Save the Gnomes
Optimal Strategy Hat Games

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Introduction

What is a Hat Game?

A Hat Game is a specific type of strategy puzzle that involves finding the 'best' outcome for a given set of parameters.

Now, these particular puzzles are called 'hat games' since all of them are placed in a context where we have a given number of players, and we place hats of varying colors on their heads.

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So our goal is to ***Save the Gnomes!***

Variations

These games have variations including, but not limited to,

- number of players
- number of hat colors
- visual information available to players
- auditory information available to players
- random uniform vs. non-uniform hat distributions
- rule of how hat colors are chosen (ie, based on fair coin flip vs. not fair coin flip)
- sequential vs. simultaneous guessing
- ability to pass or guess vs. being required to guess
- desired results (ie, majority guess correct, no one guess incorrect, highest probability to win for a specified definition of a win, etc...)
- adversarial vs. non-adversarial settings
- types of strategies allowed, etc.

So, one can imagine how wide an assortment of games can be defined.

Overview

During this presentation we will discuss several hat games and give their solutions with brief justifications and a sample game. These preliminary hat games are defined in order to construct a fourth game. For this last game we will define a strategy, sketch the proof of its optimality and provide a sample game.

Ebert's Hat Game

The Game/Rules

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There are 3 gnome prisoners (Elandria, Nisse and Mano) in a cell. Each has a hat placed on their head by the evil jailer. The hats can be one of two possible colors (black or red) and are chosen based on the outcome of a flip of a fair coin. The rules are as follows:

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All gnomes are pardoned if at least one gnome guesses correctly and no gnome guesses incorrectly, otherwise they are all executed.
What is the best strategy so the gnomes have the highest probability to survive?

The Strategy

The following guessing strategy is optimal.

Let each gnome guess 'red' if both the others are wearing black, 'black' if the others are both wearing red, and 'pass' if they are wearing opposite colors.

The following table shows how the guesses would occur for any hat configuration.

configuration	Elandria	Nisse	Mano
BBB	R	R	R
BBR	P	P	R
BRB	P	R	P
RBB	R	P	P
BRR	B	P	P
RBR	P	B	P
RRB	P	P	B
RRR	B	B	B

So, suppose the hat configuration is BBR (Elandria receives a black hat, Nisse a black hat and Mano a red hat). Then the guess configuration would be PPR and the gnomes are free.

Example Game

configuration	Elandria	Nisse	Mano	*outcome*
BBB	R	R	R	*lose*
BBR	P	P	R	*win*
BRB	P	R	P	*win*
RBB	R	P	P	*win*
BRR	B	P	P	*win*
RBR	P	B	P	*win*
RRB	P	P	B	*win*
RRR	B	B	B	*lose*

We can see in the table each gnome still has a 50% chance of guessing correctly but, all the incorrect guesses are concentrated in the BBB and RRR configurations. So, by following this strategy, the group wins with probability $\frac{6}{8} = 75\%!$

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- The minimum distance between any two codewords is 3.
- No two codewords are adjacent to each other, and any other vertex of $\{0,1\}^n$ is adjacent to exactly one codeword.
- Hamming codes only exist for $\{0,1\}^n$ where $n = 2^m - 1$ for some integer $m \geq 2$. So, Hamming codes exist for dimensions 3,7,15,31,...
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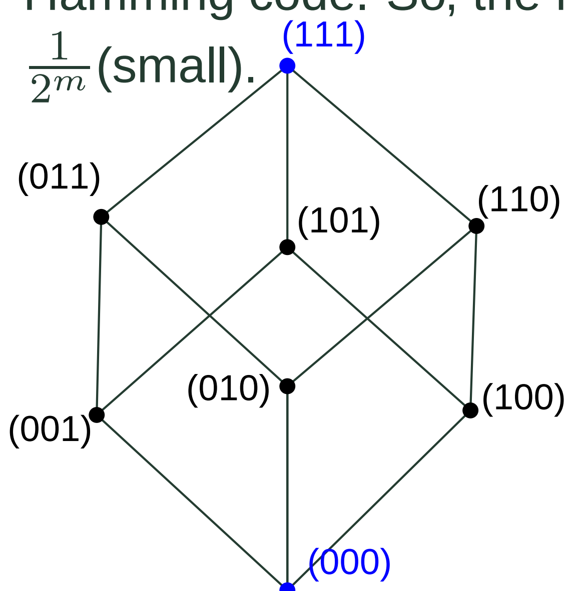

Justification

We must notice several things regarding our game and $\{0,1\}^n$:

- For 2 colors every vertex of $\{0,1\}^n$ coincides with a one of our possible hat configurations if we denote black as 0 and red as 1.
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- Every player corresponds to a unique edge of $\{0,1\}^n$.
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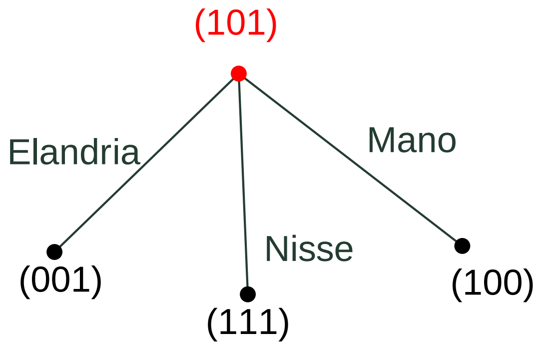
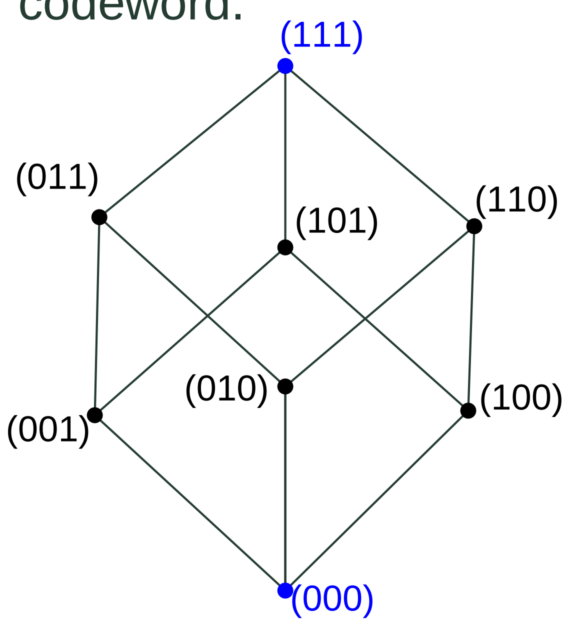
  –example: If 101 is our configuration then Elandria$\sim$ $(001, 101)$, Nisse$\sim$ $(101, 111)$, and Mano$\sim$ $(100, 101)$.

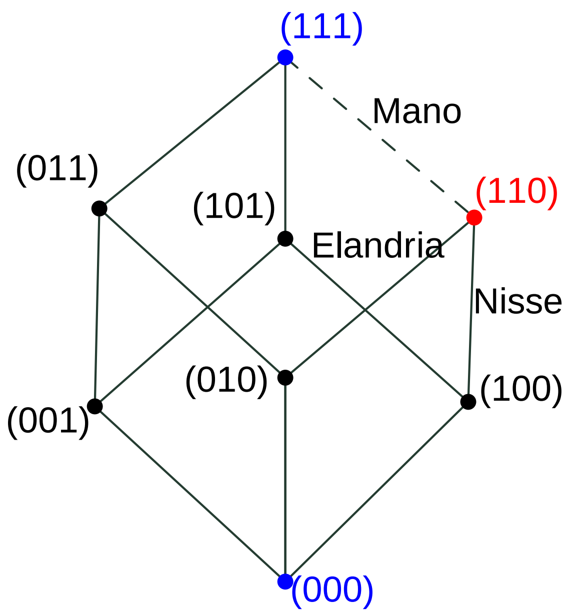


For Ebert's game we consider the hypercube $\{0, 1\}^3$ shown.
The codewords here are 000 and 111. Our strategy is equivalent to any player who sees a possible codeword to guess the color that would make the configuration not a codeword. Then the following are true,

- If our configuration is a codeword everyone will guess incorrectly.
- If our configuration is not a codeword then only the player who see a possible codeword will guess (correctly). There is only one player who see a possible codeword as every vertex is adjacent to only one codeword.



Consider a sample game with hat configuration RRB we can see that only Mano sees a possible codeword $(1, 1, 1)$, so only Mano would guess and he would guess the opposite of $1 =$Red, i.e. $0 =$ Black and be correct, freeing our captives!



Majority Hat Game

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- No player can see their own hat color but each person can see the hat color of every other person.
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- Players will all guess simultaneously.

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What is the best strategy so the gnomes have the highest probability to survive?

The Strategy

Let the players again be Elandria, Nisse and Mano. Let Elandria pick the opposite color of Nisse's hat, Nisse pick the opposite color of Mano's hat and Mano pick the opposite color of Elandria's hat.

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Then the following table shows all possible configurations of hat colors and the outcome of each game using this strategy.

$configuration$	$Elandria$	$Nisse$	$Mano$	$outcome$
BBB	R	R	R	$lose$
BBR	R	B	R	win
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Here again we see a high probability of winning, due to a concentration of multiple incorrect guesses into just 2 cases (the top and bottom rows). Indeed, we get a $\frac{3}{4} = 75\%$ chance of winning.

Justification

The strategy above can be generalized to any number of players n, for which a Hamming code exists for the n-dimensional hypercube.

The following justification, due to Elwyn Berlekamp, is based on a clever orientation of the edges of the hypercube $\{0,1\}^n$ in the case of $n = 2^m - 1$ players.

Recall from before, that every configuration is associated to a vertex and every player to a unique edge adjacent to the configuration vertex.

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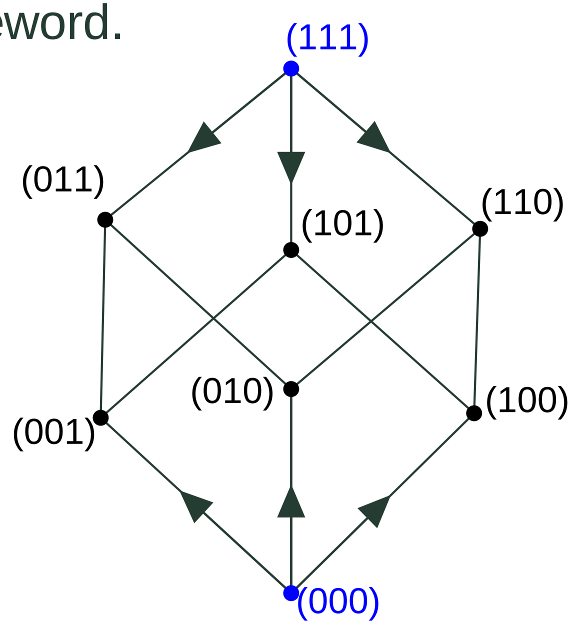

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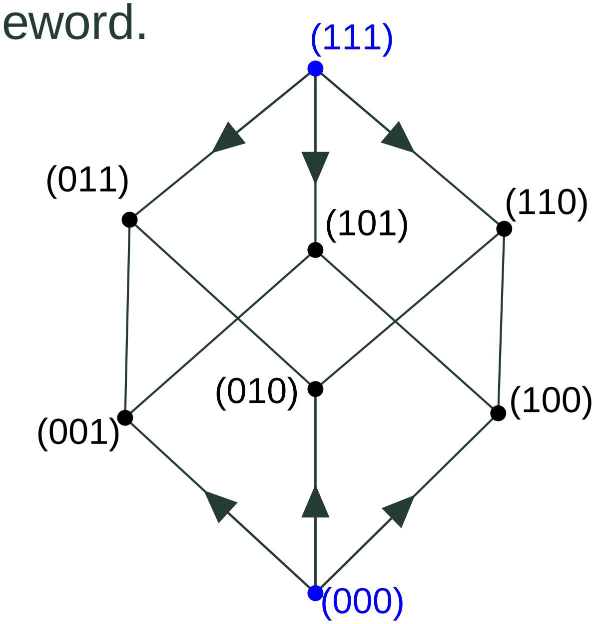

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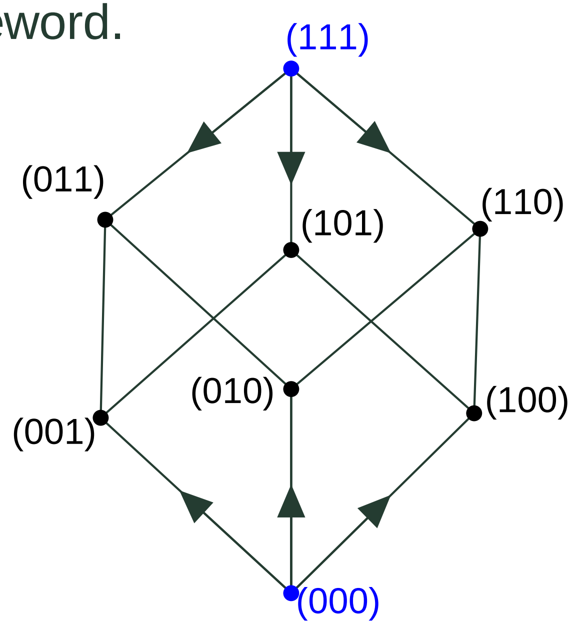

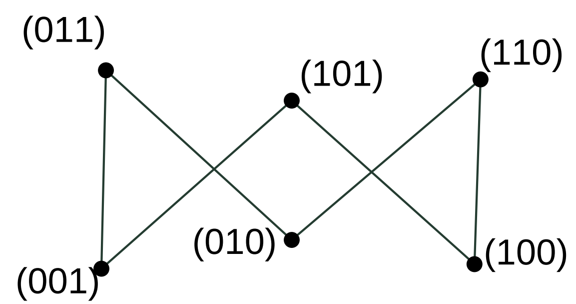

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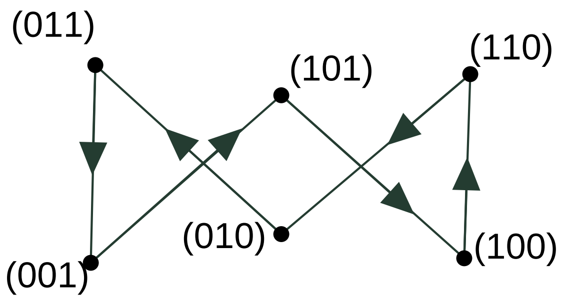
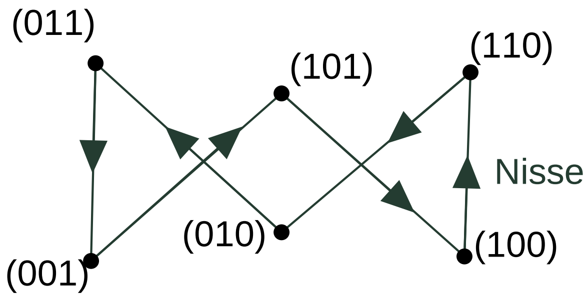
Since every vertex has even degree, the remainder of the edges can be oriented along an Eulerian circuit. (If it is not connected, direct the connected components to get a disjoint union of Eulerian cycles.)



Then each prisoner will guess the color corresponding to their position on the vertex their edge is directed towards. For example, in



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Each non-codeword vertex has 2^{m-1} edges directed towards it and $2^{m-1} - 1$ edges directed away from it. This is the case since half of the $2^m - 2$ edges from the Eulerian cycle are directed towards it and half away (that is $2^{m-1} - 1$ each direction) and has the 1 edge that connects it to a codeword directed towards it. This implies the majority of the prisoners guess correct.

Justification

With the orientations described we can see:

Each non-codeword vertex has 2^{m-1} edges directed towards it and $2^{m-1}-1$ edges directed away from it. This is the case since half of the 2^m-2 edges from the Eulerian cycle are directed towards it and half away (that is $2^{m-1}-1$ each direction) and has the 1 edge that connects it to a codeword directed towards it. This implies the majority of the prisoners guess correct.

Each codeword has 2^m-1 edges directed away from it and 0 directed towards it. This implies everyone guesses wrong and the gnomes die.

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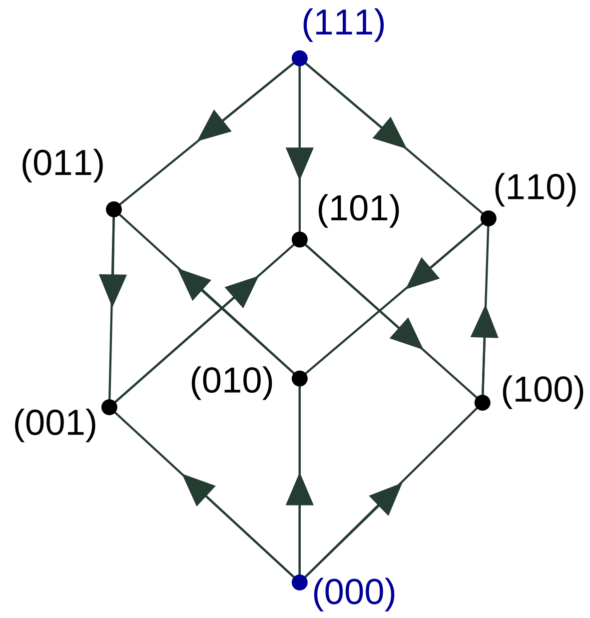
Each codeword has 2^m-1 edges directed away from it and 0 directed towards it. This implies everyone guesses wrong and the gnomes die.

Thus as long as our configuration is not one of the2^{n-m} codewords, the gnomes live!

Side note: They win with probability $\frac{2^n-2^{n-m}}{2^n}=\frac{2^m-1}{2^m}$.

Example game

For a sample game to illustrate this strategy, let us look at the 3-player game with hat configuration RRB. Letting B be denoted by 0, and R by 1, we get the 110 vertex of $\{0, 1\}^3$. Then,



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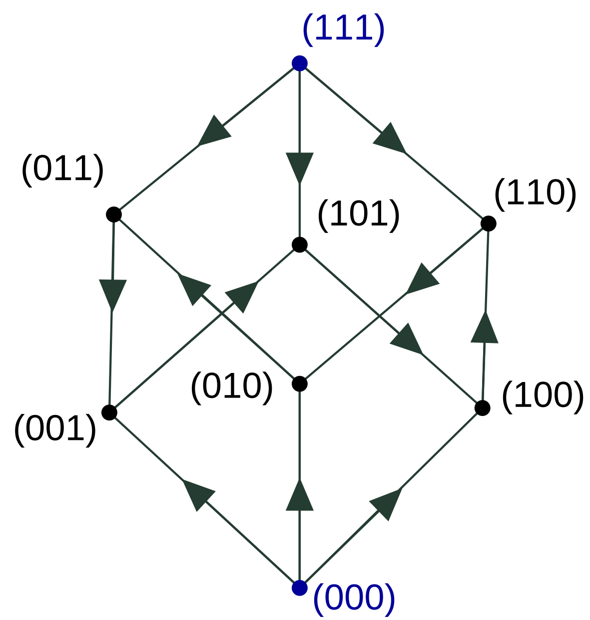

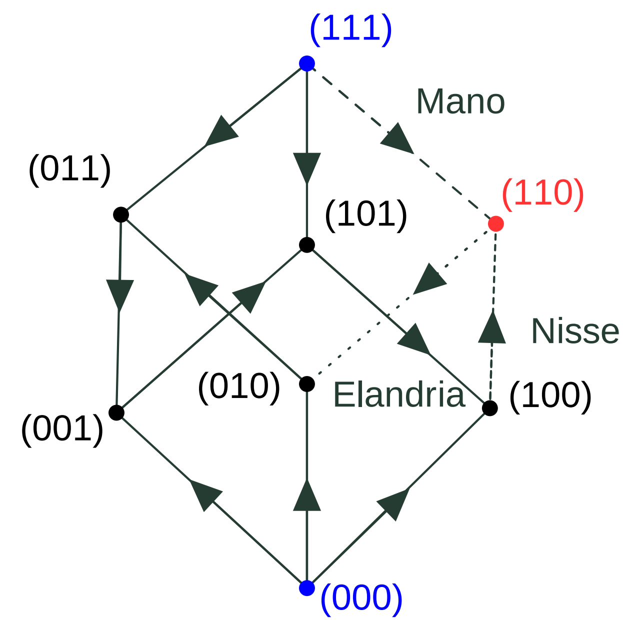

and the guess configuration would be BRB for a win.

Hats-on-a-line Game

This next game is known as the 'Hats on a line' game. For this game the jailer lines n gnomes up in a single file line. For ease, we denote gnome i by G_i and we let G_1 be the gnome at the back of the line, and G_n be the gnome at the front of the line. Each gnome has a hat placed on their head by the evil jailer. The hats can be one of two possible colors (black or red) and are chosen based on the outcome of a flip of a fair coin. The rules are as follows:

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- Each gnome must make a guess (no passing).

For this game any gnome who guesses wrong will be executed. What is the best strategy so the highest number of gnomes survive?

The Strategy

The following strategy is optimal:
Prisoner G_1 will give the others the sum/parity of their hat colors by guessing for his hat color the sum of their hat colors (with black$= 0$ and red$= 1$) $mod 2$. Then each consecutive player will use this total and subtract the sum of the hat colors they see and the sum of the hat colors that have been guessed from this total. The number they are left with taken $mod\ 2$ will be their hat color guess.

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This clever strategy uses nothing but modular arithmetic and the auditory advantages allowed by the game to guarantee everyone (but possibly prisoner G_1) wins and keeps G_1's chances of winning at $\frac{1}{q}$. The gnomes just better hope that G_1 will agree to their strategy, everyone is paying attention and everyone can add mod q, otherwise they are all in trouble!

Formally this is stated as Letting black be 0, red be 1 and c_i be the color of G_i's hat. Then G_1 guesses

$$g_1 = \sum_{i=2}^{n} c_i (\text{mod } 2)$$

(which has a 50% chance of being right). Each additional prisoner will then guess

$$g_j = g_1 - \sum_{i \in \{2,\ldots,n\} \setminus \{j\}} c_i \text{ (mod 2)}$$

which will always be correct (i.e. $g_2 = c_2$).

To see how this works, consider 5 prisoners. Suppose that the configuration of hats is $BBRBR$.

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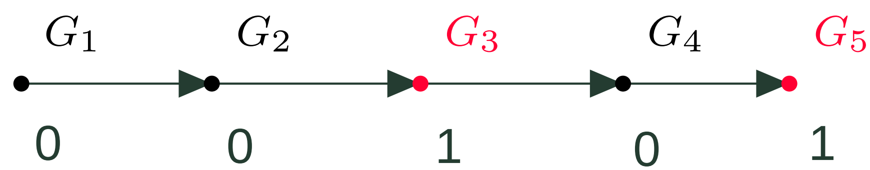

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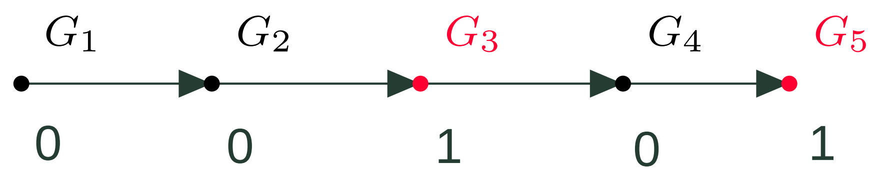

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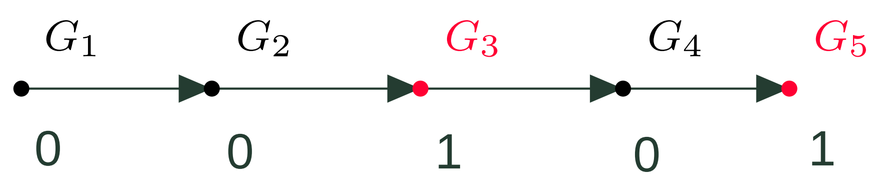

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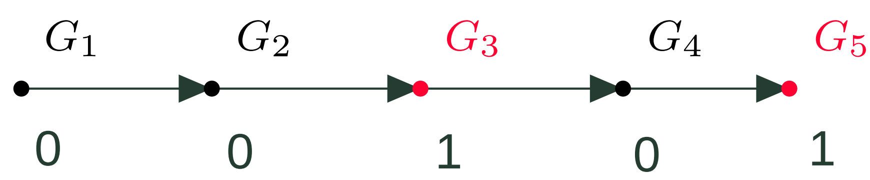

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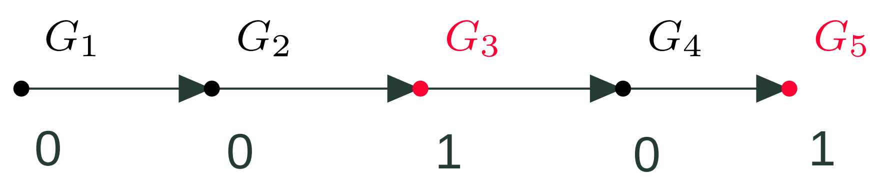

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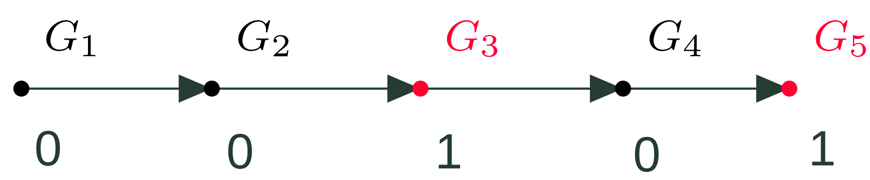

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This configuration has every gnome surviving. Notice that $RBRBR$ would have lead to the same process but G_1 would have been executed.

New Hats-on-a-Line Game

History

Finally, we come to the main game. This final game is a combination of Ebert's hat game and the Hats-on-a-line game and was proposed by Maura Paterson and Douglas Stinson in *Yet Another Hat Game* published in The Electronic Journal of Combinatorics in 2010.

The Game/Rules

For this game the jailer lines n gnomes up in a single file line. For ease, we denote gnome i by G_i and we let G_1 be the gnome at the back of the line, and G_n be the gnome at the front of the line. Each prisoner has a hat of any one of $q \geq 2$ colors placed on their head, with each color having even probability of being picked. The rules are as follows:

- No prisoner can see their own hat color.
- Gnome G_i can see the hat color of gnome G_j for all $j > i$. Note that G_1 can see everyone's hat color (except his own).
- Prisoners will guess in sequential order ($G_1, G_2, G_3, \ldots$, etc.), and all gnomes can hear the guesses made before them.
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All gnomes are pardoned if at least one gnome guesses correctly and no gnome guesses incorrectly, otherwise they are all executed.
What is the best strategy so the gnomes have the highest probability to survive?

We will call the solution to this game 'The Strategy' which is defined as follows:

Let black be one of the q hat colors. For each gnome G_i *for* $1 \leq i \leq n$*, if* G_i *sees a black hat then they pass, otherwise they guess black. In other words, the first gnome not to see a black hat guesses black. After anyone has guessed black, the rest of the prisoners will pass.*

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$$\text{Pr}(G_1 \text{ guesses correct}) \times \text{ Pr}(G_1 \text{ sees no black hats}) + \text{Pr(group wins if } G_1 \text{ pass)} \times \text{ Pr}(G_1 \text{ pass}).$$

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  –This is a contradiction proof. Assume a non-restricted strategy $\mathcal{S}$ is optimal. Modify $\mathcal{S}$ into a restricted strategy $\mathcal{S}'$ (based on G_1's knowledge of $\mathcal{S}$). Show $\mathcal{S}'$ has a higher probability of winning. Contradicts $\mathcal{S}$ was optimal.

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Thus, 'The Strategy' is an optimal strategy, since it is restricted (implying it can be optimal) and has the same probability of winning as the maximum probability for any restricted strategy.

Example Game

4 gnome, 2 hat color example

$Configuration$	gnome who guesses	gnome's guess
$BBBB$	4	B
$BBBR$	3	B
$BBRB$	4	B
$BRBB$	4	B
$RBBB$	4	B
$BBRR$	2	B
$BRBR$	3	B
$BRRB$	4	B
$RBBR$	3	B
$RBRB$	4	B
$RRBB$	4	B
$RRRB$	4	B
$RRBR$	3	B
$RBRR$	2	B
$BRRR$	1	B
$RRRR$	1	B

Example continued

Notice that the only wrong guess here is the $RRRR$ configuration. So this game wins with this strategy with probability $\frac{15}{16} = 93.75\%$.

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'The Strategy' probability for $q = 2$ colors is

$$1 - \left(\frac{2-1}{2}\right)^4 = 1 - \frac{1}{2^4} = 1 - 1\frac{1}{16} = \frac{15}{16}.$$

Conclusion

Summary

- There is not one approach to every type of hat game due to all the possible variations. We saw several different strategy/solution types.
- Theses games could help develop theory for many different branches of mathematics.
- Some of these games may not seem to have relevant applications however, that does not mean they don't! Even if their only application is to let people enjoy a good puzzle and use their brains!
- We can leave today knowing we saved as many gnomes as possible within each game!

The End