively simple set of of inequalities to verify in order to conclude that a given graphic sequence contains an $(r+1)$-clique, it is important to keep in mind that these conditions are sufficient but not necessary. Indeed, the sequence $(5, 4, 4, 3, 3, 1, 1, 1)$ realized in Figure 6, is clearly both graphic and potentially A_4-graphic. However, this sequence fails to satisfy the conditions of Theorem 4.1. In short, the theorem just given is not a simple characterization of all graphic sequences that are potentially A_{r+1}-graphic, just a tool for verifying when a graphic sequence certainly is A_{r+1}-graphic.

Next, we offer our second sufficient condition for $\pi \in GS_n$ to be potentially A_{r+1}-graphic. Again, we note the clear similarity between the result to follow and a previous theorem (Theorem 3.3).

Theorem 4.2. *Let n and r be fixed positive whole numbers such that $n \geq 2r+2$, and let $\pi = (d_1, d_2, \ldots d_n) \in GS_n$ such that $d_{r+1} \geq r$. If $d_{2r+2} \geq r-1$, then π is potentially A_{r+1}-graphic.*

Proof. We prove this theorem by induction on r as well. If $r = 1$ we once again have $d_2 \geq 1$, indicating that, regardless of our choice of n satisfying $n \geq 2r+2$, any graph realizing such a sequence $\pi \in GS_n$ contains at least one edge, hence π is clearly A_2-graphic.

Now, suppose that the theorem holds for a particular (positive) integer value $r-1$. In other words, we assume that for every $n-1$ that is at least $2(r-1)+2$ and every

$\pi \in GS_{n-1}$ such that $d_{(r-1)+1} \geq r-1$, if $d_{2(r-1)+2} \geq (r-1)-1$, then π is potentially $A_{(r-1)+1}$-graphic. We wish to verify that the theorem holds for the integer r as well, thus we let π be an arbitrary sequence of length $n \geq 2r+2$ which satisfies each of the conditions of this theorem. We break up our argument, once again, into a pair of cases.

(1) Suppose that $d_1 = n-1$ or that there exists some integer t such that $r+1 \leq t \leq d_1+1$ and $d_t > d_{t+1}$. Form $\pi_1' = (d_1', d_2', \ldots, d_{n-1}')$ by laying off d_1. Note that since $n \geq 2r+2$ by hypothesis, it follows that $n-1 \geq 2r+1 > 2(r-1)+2$, thus we satisfy the first condition of our inductive hypothesis. By Theorem 2.1, π_1' is graphic, and we thereby meet our second condition as well. Now, for reasons identical to those given in the preceding proof, we have $d_i' = d_{i+1}-1$ for $i \in [r]$. For, $r+1 \leq i \leq n-1$, however, the strongest similar claim we can make is $d_i' \geq d_{i+1}-1$, due to potential reordering.

   Next, $d'_{(r-1)+1} = d_r' = d_{r+1}-1$, and by hypothesis $d_{r+1} \geq r$. Thus $d'_{(r-1)+1} \geq r-1$, and we have met the third condition of our induction hypothesis. Finally, $d'_{2(r-1)+2} = d'_{2r} \geq d_{2r+1}-1 \geq d_{2r+2}-1$. Since $d_{2r+2} \geq r-1$ by hypothesis, we have $d'_{2(r-1)+2} \geq (r-1)-1$. Consequently, we have satisfied every condition of our inductive hypothesis, and can now assert that π_1' is potentially $A_{(r-1)+1}$-graphic. By reasons identical to those given in the preceding proof, we conclude that π is thereby potentially A_{r+1}-graphic, as desired.

(2) Suppose that $d_1 \neq n-1$ (hence $d_1 \leq n-2$) and that $d_{r+1} = \cdots = d_{d_1+2}$. In particular, we have $n-2 \geq d_1 \geq \cdots \geq d_r \geq d_{r+1} = \cdots = d_{d_1+2} \geq \cdots \geq d_n$. Since in this case we have satisfied all necessary conditions to employ Theorem 3.3, we do so, concluding that π is potentially A_{r+1}-graphic. Once again, we have completely examined every case, thus our inductive step has been successfully made. Hence, the proof of the theorem is complete.

$\square$

Consider the sixty-six term sequence $(\overbrace{11, \ldots, 11}^{\text{11 terms}}, \overbrace{10, \ldots, 10}^{\text{10 terms}}, \ldots, 3, 3, 3, 2, 2, 1)$. The presence of an even number of odd terms guarantees that our sequence has a necessarily even sum. The twelfth term is clearly 10, thus Theorem 2.3 guarantees us that we can find a graph which realizes our sequence. Since the eleventh term of our sequence is at least ten and the twenty-second term is at least nine, Theorem 4.2 assures us that we can find a graph realizing our sequence that contains an 11-clique. This author finds the existence claim of such a graph rather remarkable.

Next, consider the eleven term sequence $(8, 8, \ldots, 8)$. It is clear that this sequence is graphic by Theorem 2.3. Though it exceeds the scope of this paper, this sequence is not realizable by a graph containing a K_6 subgraph, Yin et al. in [1]. In particular, any

$2r+1$ term sequence whose terms are all equal to $2r-2$ is graphic, but not potentially K_{r+1}-graphic. Hence the condition $n \geq 2r+2$ is best possible for Theorem 4.2. Indeed, a sequence that has $2r+2$ terms whose first $2r+1$ terms are each r and last terms is $r-2$ is graphic but not potentially K_{r+1}-graphic as well, Yin [1]. Thus the condition $d_{2r+2} \geq r-1$ is best possible as well.

5. Sufficient conditions for $\pi \in GS_n$ to be *nearly* potentially K_{r+1}-graphic

In the preceding section we gave two sufficient conditions for a graphic sequence to be potentially A_{r+1}-graphic (hence, potentially K_{r+1}-graphic). In this section we consider a slightly less restrictive goal. When is a graphic sequence *practically* able to be realized with a prescribed clique size? In particular, when can we come within a single edge of a complete subgraph of some desired size? We prove two sufficient conditions for such a scenario in the next pair of theorems, both due to Yin et al. [1]. Note that, in the theorems and arguments to follow, when we write $K_{r+1}-e$ we mean a simple graph on $r+1$ vertices that is one edge shy of being complete.

Theorem 5.1. *Let n and r be fixed positive whole numbers such that $n \geq r+1$, and let $\pi = (d_1, d_2, \dots d_n) \in GS_n$ such that $d_{r+1} \geq r-1$. If $d_i \geq 2r-i$ for all $i \in [r-1]$, then π has a realization containing $K_{r+1}-e$ as a subgraph.*

Proof. If $r=1$, then by hypothesis $d_1 \geq 1$. In other words, regardless of our choice of n satisfying $n \geq r+1$, any graph realizing such a sequence $\pi \in GS_n$ necessarily contains an edge, hence at least two vertices. Since any two vertices of any graph clearly form the subgraph $K_2 - e$, we have established that the theorem holds for $r=1$. We therefore assume that $r \geq 2$ for the remainder of this proof.

Given $\pi = (d_1, d_2, \dots, d_n)$, if $d_{r+1} \geq r$, then by Theorem 3.2, π is potentially A_{r+1}-graphic. It is clear that any graph that has an induced subgraph K_{r+1} certainly contains $K_{r+1}-e$ as a subgraph. Thus, we may assume without loss of generality that $d_{r+1} \leq r-1$, hence $d_{r+1} = r-1$.

Note that since $d_i \geq 2r-i$ for all $i \in [r-1]$, we have $d_{r-1} \geq 2r-(r-1) = r+1$. In other words, $d_{r-1} > d_{r+1}$. Now, let us form $\pi'_{r+1} = (d'_1, d'_2, \dots d'_{n-1})$ by laying off $d_{r+1} = r-1$. Since each d_i for $i \in [r-1]$ is at least as large as d_{r-1}, and d_{r-1} is strictly larger than d_{r+1}, we see that any reordering that takes place in our laying off process will be restricted to only the terms $(d'_1, \dots, d'_r)$. In particular, we note that all of the $r-1$ terms that have been reduced by 1 are among the first r terms of π'_{r+1}.

Since, by hypothesis, $n \geq r+1$, we have $n-1 \geq (r-1)+1$. Furthermore, since π is graphic, Theorem 2.1 guarantees us that π'_{r+1} is graphic as well. Thus, we have satisfied the first two conditions necessary for invoking Theorem 4.1. Now, $d'_i = d_{i+1}$ for $r \leq i \leq n-1$ since the last $n-r-2$ terms of π certainly do not get reordered in passing from π to

π'_{r+1}. Thus, $d'_{(r-1)+1} = d'_r = d_{r+1} = r-1$, and we have satisfied the third condition for applying Theorem 4.1. Finally, we note that $d'_i \geq d_{i+1} - 1$ for $1 \leq i \leq r-1$ since these are precisely the terms of π that might get reordered in passing from π to π_{r+1}. By hypothesis, $d_{i+1} \geq 2r-(i+1)$ for all $i \in [r-2]$. Consequently, $d'_i \geq d_{i+1}-1 \geq 2r-(i+1)-1 = 2(r-1)-i$ for all $i \in [(r-1)-1]$, and we have now met every condition necessary for invoking Theorem 4.1. We conclude that π'_{r+1} is potentially $A_{(r-1)+1}$-graphic.

Let G be a graph realizing π'_{r+1} whose r vertices of highest degree induce the complete graph K_r. To the graph G, we add a new vertex, connecting this new vertex to each of those $r-1$ vertices whose degrees correspond to those entries of π that were reduced by 1 in passing from π to π'_{r+1}. We have already seen that these vertices are among those vertices of r largest degree. Hence the graph G^* that results, which clearly realizes π, contains a subgraph K_r along with an additional vertex that is adjacent to $r-1$ of the vertices which induce the subgraph K_r. In short, our graph G^* contains the subgraph $K_{r+1} - e$, hence π has precisely the realization desired.

□

Our second sufficient condition for $\pi \in GS_n$ to be *nearly* potentially K_{r+1}-graphic is strongly reminiscent of Theorem 4.2 in much the same way that the previous result resembled Theorem 4.1.

Theorem 5.2. *Let n and r be fixed positive whole numbers such that $n \geq 2r+2$, and let $\pi = (d_1, d_2, \ldots d_n) \in GS_n$ such that $d_{r-1} \geq r$. If $d_{2r+2} \geq r-1$, then π has a realization containing $K_{r+1} - e$ as a subgraph.*

Proof. Let $\pi = (d_1, d_2, \ldots, d_n)$ be a sequence satisfying the conditions of this theorem. If $d_{r+1} \geq r$, then by Theorem 4.2, π is potentially A_{r+1}-graphic, hence π has a realization containing K_{r+1} as a subgraph. Thus, we may assume without loss of generality that $r-1 \geq d_{r+1}$. Also, we note that one of the conditions of this theorem is $d_{r-1} \geq r$, which is undefined for $r \leq 1$. Thus we may also safely assume that $r \geq 2$.

Now $r-1 \geq d_{r+1} \geq \cdots \geq d_{2r+2}$. Since d_{2r+2} is itself greater than or equal to $r-1$ by hypothesis, we find that $d_{r+1} = \cdots = d_{2r+2} = r-1$. Let us form $\pi'_{r+1} = (d'_1, d'_2, \ldots d'_{n-1})$ by laying off d_{r+1}. Since $d_{r+1} = r-1$, it is clear that in our laying off process, we will subtract 1 from each d_i for $i \in [r-1]$. Since $d_{r-1} \geq r$, we find that $d_i - 1$ is greater than or equal to $r-1 = d_{r+2}$ for all $i \in [r-1]$. Consequently, any reordering that needs to take place in our laying off procedure will only involve the first r terms of π. In short, those terms reduced by 1 in the laying off procedure will certainly be among the first r terms of π'_{r+1}.

Since, by hypothesis, $n \geq 2r+2$, it follows that $n-1 \geq 2r+1 > 2(r-1)+2$. Since π is graphic, π'_{r+1} is graphic as well by Theorem 2.1. For identical reasons to those given in the preceding proof, $d'_i = d_{i+1}$ for $r \leq i \leq n-1$, hence $d'_{(r-1)+1} = r-1$. Finally,

$d'_{2(r-1)+2} = d'_{2r} = d_{2r+1} = r-1 > (r-1)-1$. Thus, we find that π'_{r+1} satisfies the conditions necessary for invoking Theorem 4.2, and we conclude that π'_{r+1} is potentially $A_{(r-1)+1}$-graphic. For precisely the same reasons as those given in the last paragraph of our proof of Theorem 5.1, we conclude that π has a realization that contains the subgraph $K_{r+1} - e$, as desired.

□

6. Applications

We begin this section with a pair of simple consequences of Theorems 4.1 and 4.2. The first is a result credited to Rao [13], and the second is a result due to Li et al. [10].

Theorem 6.1. *Let n and r be fixed positive whole numbers such that $n \geq r+1$, and let $\pi = (d_1, d_2, \dots d_n) \in GS_n$. If $d_{r+1} \geq 2r-1$, then π is potentially A_{r+1}-graphic.*

Proof. Note that $d_i \geq d_{r+1} \geq 2r-1 \geq 2r-i$ for all $i \in [r-1]$. Also, $2r-1 \geq r$ for all positive integers r, hence $d_{r+1} \geq r$. We have therefore satisfied every condition necessary to invoke Theorem 4.1, and we conclude that π is potentially A_{r+1}-graphic, as claimed.

□

Theorem 6.2. *Let n and r be fixed positive whole numbers such that $n \geq 2r+2$, and let $\pi = (d_1, d_2, \dots d_n) \in GS_n$ such that $d_{r+1} \geq r$. If $n-2 \geq d_1 \geq \cdots \geq d_r = d_{r+1} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n \geq r-1$, then π is potentially A_{r+1}-graphic.*

Proof. It is quite clear that $d_{2r+2} \geq d_n \geq r-1$. Since every condition necessary for invoking Theorem 4.2 has been satisfied, we conclude that π is potentially A_{r+1}-graphic, as desired.

□

We have now developed several useful tools for deciding when a given sequence of non-increasing whole numbers is graphic, and when it is able to be realized with a prescribed clique size. In a sense, these realizations depend on both the structure of our sequence and on individual term size. In other words, it is critical to know whether or not, in some sense, the terms of our sequence have a variety of different values and or wild jumps in size. Indeed, in order to have realizations with a particular desired clique, we need to know that our vertex degrees are large enough and/or plentiful enough.

Let us consider this question of graphic realizations from a slightly different direction than we have pursued thus far. Let $\sigma(K_{r+1}, n)$ be defined to mean the smallest sum that a sequence belonging to GS_n must have in order to be guaranteed to have some realization containing a K_{r+1} subgraph. Clearly, this minimum sum will depend on both r and n. It was observed by Erdös et al. [3] that the n term (non-increasing) sequence consisting of $r-1$ copies of $n-1$ followed by $n-r+1$ copies of $r-1$ is graphic, but that it is uniquely realized. Critical to our discussion, this unique realization does not contain a K_{r+1} subgraph. Thus, we learn that an n term sequence with sum $(r-1)(n-1)+(n-r+1)(r-1) = (r-1)(2n-r)$ is not guaranteed to have a K_{r+1} subgraph. In short, $\sigma(K_{r+1}, n) \geq (r-1)(2n-r)+2$

(since our sum must remain even to be graphic at all).

Progress toward pinning down $\sigma(K_{r+1}, n)$ further has been made by several contributors, as noted by Yin et al. [1]. Erdös et al. [3] conjectured that for large enough values of n, $\sigma(K_{r+1}, n) = (r-1)(2n-r)+2$. This conjecture has been demonstrated true for several specific pairs of values r and n by several contributors (see Yin [1]). The theorem that follows, first stated and proven by Yin [11], settles the conjecture once and for all. We offer (and prove) the theorem below as an application of the tools developed in this paper.

Theorem 6.3. *$\sigma(K_{r+1}, n) = (r-1)(2n-r)+2$ for $n \geq \frac{3}{2}r^2$.*

Proof. We have already seen that $\sigma(K_{r+1}, n) \geq (r-1)(2n-r)+2$ by the discussion above. Hence we will have succeeded in proving this theorem if we can show $\sigma(K_{r+1}, n) \leq (r-1)(2n-r)+2$ for $n \geq \frac{3}{2}r^2$. In other words, we wish to show if $n \geq \frac{3}{2}r^2$, then any sequence $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ such that $\sigma(\pi) \geq (r-1)(2n-r)+2$ is potentially K_{r+1}-graphic.

If $r = 1$, we must verify that every graphic sequence with at least two terms whose sum is at least two has a K_2 subgraph. This is very clearly the case, as such a sequence will always be realized by a graph with at least one edge, and each edge is a copy of K_2. Therefore, we assume without loss of generality that $r \geq 2$.

Suppose that $d_{r+1} \leq r-1$. Note that $\sigma(\pi) = \sum_{i=1}^{r} d_i + \sum_{i=r+1}^{n} d_i$, whereas by Theorem 2.2 we have $\sum_{i=1}^{r} \leq r(r-1) + \sum_{i=r+1}^{n} \min(r, d_i)$. Hence we find that $\sigma(\pi) \leq r(r-1) + 2\sum_{i=r+1}^{n} \min(r, d_i)$. Since each d_i for $r+1 \leq i \leq n$ is no larger than $r-1$, it follows that $\min(r, d_i) = d_i$ for these same values of i. Consequently the sum $\sum_{i=r+1}^{n} \min(r, d_i)$ is not larger than $n-r$ copies of $r-1$. Thus,

$$\sigma(\pi) \leq r(r-1) + 2(n-r)(r-1) = (r-1)(2n-r),$$

which contradicts our hypothesis that $\sigma(\pi)$ has sum greater than or equal to $(r-1)(2n-r)+2$. We therefore find that $d_{r+1} \geq r$.

If either $d_i \geq 2r-i$ for each $i \in [r-1]$ or $d_{2r+2} \geq r-1$, then by Theorem 4.1 or Theorem 4.2 we have that π is potentially A_{r+1}-graphic, and we are done. Hence, we consider the only remaining possibility. Namely, we assume that $d_{2r+2} \leq r-2$ and there exists some $i \in [r-1]$ such that $d_i \leq 2r-i-1$. Clearly the first $i-1$ terms of π are no larger than the first term, itself no larger than $n-1$. The $2r+2-i$ terms from d_i to d_{2r+1} are each no larger than d_i, itself no larger than $2r-i-1$. Finally the remaining $n-2r-1$ terms are each no larger than d_{2r+2}, itself no larger than $r-2$. Putting these thoughts together yields

$$\begin{aligned} \sigma(\pi) &\leq (i-1)(n-1) + (2r+2-i)(2r-i-1) + (n-2r-1)(r-2) \\ &= n(i-1+r-2) - (i-1) - (2r+1)(r-2) + (2r+2-i)(2r-i-1). \end{aligned} \tag{11}$$

Now since $1 \leq i$, we may replace every i being subtracted in (11) with a one and our inequality symbol will be pointing the desired direction. Likewise, since $i \leq r-1$, we may replace every i being added in (11) with $r-1$. The result is

$$\begin{aligned}\sigma(\pi) &\leq n(2r-4) - (2r+1)(r-2) + (2r+1)(2r-2)\\ &= n(2r-4) + (2r+1)r\\ &= (2r-2)n - 2n + (2r+1)r.\end{aligned}$$

Finally, we make use of our bound $n \geq \frac{3}{2}r^2$. Since $2n \geq 3r^2$, we have

$$\begin{aligned}\sigma(\pi) &\leq (2r-2)n - 3r^2 + (2r+1)r\\ &= (2n-r)(r-1)\\ &< (2n-r)(r-1) + 2,\end{aligned}$$

which is a contradiction. Since we have now considered every possible case, our proof is complete.

□

Yin et al. [13] posed a similar question to the one just considered. Instead of insisting that our graph have a copy of K_{r+1} as a subgraph, suppose we only require that it *nearly* has a copy of K_{r+1}. In other words, how large of a sum must a graphic sequence of length n have in order to have a realization that contains $K_{r+1} - e$ as a subgraph? Once more, it is clear that the answer to this question will depend on both n and r. Our intuition would lead us to expect the lower bound on this sum to be no greater than $(r-1)(2n-r)+2$, since this lower bound guarantees a K_{r+1} subgraph. Indeed this is the case. Several solutions of $\sigma(K_{r+1} - e, n)$ have been found for small values of n and r by a number of contributors, (see [1]). Yin et al. [14], established that for $r \geq 2$ and $n \geq r+1$, we have

$$\sigma(K_{r+1} - e, n) \geq \begin{cases}(r-1)(2n-r) + 2 - (n-r) & \text{if } n-r \text{ is even,}\\ (r-1)(2n-r) + 1 - (n-r) & \text{if } n-r \text{ is odd.}\end{cases}$$

Yin conjectured that these lower bounds are realized for large enough n. This conjecture is proven by Yin et al. in [1]. We offer both the theorem and proof below.

Theorem 6.4. *If $r \geq 2$ and $n \geq 3r^2 - r - 1$, then*

$$\sigma(K_{r+1} - e, n) = \begin{cases}(r-1)(2n-r) + 2 - (n-r) & \textit{if } n-r \textit{ is even,}\\ (r-1)(2n-r) + 1 - (n-r) & \textit{if } n-r \textit{ is odd.}\end{cases}$$

Proof. Let n and r (≥ 2) be positive whole numbers such that $n \geq 3r^2 - r - 1$ and take $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ such that $\sigma(\pi) \geq (r-1)(2n-r) + 2 - (n-r)$. We demonstrate below that such a sequence must necessarily have a realization containing $K_{r+1} - e$ as a subgraph.

Let $g(x) = 3x^2 - 2x - 2$. Note that g is a strictly increasing function on the domain $(\frac{1}{3}, \infty)$. Since $r \geq 2$ and $g(2) = 6$, we conclude that $3r^2 - 2r - 2 \geq 0$. It follows that $3r^2 - r - 1 \geq r + 1$, hence $n \geq r + 1$.

Now, let us suppose that $d_{r+1} \leq r - 2$. By Theorem 2.2, we have $\sigma(\pi) = \sum_{i=1}^{r} d_i + \sum_{i=r+1}^{n} d_i \leq r(r-1) + \sum_{i=r+1}^{n} \min(r, d_i) + \sum_{i=r+1}^{n} d_i$. Since $d_i \leq d_{r+1} \leq r - 2$ for $r + 1 \leq i \leq n$, these last two sums are identical. Indeed the $n - r$ terms of each sum are each no greater than d_{r+1}, itself no greater than $r-2$. Thus, $\sigma(\pi) \leq r(r-1)+2(n-r)(r-2)$. Simple algebra thus reveals

$$\begin{aligned}
\sigma(\pi) &\leq r(r-1) + 2(n-r)(r-2) \\
&= r(r-1) + (2n-2r)(r-1) - (2n-2r) \\
&= (r-1)(2n-r) - 2(n-r) \\
&< (r-1)(2n-r) - (n-r) + 2,
\end{aligned}$$

a contradiction. Thus it must be the case that $d_{r+1} \geq r - 1$.

Next, we suppose that $d_{r-1} \leq r - 1$. Once again invoking Theorem 2.2, we have $\sigma(\pi) = \sum_{i=1}^{r-2} d_i + \sum_{i=r-1}^{n} d_i \leq (r-2)(r-3) + \sum_{i=r-1}^{n} \min(r-2, d_i) + \sum_{i=r-1}^{n} d_i$. Since $r - 2 \geq \min(r-2, d_i)$ for all relevant i, the second to last sum is no greater than $n - r + 2$ copies of $r - 2$. Meanwhile, since $d_{r-1} \leq r - 1$, the final sum is clearly no larger than $n - r + 2$ copies of $r - 1$. Thus $\sigma(\pi) \leq (r-2)(r-3) + (n-r+2)(r-2) + (n-r+2)(r-1) = (r-2)(n-1) + (n-r+2)(r-1)$. Consequently

$$\begin{aligned}
\sigma(\pi) &\leq (r-1)(n-1) - (n-1) + (n-r+1)(r-1) + (r-1) \\
&= (r-1)(2n-r) - (n-r) \\
&< (r-1)(2n-r) - (n-r) + 2,
\end{aligned}$$

a contradiction. We conclude that $d_{r+1} \geq r$.

Now, if either $d_i \geq 2r - i$ for all $i \in [r-1]$ or $d_{2r+2} \geq r - 1$, then we will have met all the conditions necessary to employ Theorem 5.1 or Theorem 5.2. Let us assume to the contrary that $d_{2r+2} \leq r - 2$ and that there exists some $i \in [r-1]$ such that $d_i \leq 2r - i - 1$. By an argument identical to the one given in the preceding proof, we have that $\sigma(\pi) \leq (2r-2)n - 2n + (2r+1)r$. Note that since $n \geq 3r^2 - r - 1$,

$$\begin{aligned}
\sigma(\pi) &\leq (2r-2)n - 2n + (2r+1)r \\
&\leq (2r-2)n - n - (3r^2 - r - 1) + (2r+1)r \\
&= (r-1)(2n-r) + 1 - (n-r) \\
&< (r-1)(2n-r) + 2 - (n-r),
\end{aligned}$$

a contradiction. We conclude that the conditions necessary for employing either Theorem 5.1 or Theorem 5.2 hold, hence π has a realization containing $K_{r+1} - e$ as a subgraph.

Since we have shown that a graphic sequence with sum at least $(r-1)(2n-r)+2-(n-r)$ has the desired subgraph $K_{r+1}-e$, we conclude that $\sigma(K_{r+1}-e,n)$ is no greater than this figure. Note that $(r-1)(2n-r) = 2n(r-1) - r(r-1)$ is certainly an even integer. In particular, since every graphic sequence has an even degree sum, if $n-r$ is odd, then $\sigma(K_{r+1}-e,n)$ must be strictly less than $(r-1)(2n-r)+2-(n-r)$. We conclude that

$$\sigma(K_{r+1}-e,n) \leq \begin{cases} (r-1)(2n-r)+2-(n-r) & \text{if } n-r \text{ is even,} \\ (r-1)(2n-r)+1-(n-r) & \text{if } n-r \text{ is odd.} \end{cases}$$

This result, along with the one given in the discussion prior to the statement of Theorem 6.4, concludes this proof.

□

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